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GENERALIZED BERNOULLI POLYNOMIALS REVISITED AND SOME OTHER APPELL SEQUENCES

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Abstract. We study the problem posed by Nörlund in terms of dual sequences. We determine the functional equation fulfilled by the canonical form of any generalized Bernoulli sequence. Surprisingly these canonical forms are positive definite. Some results are given for an Euler sequence.

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Introduction

The Appell polynomials and their generalization were studied by many authors [7]–[12], [18]–[20], [22]. At the present time there are papers dealing with Appell polynomials related to quadrature rules [2], [3], some generalizations of Appell polynomials [4], [6] and Bernoulli polynomials [5]. The present paper deals with some Appell (in particular Bernoulli and Euler) sequences in terms of dual sequences. In fact we study the following problem:

Let $\{C_n\}_{n\geq 0}$ be an Appell sequence, determine all sequences $\{B_n\}_{n\geq 0}$ such that

$$\begin{aligned} \left(\frac{B_{n+1}}{n+1}\right)'(x) &= B_n(x), \quad n \geq 0, \\ \left(D_1 \frac{B_{n+1}}{n+1}\right)(x) &= C_n(x), \quad n \geq 0, \end{aligned} \tag{1}$$

with D_ω is the Hahn operator defined by

$$(D_\omega f)(x) := \frac{f(x+\omega) - f(x)}{\omega}, \quad \omega \neq 0, \quad f \in \mathcal{P}.$$

When $C_n(x) = B_n(x;k)$ (generalized Bernoulli polynomial), $k \geq 0$, with $B_n(x;0) = x^n$, this problem was studied by Nörlund [18]. In this paper we give new results concerning the dual sequence of generalized Bernoulli sequences. In particular, it is shown that the canonical form $u_0(k)$ of a generalized Bernoulli sequence of order $k \geq 1$ satisfies a homogeneous kth-order linear equation with polynomial coefficients and is a positive definite form. Similar results are obtained when $C_n(x) = \hat{H}_n$ (monic Hermite polynomial). Analogous considerations are possible if, in (1), we take T_1 instead of D_1 where T_ω is defined by

$$(T_\omega f)(x) := \frac{f(x+\omega) + f(x)}{2}, \quad f \in \mathcal{P}.$$

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In this case, for $C_n(x) = E_n(x;k)$ (generalized Euler polynomial), $k \geq 0$ with $E_n(x;0) = x^n$, we show that the canonical form $e_0(k)$ of the generalized Euler sequence of order k is not regular for $k \geq 0$.

Section 1 contains the material of a preliminary and introductory character. In Section 2 we study the Bernoulli sequence. Section 3 deals with the general case. I Section 4 we study the Nörlund case in terms of dual sequences and the case where $C_n(x) = \hat{H}_n(x)$. In Section 5 we study the Euler sequence and we determine the canonical form $e_0(k)$ of a generalized Euler sequence of order k.

1. Preliminaries and the Notation

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$, and $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, the moments of u. For any form u, any polynomial h, we let hu and u' be the forms defined by the duality

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle hu, f\rangle := \langle u, hf\rangle, \quad f \in \mathcal{P}.$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials, $\deg P_n = n$, $n \geq 0$ (monic polynomial sequence: MPS) and $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}$ defined by $\langle u_n, P_m\rangle := \delta_{n,m}$, $n, m \geq 0$.

The dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]}(x) := (n+1)^{-1} P'_{n+1}(x)$, $n \geq 0$, is given by [14], [15], [17].

$$(u_n^{[1]})' = -(n+1) u_{n+1}, \quad n \geq 0. \tag{1.1}$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\tilde{P}_n\}_{n\geq 0}$ with $\tilde{P}_n(x) := a^{-n} P_n(ax+b)$, $n \geq 0$, $a \neq 0$ is given by [14], [15], [16]

$$\tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n, \quad n \geq 0, \tag{1.2}$$

where

$$\begin{aligned} \langle \tau_{-b} u, f\rangle &:= \langle u, \tau_b f\rangle = \langle u, f(x-b)\rangle, \quad u \in \mathcal{P}', \ f \in \mathcal{P}, \ b \in \mathbb{C}, \\ \langle h_a u, f\rangle &:= \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \quad u \in \mathcal{P}', \ f \in \mathcal{P}, \ a \in \mathbb{C} - \{0\}. \end{aligned}$$

A form u is called regular if we can associate with it a polynomial sequence $\{P_n\}_{n\geq 0}$ such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m}, \quad n, m \geq 0; \quad r_n \neq 0, \quad n \geq 0.$$

This sequence is orthogonal with respect to u. Since we have $u = \lambda u_0$, $\lambda \neq 0$, and $\deg P_n = n$, $n \geq 0$, it is always possible to suppose that $\{P_n\}_{n\geq 0}$ is a (MPS); $\{P_n\}_{n\geq 0}$ is unique and satisfies the recurrence relation

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1} - \gamma_{n+1} P_n(x), \quad \gamma_{n+1} \neq 0, \quad n \geq 0. \end{cases} \tag{1.3}$$

Let ϕ and ψ be two monic polynomials, $\deg \phi = t$, $\deg \psi = p \geq 1$. Writing $\psi = a_p x^p + \cdots$ we say the pair (ϕ, ψ) is admissible; in the case $p = t-1$, a_p is not a positive integer.

Definition 1.1 ([14], [17]). A form u is called semi-classical when it is regular and satisfies the equation

$$(\phi u)' + \psi u = 0, \tag{1.4}$$

where the pair (ϕ, ψ) is admissible. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called semi-classical.

Remark 1.2 ([14], [16], [17]). When $\deg\phi \leq 2$ and $\deg\psi = 1$, u is called a classical form.

Lemma 1.3. *Consider the sequence $\{\tilde{P}_n\}_{n\geq 0}$ obtained by shifting P_n, i.e., $\tilde{P}_n(x) = a^{-n}P_n(ax+b)$, $n \geq 0$, $a \neq 0$. When u_0 satisfies (1.4), then $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ fulfils the following equation*

$$\tilde{\phi}\tilde{u}_0 + \tilde{\psi}\tilde{u}_0 = 0, \tag{1.5}$$

where $\tilde{\phi}(x) = a^{-\deg(\phi)}\phi(ax+b)$, $\tilde{\psi}(x) = a^{1-\deg\phi}\psi(ax+b)$.

Definition 1.4. A sequence $\{P_n\}_{n\geq 0}$ is called an Appell sequence when $P_n^{[1]}(x) = P_n(x)$, $n \geq 0$.

Lemma 1.5 ([16]). *Let $\{P_n\}_{n\geq 0}$ be a MPS and let $\{u_n\}_{n\geq 0}$ be its dual sequence, then $\{P_n\}_{n\geq 0}$ is an Appell sequence if and only if*

$$u_n = \frac{(-1)^n}{n!}u_0^{(n)}, \quad n \geq 0, \tag{1.6}$$

where $u_0^{(n)}$ is the nth derivative of u_0.

Lemma 1.6 ([7], [16], [20]). *If $\{P_n\}_{n\geq 0}$ is an Appell sequence, then $\{P_n\}_{n\geq 0}$ is orthogonal if and only if P_n is obtained by shifting $\hat{H}_n$, where $\{\hat{H}_n\}_{n\geq 0}$ is a monic Hermite sequence.*

Finally, we introduce the Hahn's operator

$$\big(D_\omega f\big)(x) := \frac{f(x+\omega) - f(x)}{\omega}, \quad f \in \mathcal{P}, \quad \omega \neq 0.$$

We have $D_\omega = \frac{1}{\omega}(\tau_\omega - I_{\mathcal{P}})$ where $I_{\mathcal{P}}$ is the identity operator in $\mathcal{P}$. The transposed ${}^tD_\omega$ of D_ω is ${}^tD_\omega = \frac{1}{\omega}(\tau_\omega - I_{\mathcal{P}'}) = -D_{-\omega}$.

2. Connection between the Legendre Sequence and the Bernoulli Sequence

Let $\{B_n\}_{n\geq 0}$ be the Bernoulli sequence and $\{u_n\}_{n\geq 0}$ be its dual sequence.

Definition 2.1 ([9], [18], [19]). The Bernoulli sequence $\{B_n\}_{n\geq 0}$ is defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}. \tag{2.1}$$

Lemma 2.2. *The following relations define the Bernoulli sequence*

$$B_n^{[1]}(x) = B_n(x), \quad n \geq 0, \tag{2.2}$$

$$D_1\frac{B_{n+1}}{n+1}(x) = x^n, \quad n \geq 0. \tag{2.3}$$

Proof. By virtue of the relation (2.1) we get relations (2.2)–(2.3) and vice versa. □

Lemma 2.3. *The canonical form u_0 of Bernoulli sequence satisfies*

$$(u_0)' = D_{-1}\delta = \delta - \delta_1. \tag{2.4}$$

Proof. On account of relation (2.2), by Lemma 1.6 we get

$$u_n = \frac{(-1)^n}{n!}u_0^{(n)}, \quad n \geq 0. \tag{2.5}$$

Let $\{w_n\}_{n\geq 0}$ be the dual sequence of $\{x^n\}_{n\geq 0}$. We have

$$w_n = \frac{(-1)^n}{n!}\delta^{(n)}, \quad n \geq 0. \tag{2.6}$$

Taking relation (2.3) and the definition of $\{w_n\}_{n\geq 0}$ into account, we obtain

$$\langle w_n, D_1B_{m+1}\rangle = (m+1)\delta_{n,m}, \quad n, m \geq 0.$$

Therefore

$$\langle D_{-1}w_n, B_m\rangle = 0, \quad m \geq n+2,$$
$$\langle D_{-1}w_n, B_{n+1}\rangle = -(n+1).$$

By virtue of Lemma 1.1

$$D_{-1}w_n = \sum_{\nu=0}^{n+1}\lambda_{n,\nu}u_\nu, \quad n \geq 0.$$

But $\langle D_{-1}w_n,\ B_\mu\rangle = \lambda_{n,\mu}$, $0 \leq \mu \leq n+1$ and $\lambda_{n,\mu} = 0$, $0 \leq \mu \leq n$, $\lambda_{n,n} = -(n+1)$, $n \geq 0$.

Hence

$$D_{-1}w_n = -(n+1)u_{n+1}, \quad n \geq 0. \tag{2.7}$$

Setting $n = 0$ in this equation, we get $D_{-1}w_0 = -u_1$. Now from (2.5) and (2.6) we can deduce (2.4). □

Proposition 2.4. *The form u_0 satisfies the following properties*

$$\big(\phi u_0\big)' + \psi u_0 = 0, \tag{2.8}$$

with $\phi(x) = x(x-1)$, $\psi(x) = -(2x-1)$.

$$u_0 = \big(\tau_{\frac{1}{2}} \circ h_{\frac{1}{2}}\big)\mathcal{L}, \tag{2.9}$$

where $\mathcal{L}$ is the Legendre form.

$$\langle u_0, f\rangle = \int_0^1 f(x)dx, \quad f \in \mathcal{P}. \tag{2.10}$$

Proof. We need the following results [13], [16].

The Legendre form $\mathcal{L}$ satisfies

$$\Big((x^2-1)\mathcal{L}\Big)'-2x\mathcal{L}=0, \tag{2.11}$$

$$\langle \mathcal{L}, f\rangle = \frac{1}{2}\int_{-1}^{1} f(x)dx, \quad f\in\mathcal{P}. \tag{2.12}$$

From Lemma 2.3 we have $(u_0)'=\delta-\delta_1$. Hence

$$x(x-1)(u_0)'=0, \tag{2.13}$$

from which we get (2.8).

Let $\tilde{u}=h_{a^{-1}}\circ\tau_{-b}u$. By virtue of Lemma 1.3, equation (2.8) becomes

$$(\tilde{\phi}\tilde{u})'+\tilde{\psi}\tilde{u}=0,$$

where $\tilde{\phi}(x)=(x+a^{-1}b)(x+a^{-1}b-a^{-1})$, $\tilde{\psi}(x)=-(2x+2a^{-1}b-a^{-1})$. Choosing $a=b=\frac{1}{2}$ we get $\tilde{u}=\mathcal{L}$. Hence follows (2.9).

Relations (2.9) and (2.12) imply (2.10). □

Remark 2.5. Denote

$$\hat{B}_n(x)=2^nB_n\left(\frac{x+1}{2}\right), \quad n\geq 0. \tag{2.14}$$

Then $\{\hat{B}_n\}_{n\geq 0}$ is the symmetric Bernoulli sequence satisfying

$$\frac{te^{tx}}{\sinh t}=\sum_{n\geq 0}\hat{B}_n(x)\,\frac{t^n}{n!}\,.$$

We have

$$\hat{u}_0=h_2\circ\tau_{-1/2}u_0=\mathcal{L}. \tag{2.15}$$

Proposition 2.6 (c.f. [22])**.** *Let $\{L_n\}_{n\geq 0}$ be the monic orthogonal sequence of Legendre. The following formula holds:*

$$L_{n+1}(x)=\frac{\sqrt{\pi}}{2^{n+1}}\,\frac{\Gamma(n+2)}{(2n+1)\Gamma(n+\frac{1}{2})} \times\sum_{\nu=0}^{n}\frac{1+(-1)^{\nu+n}}{2^{\nu}\Gamma(\nu+1)\Gamma(\nu+2)}\,\frac{\Gamma(n+2+\nu)}{\Gamma(n+2-\nu)}\hat{B}_{\nu+1}(x), \quad n\geq 0. \tag{2.16}$$

Proof. We need the formulas [13], [16]

$$(x^2-1)L'_{n+1}(x)=(n+1)xL_{n+1}(x)-\frac{(n+1)^2}{2n+1}L_n(x), \quad n\geq 0, \tag{2.17}$$

$$(x^2-1)L''_{n+1}(x)+2xL'_{n+1}(x)-(n+1)(n+2)L_{n+1}(x)=0, \quad n\geq 0. \tag{2.18}$$

On account of (2.18) we can prove by induction the relation

$$(x^2-1)L^{(m+2)}_{n+1}(x)+2(m+1)xL^{(m+1)}_{n+1}(x) +(m^2+m-(n+1)(n+2))L^{(m)}_{n+1}(x)=0, \quad n,m\geq 0, \tag{2.19}$$

where $P^{(k)}$ is the kth derivative of $P\in\mathcal{P}$.

We can write

$$L_{n+1}(x)=\sum_{\nu=0}^{n+1}\lambda_{n,\nu}\hat{B}_\nu(x), \quad n\geq 0, \tag{2.20}$$

where $\lambda_{n,m}=\langle\hat{u}_m, L_{n+1}\rangle$, $0\leq m\leq n+1$. On account of (2.15) we get

$$\lambda_{n,m}=\frac{1}{m!}\langle\mathcal{L}, L^{(m)}_{n+1}\rangle, \quad 0\leq m\leq n+1. \tag{2.21}$$

Putting $m=0$ into (2.21), we get

$$\lambda_{n,0}=\langle\mathcal{L}, L_{n+1}\rangle=0, \quad n\geq 0. \tag{2.22}$$

Taking (2.12) into account, (2.21) can be written as

$$\lambda_{n,m+1}=\frac{1}{(m+1)!}\Big\{L^{(m)}_{n+1}(1)-L^{(m)}_{n+1}(-1)\Big\}, \quad 0\leq m\leq n. \tag{2.23}$$

Let $\varepsilon=\pm 1$, from (2.17) we get

$$\frac{L_{n+1}(\varepsilon)}{L_n(\varepsilon)}=\frac{1}{2\varepsilon}\frac{n+1}{n+\frac{1}{2}}, \quad n\geq 0,$$

hence

$$L_{n+1}(\varepsilon)=\frac{2\sqrt{\pi}}{(2\varepsilon)^{n+1}}\,\frac{\Gamma(n+2)}{(2n+1)\Gamma(n+\frac{1}{2})}, \quad n\geq 0. \tag{2.24}$$

By virtue of relation (2.19) we have

$$\frac{L^{(m+1)}(\varepsilon)}{L^{(m)}(\varepsilon)}=-\frac{(m-n-1)(m+n+2)}{2\varepsilon(m+1)}, \quad 0\leq m\leq n+1,$$

which implies

$$L^{(m)}_{n+1}(\varepsilon)=\frac{1}{(2\varepsilon)^m\Gamma(m+1)}\,\frac{\Gamma(n+2+m)}{\Gamma(n+2-m)}\,L_{n+1}(\varepsilon), \quad 0\leq m\leq n+1.$$

On account of (2.24) we obtain

$$L^{(m)}_{n+1}(\varepsilon)=\frac{2\sqrt{\pi}}{(2n+1)(2\varepsilon)^{m+n+1}}\,\frac{\Gamma(n+2)}{\Gamma(m+1)}\,\frac{\Gamma(n+2+m)}{\Gamma(n+2-m)\Gamma(n+\frac{1}{2})}, \quad 0\leq m\leq n+1. \tag{2.25}$$

Finally, from relations (2.22) and (2.25) we can deduce (2.16). □

3. General Results

Let $\{B_n\}_{n\geq 0}$ be a MPS and $\{u_n\}_{n\geq 0}$ be its dual sequence.

Lemma 3.1. *Let $\{v_n\}_{n\geq 0}$ be the dual sequence of $\big\{D_1\frac{B_{n+1}}{n+1}\big\}_{n\geq 0}$. Then we have*

i) *The dual sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ satisfy the relation*

$$D_{-1}v_n=-(n+1)u_{n+1}, \quad n\geq 0. \tag{3.1}$$

ii) *If* $\{B_n\}_{n\geq 0}$ *is an Appell sequence, then* $\{D_1 \frac{B_{n+1}}{n+1}\}_{n\geq 0}$ *is an Appell one and we have the functional relation*

$$D_{-1}v_0 = (u_0)'. \tag{3.2}$$

Proof. i) The proof of (3.1) is similar to that of (2.7).

ii) Indeed, from the definition $D_1 \frac{B_{n+2}}{n+2}(x) = \frac{B_{n+2}(x+1)-B_{n+2}(x)}{n+2}$, $n \geq 0$, differentiating its both sides we obtain

$$\left(D_1 \frac{B_{n+2}}{n+2}\right)'(x) = \frac{B'_{n+2}(x+1) - B'_{n+2}(x)}{n+2}, \quad n \geq 0,$$

but $\{B_n\}_{n\geq 0}$ is an Appell sequence and thus we have

$$\left(D_1 \frac{B_{n+2}}{n+2}\right)'(x) = B_{n+1}(x+1) - B_{n+1}(x), \quad n \geq 0,$$

which implies

$$\left(D_1 \frac{B_{n+2}}{n+2}\right)^{[1]}(x) = D_1 \frac{B_{n+1}}{n+1}, \quad n \geq 0.$$

Therefore $\{D_1 \frac{B_{n+1}}{n+1}\}_{n\geq 0}$ is an Appell sequence.

Since $\{D_1 \frac{B_{n+1}}{n+1}\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ are two Appell sequences, by virtue of (1.6) we get

$$v_n = \frac{(-1)^n}{n!} v_0^{(n)}, \quad u_n = \frac{(-1)^n}{n!} u_0^{(n)}, \quad n \geq 0.$$

On account of (3.1) we infer that

$$D_{-1}\left(v_0^{(n)}\right) = u_0^{(n+1)}, \quad n \geq 0.$$

Setting $n = 0$ in the latter relation we obtain (3.2). □

Proposition 3.2. *If* $\{C_n\}_{n\geq 0}$ *is an Appell sequence, than there exists a unique sequence, then* $\{B_n\}_{n\geq 0}$ *satisfying*

$$B_n^{[1]}(x) = B_n(x), \quad n \geq 0, \tag{3.3}$$

$$D_1 \frac{B_{n+1}}{n+1}(x) = C_n(x), \quad n \geq 0. \tag{3.4}$$

Proof. Let $\{B_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two sequences satisfying (3.3) and (3.4). The sequence $\{R_n\}_{n\geq 0}$ defined by $R_n(x) = B_n(x) - Q_n(x)$, $n \geq 0$, satisfies

$$R_0(x) = 0, \tag{3.5}$$

$$\frac{R'_{n+1}}{n+1}(x) = R_n(x), \quad n \geq 0, \tag{3.6}$$

$$R_{n+1}(x+1) = R_{n+1}(x), \quad n \geq 0. \tag{3.7}$$

We will prove by induction that $R_n(x) = 0, \quad n \geq 0.$

Suppose

$$R_\nu(x) = 0 \quad \text{for} \quad 0 \leq \nu \leq n.$$

From (3.6) we can deduce that $R_{n+1}(x) = a_{n+1}$.

So $\frac{R'_{n+2}}{n+2}(x) = R_{n+1}(x) = a_{n+1}$, then

$$R_{n+2}(x) = (n+2)a_{n+1}x + a_{n+2}.$$

On account of (3.7) we get

$$(n+2)a_{n+1}(x+1) + a_{n+2} = (n+2)a_{n+1}x + a_{n+2}.$$

Now we obtain $a_{n+1} = 0$, which implies $R_{n+1}(x) = 0$. The uniqueness is proved.

For the existence we introduce the sequence $\{B_n\}_{n\geq 0}$ defined by the recurrence formulas

$$B_0(x) = 1, \tag{3.8}$$

$$B_n(x) = n\int_0^x B_{n-1}(t)dt - n\int_0^1\int_0^x B_{n-1}(t)dt\,dx + C_n(0), \quad n \geq 1. \tag{3.9}$$

From relations (3.8) and (3.9) we can deduce that $\{B_n\}_{n\geq 0}$ satisfies (3.3).

We will prove by induction that $\{B_n\}_{n\geq 0}$ satisfies (3.4).

From (3.9) we get $B_1(x) = x - \frac{1}{2} + C_1(0)$, which implies $D_1B_1(x) = 1 = C_0(x)$.

Suppose

$$D_1B_n(x) = nC_{n-1}(x). \tag{3.10}$$

By relation (3.9) we have

$$D_1 \frac{B_{n+1}}{n+1}(x) = \int_0^{x+1} B_n(t)dt - \int_0^x B_n(t)dt = \int_1^{x+1} B_n(t)dt - \int_0^x B_n(t)dt$$
$$+ \int_0^1 B_n(t)dt = \int_0^x \left(D_1B_n\right)(t)dt + \int_0^1 B_n(t)dt.$$

Using (3.10) we get $D_1B_n(t) = nC_{n-1}(t)$, and on account of (3.9) we obtain $\int_0^1 B_n(t)dt = C_n(0)$, whence

$$D_1 \frac{B_{n+1}}{n+1}(x) = n\int_0^x C_{n-1}(t)dt + C_n(0),$$

but $\{C_n\}_{n\geq 0}$ is an Appell sequence, so $C'_n(t) = nC_{n-1}(t)$ and

$$D_1 \frac{B_{n+1}}{n+1}(x) = \int_0^x C'_n(t)dt + C_n(0) = C_n(x).$$

Hence follows the desired result. □

Proposition 3.3. *If $\{C_n\}_{n\geq 0}$ is an Appell sequence defined by the generating function*

$$G(x,t)=C(t)e^{tx}=\sum_{n\geq 0}C_n(x)\frac{t^n}{n!},\tag{3.11}$$

then the sequence $\{B_n\}_{n\geq 0}$ defined by (3.3)–(3.4) has the following generating function

$$F(x,t)=\frac{tC(t)}{e^t-1}e^{tx}=\sum_{n\geq 0}B_n(x)\frac{t^n}{n!}.\tag{3.12}$$

Proof. Let

$$F(x,t)=A(t)e^{tx}=\sum_{n\geq 0}B_n(x)\frac{t^n}{n!}.$$

We have

$$F(x+1,t)-F(x,t)=A(t)(e^t-1)e^{tx}=t\sum_{n\geq 0}D_1\frac{B_{n+1}}{n+1}(x)\frac{t^n}{n!}.$$

On account of (3.4) this equation can be written as

$$A(t)e^{tx}(e^t-1)=t\sum_{n\geq 0}C_n(x)\frac{t^n}{n!}.$$

By relation (3.11) we can deduce that $A(t)e^{tx}(e^t-1)=tC(t)e^{tx}$. Hence follows the desired result. □

4. The Particular Cases

$\mathbf{A}_1$. Nörlund case. We denote by $\{B_n(.;1)\}_{n\geq 0}$ the Bernoulli sequence and assume that $\{u_n(1)\}_{n\geq 0}$ is its dual sequence.

Definition 4.1 ([18]). The polynomial sequence $\{B_n(x;2)\}_{n\geq 0}$ defined by

$$B_n^{[1]}(x;2)=B_n(x;2),\quad n\geq 0,\tag{4.1}$$

$$D_1\frac{B_{n+1}(x;2)}{n+1}=B_n(x;1),\quad n\geq 0,\tag{4.2}$$

is called the generalized Bernoulli sequence of order 2.

Lemma 4.2. *The polynomial sequence $\{B_n(x;2)\}_{n\geq 0}$ has the following generating function*

$$\left(\frac{t}{e^t-1}\right)^2e^{xt}=\sum_{n\geq 0}B_n(x;2)\frac{t^n}{n!}.\tag{4.3}$$

Proof. Taking relations (4.1)–(4.2) and (2.1) into account, Proposition 3.3 implies (4.3). □

Denote by $\{u_n(2)\}_{n\geq 0}$ the dual sequence of $\{B_n(.;2)\}_{n\geq 0}$.

Proposition 4.3. *Form $u_0(2)$ satisfies the properties*

$$\Big(\big(\phi u_0(2)\big)'+\psi u_0(2)\Big)'+\chi u_0(2)=0,\tag{4.4}$$

with $\phi(x)=x(x-1)(x-2)$, $\psi(x)=-2(3x^2-6x+2)$, $\chi(x)=6(x-1)$,

$$\langle u_0(2),f\rangle=\int_0^1 xf(x)dx+\int_1^2(2-x)f(x)dx,\quad f\in\mathcal{P}.\tag{4.5}$$

Proof. On account of (4.1) and (4.2), relation (3.2) of Lemma 3.1 implies

$$\big(u_0(2)\big)'=D_{-1}u_0(1)=u_0(1)-\tau_1u_0(1).\tag{4.6}$$

Differentiating this equation we obtain

$$\big(u_0(2)\big)''=\big(u_0(1)\big)'-\tau_1\big(u_0(1)\big)'.$$

From relation (2.4) we have $u_0(1)'=\delta-\delta_1$, then we get

$$\big(u_0(2)\big)''=\delta-2\delta_1+\delta_2.$$

This yields

$$x(x-1)(x-2)\Big(u_0(2)\Big)''=0.\tag{4.7}$$

We can evidently deduce (4.4).

Let

$$\langle u_0(2),f\rangle=\int_\alpha^\beta U(x)f(x)dx,\quad f\in\mathcal{P}.$$

Taking relations (4.6) and (2.10) into account we can deduce that

$$\int_\alpha^\beta U'(x)f(x)dx-[U(x)f(x)]_\alpha^\beta=\int_0^1 f(x)dx-\int_1^2 f(x)dx,\quad f\in\mathcal{P}.$$

Let us choose $\alpha=0$, $\beta=2$, then this condition can be written as

$$\int_0^2(U'(x)-W(x))f(x)dx=U(x)f(x)]_0^2,\quad f\in\mathcal{P},$$

with

$$W(x)=\begin{cases}1, & 0<x<1,\\ -1, & 1<x<2.\end{cases}$$

Setting

$$U(2)f(2)-U(0)f(0)=0,\quad f\in\mathcal{P},\tag{4.8}$$

we get

$$U'(x)-W(x)=0,$$

this yields

$$W(x)=\begin{cases}x+a, & 0\le x\le 1,\\ -x+b, & 1\le x\le 2,\end{cases}$$

and (4.8) gives $a=0$, $b=2$. Consequently $U(1-0)=U(1+0)$ and (4.5) holds. □

Corollary 4.4. *Let* $\hat{u}_0(2)=\tau_{-1}u_0(2)$, $\hat{u}_0(2)$ *satisfy*

$$\Big(\big(\hat{\phi}\hat{u}_0(2)\big)'+\hat{\psi}\hat{u}_0(2)\Big)'+\hat{\chi}\hat{u}_0(2)=0 \tag{4.9}$$

with $\hat{\phi}(x)=x(x^2-1)$, $\hat{\psi}(x)=-2(3x^2+5)$, $\hat{\chi}(x)=6x$,

$$\langle \hat{u}_0(2),f\rangle=\int_{-1}^{1}(1-|x|)f(x)dx,\quad f\in\mathcal{P}, \tag{4.10}$$

$$\big(\hat{u}_0(2)\big)_{2n}=\frac{1}{(n+1)(2n+1)},\quad \big(\hat{u}_0(2)\big)_{2n+1}=0,\quad n\ge 0. \tag{4.11}$$

Proof. The proof is evident from Proposition 4.3 and the definition of $\hat{u}_0(2)$. □

Remark 4.5.
1. The form $\hat{u}_0(2)$ is symmetric and positive definite.
2. The study of the sequences $\{B_n(.;1)\}_{n\ge 0}$, $\{B_n(.;2)\}_{n\ge 0}$ suggests the study of the Nörlund problem [18] in terms of dual sequences.

Definition 4.6 ([10], [11], [18], [19])**.** The polynomial sequence $\{B_n(.;k)\}_{n\ge 0}$, $k\ge 0$ defined by

$$B_n^{[1]}(x;k+1)=B_n(x;k+1),\quad n,k\ge 0, \tag{4.12}$$

$$D_1\frac{B_{n+1}(x;k+1)}{n+1}=B_n(x;k),\quad n,k\ge 0, \tag{4.13}$$

with $B_n(x;0)=x^n$, $n\ge 0$, is called the generalized Bernoulli sequence of order k.

Lemma 4.7. *The polynomials of the sequence* $\{B_n(.;k)\}_{n\ge 0}$ *have the following generating function*

$$\left(\frac{t}{e^t-1}\right)^k e^{xt}=\sum_{n\ge 0}B_n(x;k)\frac{t^n}{n!},\quad k\ge 0. \tag{4.14}$$

Proof. On account of relations (4.12), (4.13) and Proposition 3.3, we can deduce the desired result by induction with respect to k. □

Denote by $\{u_n(k)\}_{n\ge 0}$ the dual sequence of $\{B_n(.;k)\}_{n\ge 0}$, $k\ge 0$, so $u_0(0)=\delta$.

Proposition 4.8. *The forms* $\{u_0(k)\}_{k\ge 1}$ *satisfy the relations*

$$\big(u_0(k)\big)'=D_{-1}u_0(k-1),\quad k\ge 1. \tag{4.15}$$

$$\big(\phi_{k+1}u_0(k)\big)^{(k)}-\sum_{\nu=1}^{k}\binom{k}{\nu}\phi_{k+1}^{(\nu)}\big(u_0(k)\big)^{(k-\nu)}=0,\quad k\ge 1, \tag{4.16}$$

with $\phi_{k+1}(x)=\prod_{\nu=0}^{k}(x-\nu)$.

Proof. By virtue of relations (4.12), (4.13) and (3.2) of Lemma 3.1 we obtain (4.15).
Taking relation (4.15) into account, we can prove by induction with respect to m that

$$u_0(k)^{(m)}=D_{-1}^m u_0(k-m),\quad 1\le m\le k.$$

For $m=k$,

$$u_0(k)^{(k)}=D_{-1}^k u_0(0)=D_{-1}^k\delta,$$

and, obviously, $D_{-1}^k\delta=\sum_{\nu=0}^{k}\binom{k}{\nu}(-1)^\nu\delta_\nu$ [1], which implies $\prod_{\nu=0}^{k}(x-\nu)u_0(k)^{(k)}=0$, therefore we can deduce (4.16). □

Proposition 4.9. *For any* $k\ge 0$*, the form* $u_0(k+1)$ *is positive definite.*

First we will prove a lemma.

Lemma 4.10. *There exist polynomials* Ω_k^ν, $0\le\nu\le k$*, such that*

$$\langle u_0(k+1),f\rangle=\sum_{\nu=0}^{k}\int_{\nu}^{\nu+1}\Omega_k^\nu(x)f(x)dx,\quad f\in\mathcal{P},\quad k\ge 0, \tag{4.17}$$

with

$$\begin{cases}\Omega_0^0(x)=1,\\ \Omega_k^\nu(x)=\displaystyle\int_\nu^x\Omega_{k-1}^\nu(\xi)d\xi+\int_{x-1}^{\nu}\Omega_{k-1}^{\nu-1}(\xi)d\xi,\\ \qquad \nu\le x\le\nu+1,\quad 0\le\nu\le k-1,\\ \Omega_{k-1}^{-1}(x)=0,\quad k\ge 1,\\ \Omega_k^k(x)=\displaystyle\int_{x-1}^{k}\Omega_{k-1}^{k-1}(\xi)d\xi,\quad 1\le k\le x\le k+1.\end{cases} \tag{4.18}$$

Proof. For $k=0$, (4.17) is true. For $k\ge 1$, we have from (4.15)

$$(u_0(k+1))'=u_0(k)-\tau_1u_0(k),$$

therefore

$$\langle (u_0(k+1))',f\rangle=\langle u_0(k),f\rangle-\langle u_0(k),f(x+1)\rangle.$$

Supposing

$$\langle u_0(k), f\rangle = \int_0^k V_k(x) f(x)dx,$$

we get

$$- f(x)V_{k+1}(x)]_0^{k+1} + \int_0^{k+1} V'_{k+1}(x)f(x)dx$$
$$= \sum_{\nu=0}^{k-1}\int_\nu^{\nu+1} \Omega_{k-1}^{\nu}(x)f(x)dx - \sum_{\nu=0}^{k-1}\int_\nu^{\nu+1} \Omega_{k-1}^{\nu}(x)f(x+1)dx$$
$$= \int_0^1 \Omega_{k-1}^0(x)f(x)dx + \sum_{\nu=1}^{k-1}\int_\nu^{\nu+1}\Big\{\Omega_{k-1}^{\nu}(x) - \Omega_{k-1}^{\nu-1}(x-1)\Big\}f(x)dx$$
$$- \int_k^{k+1} \Omega_{k-1}^{k-1}(x-1)f(x)dx.$$

Putting

$$V_{k+1}(k+1)f(k+1) - V_{k+1}(0)f(0) = 0, \qquad (4.19)$$

we obtain

$$V'_{k+1}(x) - W_{k+1}(x) = 0, \text{a.e.},$$

with

$$W_{k+1}(x) = \begin{cases} \Omega_{k-1}^0(x), & 0 < x < 1, \\ \Omega_{k-1}^{\nu}(x) - \Omega_{k-1}^{\nu-1}(x-1), & \nu < x < \nu+1, \quad 1 \le \nu \le k-1, \\ -\Omega_{k-1}^{k-1}(x-1), & k < x < k+1. \end{cases}$$

Therefore

$$V_{k+1}(x) = \begin{cases} \int_0^x \Omega_{k-1}^0(\xi)d\xi + a_0(k), \quad 0 \le x \le 1, \\ \int_\nu^x \{\Omega_{k-1}^{\nu}(\xi) - \Omega_{k-1}^{\nu-1}(\xi-1)\}d\xi + a_\nu(k), \\ \qquad \nu \le x \le \nu+1, \quad 1 \le \nu \le k-1, \\ -\int_k^x \Omega_{k-1}^{k-1}(\xi-1)d\xi + a_k(k), \quad k \le x \le k+1. \end{cases}$$

But (4.19) implies $V_{k+1}(0) = 0$, $V_{k+1}(k+1) = 0$, consequently $a_0(k) = 0$ and

$$a_k(k) = \int_k^{k+1} \Omega_{k-1}^{k-1}(\xi-1)d\xi = \int_{k-1}^{k} \Omega_{k-1}^{k-1}(\xi)d\xi.$$

The continuity conditions $V_{k+1}(\nu-0) = V_{k+1}(\nu+0)$, $1 \le \nu \le k-1$ $(k \ge 2)$ yield

$$\int_\nu^{\nu+1}\Big\{\Omega_{k-1}^{\nu}(\xi) - \Omega_{k-1}^{\nu-1}(\xi-1)\Big\}d\xi + a_\nu(k) = a_{\nu+1}(k), \quad 0 \le \nu \le k-1, \qquad (4.20)$$

with $\Omega_{k-1}^{-1}(x) = 0$.

It can readily be seen on using (4.20) that

$$a_\nu(k) = \int_{\nu-1}^{\nu} \Omega_{k-1}^{\nu-1}(\xi)d\xi, \quad 0 \le \nu \le k.$$

Therefore

$$V_{k+1}(x) = \begin{cases} \int_\nu^x \Omega_{k-1}^{\nu}(\xi)d\xi + \int_{x-1}^{\nu} \Omega_{k-1}^{\nu-1}(\xi)d\xi, \quad \nu \le x \le \nu+1, \quad 0 \le \nu \le k-1, \\ \int_{x-1}^{k} \Omega_{k-1}^{k-1}(\xi)d\xi, \quad k \le x \le k+1. \end{cases}$$

Hence follows (4.18). □

Proof of Proposition 4.9. It is sufficient to see that $\Omega_k^{\nu}(x) \ge 0$ for $0 \le \nu \le k$. This is evident by induction from (4.18), since $\Omega_0^0(x) = 1$. □

A$_2$. The Case where $C_n(x) = \hat{H}_n(x)$, $n \ge 0$. Let us recall that $\{\hat{H}_n\}_{n\ge0}$ is the monic Hermite sequence and let $\{v_n\}_{n\ge0}$ be its dual sequence.

Let $\{B_n\}_{n\ge0}$ the polynomial sequence defined by

$$B_n^{[1]}(x) = B_n(x), \quad n \ge 0, \qquad (4.21)$$

$$D_1\frac{B_{n+1}}{n+1}(x) = \hat{H}_n(x), \quad n \ge 0. \qquad (4.22)$$

Let $\{u_n\}_{n\ge0}$ be its dual sequence.

Lemma 4.11. *The sequence $\{B_n\}_{n\ge0}$ has the following generating function*

$$\frac{te^{xt-\frac{t^2}{4}}}{e^t-1} = \sum_{n\ge0} B_n(x)\frac{t^n}{n!}. \qquad (4.23)$$

Proof. By virtue of (4.21)–(4.22) and on account of the generating function of $\{\hat{H}_n\}_{n\geq 0}$ [13] we obtain

$$e^{xt-\frac{t^2}{4}} = \sum_{n\geq 0} \hat{H}_n(x)\frac{t^n}{n!}.$$

The desired result can be deduced from Proposition 3.3. □

Proposition 4.12. *The form u_0 satisfies*

$$\Big(\big((u_0)' + 2(2x-1)u_0\big)' + (4x^2-4x-6)u_0\Big)' - 4(2x-1)u_0 = 0. \tag{4.24}$$

$$4(n+2)(u_0)_{n+1} - 4(n+1)(u_0)_n - 2n(2n+1)(u_0)_{n-1} + n(n-1)(n-2)(u_0)_{n-3} = 0, \quad n \geq 3, \tag{4.25}$$

with $(u_0)_0 = 1, (u_0)_1 = \frac{1}{2}, (u_0)_2 = \frac{5}{6}, (u_0)_3 = \frac{7}{8}$.

Moreover,

$$\langle u_0, f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)dx, \quad f \in \mathcal{P}, \tag{4.26}$$

where for any $\lambda \in \mathbb{C}$,

$$U(x) = \frac{1}{\sqrt{\pi}}\int_{x-1}^{x} e^{-t^2}dt + \lambda\int_{-\infty}^{x} S(t)dt, \quad x \in \mathbb{R}, \tag{4.27}$$

$$S(x) = \begin{cases} 0, & x \leq 0, \\ \exp(-x^{1/4})\sin(x^{1/4}), & x > 0, \end{cases} \tag{4.28}$$

in such a way that u_0 is positive definite.

Proof. We need the following relation satisfied by the Hermite form [16]

$$(v_0)' + 2xv_0 = 0. \tag{4.29}$$

On account of (4.24)–(4.25), relation (3.2) in Lemma 3.1 implies

$$(u_0)' = D_{-1}v_0 = v_0 - \tau_1 v_0. \tag{4.30}$$

Differentiating the latter relation we get

$$\begin{aligned}(u_0)'' &= (v_0)' - \tau_1(v_0)' = -2xv_0 - \tau_1(-2xv_0) \quad (\text{from (4.29)})\\ &= -2xv_0 + 2(x-1)\tau_1 v_0.\end{aligned}$$

By (4.30), $\tau_1 v_0 = v_0 - (u_0)'$, which implies $(u_0)'' = -2v_0 - 2(x-1)(u_0)'$. This equation is equivalent to

$$v_0 = -(x-1)(u_0)' - \frac{1}{2}(u_0)''. \tag{4.31}$$

Differentiating both sides of (4.31) we obtain

$$(v_0)' = -(u_0)' - (x-1)(u_0)'' - \frac{1}{2}(u_0)^{(3)}.$$

Indeed, by the latter relation and (4.31), relation (4.29) implies

$$(u_0)^{(3)} + 2(2x-1)(u_0)'' + 2(2x^2-2x+1)(u_0)' = 0.$$

Hence we obtain (4.24).

The recurrence relation (4.25) is equivalent to (4.24) and is easily obtained.

Next, suppose

$$\langle u_0, f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)dx, \quad f \in \mathcal{P}.$$

Taking into account the result [13], [16]

$$\langle v_0, f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}f(x)dx, \quad f \in \mathcal{P},$$

and by virtue of (4.30), we get

$$\int_{-\infty}^{+\infty} U'(x)f(x)dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\big(e^{-x^2} - e^{-(x-1)^2}\big)f(x)dx.$$

Consequently, for any $\lambda \in \mathbb{C}$

$$U'(x) = \frac{1}{\sqrt{\pi}}\Big(e^{-x^2} - e^{-(x-1)^2}\Big) + \lambda S(x),$$

where the function S defined by (4.28) is an integral representation of the null form [21]. Hence we obtain (4.27).

Since

$$\begin{aligned}\int_{-\infty}^{+\infty} x^n\Big(\int_{-\infty}^{x} S(t)dt\Big)dx &= \int_{0}^{+\infty} x^n\Big(\int_{0}^{x} S(t)dt\Big)dx\\ &= (n+1)^{-1}\lim_{x\to+\infty} x^{n+1}\int_0^x S(t)dt = 0, \quad n \geq 0,\end{aligned}$$

we have

$$\langle u_0, f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\Big(\int_{x-1}^{x} e^{-t^2}dt\Big)f(x)dx > 0$$

for any polynomial $f(x) \geq 0$. Thus u_0 is positive definite. In particular

$$\langle u_0, 1\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\Big(\int_{x-1}^{x} e^{-t^2}dt\Big)dx = 1,$$

as it is easy to see. □

5. Some Results on the Euler Sequence

Let us introduce the operator

$$\Big(T_\omega f\Big)(x) := \frac{f(x+\omega)+f(x)}{2}, \quad f \in \mathcal{P}, \quad \omega \in \mathbb{C}. \tag{5.1}$$

We have $T_\omega = \frac{\tau_{-\omega}+I_{\mathcal{P}}}{2}$, where $I_{\mathcal{P}}$ is the identity operator in $\mathcal{P}$. The transposed ${}^tT_\omega$ of T_ω is ${}^tT_\omega = T_{-\omega}$.

In this section we denote by $\{E_n\}_{n\geq 0}$ the monic Euler polynomial sequence and by $\{e_n\}_{n\geq 0}$ its dual sequence.

Definition 5.1 ([8], [11])**.** The Euler sequence $\{E_n\}_{n\geq 0}$ is defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n\geq 0} E_n(x)\frac{t^n}{n!}. \tag{5.2}$$

Lemma 5.2. *The following relations define the Euler sequence*

$$E_n^{[1]}(x) = E_n(x), \quad n \geq 0, \tag{5.3}$$

$$(T_1 E_n)(x) = x^n, \quad n \geq 0. \tag{5.4}$$

Proof. By virtue of (5.2) we obtain (5.3)–(5.4) and vice versa. □

Lemma 5.3. *We have*

$$e_0 = T_{-1}\delta = \frac{\delta+\delta_1}{2}. \tag{5.5}$$

Proof. The proof is analogous to that of Lemma 2.3, where D_1 is replaced by T_1. □

Remark 5.4. We denote by

$$\hat{E}_n(x) = 2^n E_n\left(\frac{x+1}{2}\right), \quad n \geq 0. \tag{5.6}$$

Then $\{\hat{E}_n\}_{n\geq 0}$ is the symmetric Euler sequence satisfying

$$\frac{e^{tx}}{\cosh t} = \sum_{n\geq 0} \hat{E}_n(x)\frac{t^n}{n!}$$

We have

$$\hat{e}_0 = (h_2 \circ \tau_{-1/2})e_0 = \frac{\delta_{-1}+\delta_1}{2}. \tag{5.7}$$

Proposition 5.5. *Let $\{P_n\}$ be a MPS, then we have*

$$P_n(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu}\hat{E}_\nu(x), \quad n \geq 0, \tag{5.8}$$

with

$$\lambda_{n,m} = \frac{1}{2\Gamma(m+1)}\Big\{P_n^{(m)}(-1)+P_n^{(m)}(1)\Big\}, \quad 0 \leq m \leq n.$$

Proof. Writing

$$P_n(x) = \sum_{\nu=0}^{n} \lambda_{n,\nu}\hat{E}_\nu(x), n \geq 0,$$

we have $\lambda_{n,m} = \langle \hat{e}_m, P_n\rangle$, $0 \leq m \leq n$.

Then $\lambda_{n,m} = \frac{1}{m!}\langle \hat{e}_0, P_n^{(m)}\rangle$, $0 \leq m \leq n$, by (5.7) we get the desired result. □

Corollary 5.6 (c.f. [22])**.** *Let $\{L_n\}_{n\geq 0}$ be a Legendre sequence, then the following equality holds:*

$$L_n(x) = \frac{\sqrt{\pi}}{2^{n+1}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \times \sum_{\nu=0}^{n} \frac{1+(-1)^{n+\nu}}{2^\nu}\frac{\Gamma(n+1+\nu)}{\Gamma^2(\nu+1)\Gamma(n+1-\nu)}\hat{E}_\nu(x) \quad n \geq 0. \tag{5.9}$$

Proof. Taking Proposition 5.5 and relation (2.25) into account we get (5.9). □

Proposition 5.7. *Let $\{C_n\}_{n\geq 0}$ be an Appell sequence and $\{\sigma_n\}_{n\geq 0}$ be its dual sequence, then there exists a unique sequence $\{P_n\}_{n\geq 0}$ satisfying*

$$P_n^{[1]}(x) = P_n(x), \quad n \geq 0, \tag{5.10}$$

$$\Big(T_1 P_n\Big)(x) = C_n(x), \quad n \geq 0. \tag{5.11}$$

If u_0 is the canonical form of $\{P_n\}_{n\geq 0}$, then we have

$$T_{-1}\sigma_0 = u_0. \tag{5.12}$$

Proof. The proposition is proved by using (3.3)–(3.4) and (3.2), where D_1 is replaced by T_1. □

Definition 5.8 ([12])**.** The polynomial sequence $\{E_n(.;k)\}_{n\geq 0}$, $k \geq 0$ defined by

$$E_n^{[1]}(x;k+1) = E_n(x;k), \quad n,k \geq 0, \tag{5.13}$$

$$T_1 E_n(x;k+1) = E_n(x;k), \quad n,k \geq 0, \tag{5.14}$$

with $E_n(x;0) = x^n$, $n \geq 0$, is called the generalized Euler sequence of order k.

Let us denote by $\{e_n(k)\}_{n\geq 0}$ the dual sequence of $\{E_n(.;k)\}_{n\geq 0}$, then $e_0(0)=\delta$.

Proposition 5.9. *We have*

$$e_0(k) = 2^{-k}\sum_{\nu=0}^{k}\binom{k}{\nu}\delta_\nu. \tag{5.15}$$

Proof. Using (5.13)–(5.14) and relation (5.12) in Proposition 5.7, we get

$$e_0(k) = T_{-1}e_0(k-1), \quad k \geq 1.$$

With the latter relation we can prove by induction with respect to m that

$$e_0(k) = T_{-1}^m e_0(k-m), \quad 1 \leq m \leq k.$$

Then we obtain

$$e_0(k) = T_{-1}^k e_0(0) = 2^{-k}(\tau_1 + I_{\mathcal{P}'})^k \delta = 2^{-k} \sum \nu = 0^k \binom{k}{\nu} \delta_\nu. \qquad \square$$

Hence follows the desired result.

Remark 5.10. Relation (5.15) shows that $e_0(k)$ is not regular for any integer k.

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