On a class of q-Bernoulli, q-Euler and q-Genocchi polynomials

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Abstract

The main purpose of this paper is to introduce and investigate a class of q-Bernoulli, q-Euler and q-Genocchi polynomials. The q-analogues of well-known formulas are derived. The q-analogue of the Srivastava–Pintér addition theorem is obtained. Some new identities involving q-polynomials are proved.

1 Introduction

Throughout this paper, we always make use of the classical definition of quantum concepts as follows:

The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1} \left(1 - q^j a\right), \quad n \in \mathbb{N},$$

$$(a;q)_\infty = \prod_{j=0}^{\infty} \left(1 - q^j a\right), \quad |q| < 1, \quad a \in \mathbb{C}.$$

It is known that

$$(a;q)_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k.$$

The q-numbers and q-numbers factorial and their improved forms are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad (q \neq 1, \ a \in \mathbb{C});$$
$$[0]_q! = 1, [n]_q! = [n]_q [n-1]_q! \quad , \ .$$

The q-polynomail coefficient and improved type of them are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q;q)_n}{(q;q)_{n-k} (q;q)_k}, \qquad (k \leqslant n, k, n \in \mathbb{N})$$

In the standard approach to the q-calculus two exponential function are used, these q-exponential and improved type (see [2]) of it are defined as follows:

$$
\begin{aligned}
e_q(z) &= \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)\,q^k z)}, \quad 0<|q|<1,\ |z|<\frac{1}{|1-q|},\\
E_q(z) &= e_{1/q}(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1+(1-q)\,q^k z\right), \qquad 0<|q|<1,\ z\in\mathbb{C},\\
\mathcal{E}_q(z) &= e_q\left(\frac{z}{2}\right) E_q\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1,q)_n}{2^n} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{z^n}{\{n\}_q!}\\
&= \prod_{k=0}^{\infty} \frac{\left(1+(1-q)\,q^k \frac{z}{2}\right)}{\left(1-(1-q)\,q^k \frac{z}{2}\right)}, 0<q<1, |z|<\frac{2}{1-q}.
\end{aligned}
$$

The form of improved type of q-exponential function $\mathcal{E}_q(z)$, motivate us to define a new q-addition and q-substraction as follows:

$$
\begin{aligned}
(x \oplus_q y)^n &:= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_k(-1,q)_{n-k}}{2^n} x^k y^{n-k}, \quad n=0,1,2,...,\\
(x \ominus_q y)^n &:= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_k(-1,q)_{n-k}}{2^n} x^k (-y)^{n-k}, \quad n=0,1,2,...
\end{aligned}
$$

It follows that

$$
\mathcal{E}_q(tx)\,\mathcal{E}_q(ty) = \sum_{n=0}^{\infty} (x \oplus_q y)^n \frac{t^n}{[n]_q!}.
$$

The Bernoulli numbers $\{B_m\}_{m\geq 0}$ are rational numbers in a sequence defined by the binomial recursion formula

$$
\sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array}\right) B_k - B_m = \begin{cases} 1, & m=1,\\ 0, & m>1, \end{cases} \tag{1}
$$

or equivalently, the generating function

$$
\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t-1}.
$$

q-analogues of the Bernoulli numbers were first studied by Carlitz [?] in the middle of the last century when he introduced a new sequence $\{\beta_m\}_{m\geqslant 0}$:

$$
\sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array}\right) \beta_k q^{k+1} - \beta_m = \begin{cases} 1, & m=1,\\ 0, & m>1. \end{cases} \tag{2}
$$

Here, and in the remainder of the paper, the parameter we make the assumption that $|q|<1$.Clearly we recover (1) if we let $q\to 1$ in (2).The q-binomial formula is known as

$$
\begin{aligned}
(1 \ominus_q x)^n &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_k}{2^k} (-x)^k = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(1+1)(1+q)...(1+q^{k-1})x^k}{2^k} (-1)^k\\
(1-a)_q^n &= (a;q)_n = \prod_{j=0}^{n-1} \left(1-q^j a\right) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k.
\end{aligned}
$$

The above q-standard notation can be found in [1].

Carlitz has introduced the q-Bernoulli numbers and polynomials in [?]. Srivastava and Pintér proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [18]. They also gave some generalizations of these polynomials. In [9]-[20], Kim et al. investigated some properties of the q-Euler polynomials and Genocchi polynomials. They gave some recurrence relations. In [3], Cenkci et al. gave the q-extension of Genocchi numbers in a different manner. In [21], Kim gave a new concept for the q-Genocchi numbers and polynomials. In [25], Simsek et al. investigated the q-Genocchi zeta function and l-function by using generating functions and Mellin transformation. There are numerous recent studies on this subject by among many other authors: Cenkci et al. [3], [4], Choi et al [6], Cheon [5], Luo and Srivastava [12], [13], [14], Srivastava et al.[18], [26], Nalci and Pashaev [24] Gabouary and Kurt B., [7], Kim et al. [23], Kurt V. [22].

We first give here the definitions of the q-numbers and q-polynomials as follows. It should be mentioned that the definition of q-Bernoulli numbers in Definition 1 can br found in [24].

Definition 1 *Let $q\in\mathbb{C},\ 0<|q|<1$. The q-Bernoulli numbers $\mathfrak{b}_{n,q}$ and polynomials $\mathfrak{B}_{n,q}(x,y)$ are defined by the means of the generating functions:*

$$\widehat{\mathfrak{B}}(t):=\frac{te_q\left(-\frac{t}{2}\right)}{e_q\left(\frac{t}{2}\right)-e_q\left(-\frac{t}{2}\right)}=\frac{t}{\mathcal{E}_q(t)-1}=\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!},\quad |t|<2\pi,$$

$$\frac{t}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)\mathcal{E}_q(ty)=\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!},\quad |t|<2\pi.$$

Definition 2 *Let $q\in\mathbb{C},\ 0<|q|<1$. The q-Euler numbers $\mathfrak{e}_{n,q}$ and polynomials $\mathfrak{E}_{n,q}(x,y)$ are defined by the means of the generating functions:*

$$\widehat{\mathfrak{E}}(t):=\frac{2e_q\left(-\frac{t}{2}\right)}{e_q\left(\frac{t}{2}\right)+e_q\left(-\frac{t}{2}\right)}=\frac{2}{\mathcal{E}_q(t)+1}=\sum_{n=0}^{\infty}\mathfrak{e}_{n,q}\frac{t^n}{[n]_q!},\quad |t|<\pi,$$

$$\frac{2}{\mathcal{E}_q(t)+1}\mathcal{E}_q(tx)\mathcal{E}_q(ty)=\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(x,y)\frac{t^n}{[n]_q!},\quad |t|<\pi.$$

Definition 3 *Let $q\in\mathbb{C},\ 0<|q|<1$. The q-Genocchi numbers $\mathfrak{g}_{n,q}$ and polynomials $\mathfrak{G}_{n,q}(x,y)$ are defined by the means of the generating functions:*

$$\widehat{\mathfrak{G}}(t):=\frac{2te_q\left(-\frac{t}{2}\right)}{e_q\left(\frac{t}{2}\right)+e_q\left(-\frac{t}{2}\right)}=\frac{2t}{\mathcal{E}_q(t)+1}=\sum_{n=0}^{\infty}\mathfrak{g}_{n,q}\frac{t^n}{[n]_q!},\quad |t|<\pi,$$

$$\frac{2t}{\mathcal{E}_q(t)+1}\mathcal{E}_q(tx)\mathcal{E}_q(ty)=\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!},\quad |t|<\pi.$$

Definition 4 *Let $q\in\mathbb{C},\ 0<|q|<1$. The q-tangent numbers $\mathfrak{T}_{n,q}$ are defined by the means of the generating functions:*

$$\begin{aligned}\tanh_q t=-i\tan_q(it)&=\frac{e_q(t)-e_q(-t)}{e_q(t)+e_q(-t)}=\frac{\mathcal{E}_q(2t)-1}{\mathcal{E}_q(2t)+1}\\&=\sum_{n=1}^{\infty}\mathfrak{T}_{2n+1,q}\frac{(-1)^k t^{2n+1}}{[2n+1]_q!}.\end{aligned}$$

It is obvious that by tending q to 1 from the left side, we lead to the classic definition of these polynomials:

$$\begin{aligned}&\mathfrak{b}_{n,q}:=\mathfrak{B}_{n,q}(0),\quad \lim_{q\to1^-}\mathfrak{B}_{n,q}(x)=B_n(x),\quad \lim_{q\to1^-}\mathfrak{B}_{n,q}(x,y)=B_n(x+y),\quad \lim_{q\to1^-}\mathfrak{b}_{n,q}=B_n,\\&\mathfrak{e}_{n,q}:=\mathfrak{E}_{n,q}(0),\quad \lim_{q\to1^-}\mathfrak{E}_{n,q}(x)=E_n(x),\quad \lim_{q\to1^-}\mathfrak{E}_{n,q}(x,y)=E_n(x+y),\quad \lim_{q\to1^-}\mathfrak{e}_{n,q}=E_n,\\&\mathfrak{g}_{n,q}:=\mathfrak{G}_{n,q}(0),\quad \lim_{q\to1^-}\mathfrak{G}_{n,q}(x)=G_n(x),\quad \lim_{q\to1^-}\mathfrak{G}_{n,q}(x,y)=G_n(x+y)\quad \lim_{q\to1^-}\mathfrak{g}_{n,q}=G_n.\end{aligned}$$

Here $B_n(x)$, $E_n(x)$ and $G_n(x)$ denote the classical Bernoulli, Euler and Genocchi polynomials which are defined by

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\qquad \frac{2}{e^t+1}e^{tx}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\ \text{ and }\ \frac{2t}{e^t+1}e^{tx}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}.$$

The aim of the present paper is to obtain some results for the above newly defined q-polynomials. It should be mentioned that q-Bernoulli and q-Euler polynomials in our definitions are polynomials of x and y and when $y=0$ they are polynomials of x, but in other definitions they respect to q^x. First advantage of this approach is that for $q\to 1^-$,$\mathfrak{B}_{n,q}(x,y)$ $(\mathfrak{E}_{n,q}(x,y)$, $\mathfrak{G}_{n,q}(x,y))$ becomes the classical Bernoulli $\mathfrak{B}_n(x+y)$ (Euler $\mathfrak{E}_n(x+y)$, Genocchi $\mathfrak{G}_{n,q}(x,y)$) polynomial and we may obtain the q-analogues of well-known results, for example Srivastava and Pintér [?], Cheon [5], etc. Second advantage is that similar to the classical case odd numbered terms of the Bernoulli numbers $\mathfrak{b}_{k,q}$ and the Genocchi numbres $\mathfrak{g}_{k,q}$are zero, and even numbered terms of the Euler numbers $\mathfrak{e}_{n,q}$ are zero.

2 Preliminary results

In this section we shall provide some basic formulas for the q-Bernoulli, q-Euler and q-Genocchi numbers and polynomials in order to obtain the main results of this paper in the next section.

Lemma 5 *The q-Bernoulli numbers $\mathfrak{b}_{n,q}$ satisfy the following q-binomial recurrence:*

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{b}_{k,q}-\mathfrak{b}_{n,q}=\begin{cases}1, & n=1,\\ 0, & n>1.\end{cases}\tag{3}$$

Proof. By simple multiplication on (1) we see that

$$\widehat{\mathfrak{B}}(t)\,\mathcal{E}_q(t)=t+\widehat{\mathfrak{B}}(t).$$

So

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{b}_{k,q}\frac{t^n}{[n]_q!}=t+\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}.$$

The statement follows by comparing t^m-coefficients. ■

We use this formula to calculate the first few $\mathfrak{b}_{k,q}$.

$$\begin{aligned}\mathfrak{b}_{0,q}&=1,\\ \mathfrak{b}_{1,q}&=-\tfrac{1}{2}=-\tfrac{1}{\{2\}_q},\\ \mathfrak{b}_{2,q}&=\tfrac{1}{4}\frac{q(q+1)}{q^2+q+1}=\tfrac{q[2]_q}{4[3]_q},\\ \mathfrak{b}_{3,q}&=0.\end{aligned}$$

The similar property can be proved for q-Euler numbers

$$\sum_{k=0}^{m}\begin{Bmatrix} m\\ k\end{Bmatrix}_q\mathfrak{e}_{k,q}+\mathfrak{e}_{m,q}=\begin{cases}2, & m=0,\\ 0, & m>0.\end{cases}\tag{4}$$

and q-Genocchi numbers

$$\sum_{k=0}^{m}\begin{Bmatrix} m\\ k\end{Bmatrix}_q\mathfrak{g}_{k,q}+\mathfrak{g}_{m,q}=\begin{cases}2, & m=1,\\ 0, & m>1.\end{cases}\tag{5}$$

Using the above recurrence formulae we calculate the first few $\mathfrak{e}_{n,q}$ and $\mathfrak{g}_{n,q}$as well.

$$
\begin{aligned}
\mathfrak{e}_{0,q} &= 1, & \mathfrak{g}_{0,q} &= 0, \\
\mathfrak{e}_{1,q} &= -\frac{1}{2}, & \mathfrak{g}_{1,q} &= 1, \\
\mathfrak{e}_{2,q} &= 0, & \mathfrak{g}_{2,q} &= -\tfrac{[2]_q}{2} = -\frac{q+1}{2}, \\
\mathfrak{e}_{3,q} &= \tfrac{[3]_q[2]_q-[4]_q}{8} = \frac{q(1+q)}{8}, & \mathfrak{g}_{3,q} &= 0.
\end{aligned}
$$

Remark 6 *The first advantage of the new q-numbers $\mathfrak{b}_{k,q}$, $\mathfrak{e}_{k,q}$ and $\mathfrak{g}_{k,q}$ is that similar to classical case odd numbered terms of the Bernoulli numbers $\mathfrak{b}_{k,q}$ and the Genocchi numbres $\mathfrak{g}_{k,q}$are zero, and even numbered terms of the Euler numbers $\mathfrak{e}_{n,q}$ are zero.*

Next lemma gives the relationsheep between q-Genocchi numbers and q-Tangent numbers.

Lemma 7 *Fro any $n \in \mathbb{N}$ we have*

$$\mathfrak{T}_{2n+1,q} = \mathfrak{g}_{2n+2,q} \frac{(-1)^{k-1} 2^{2n+1}}{[2n+2]_q}.$$

Proof. First we recall the definition of q-trigonometric functions.

$$
\begin{aligned}
\cos_q t &= \frac{e_q(it) + e_q(-it)}{2}, & \sin_q t &= \frac{e_q(it) - e_q(-it)}{2i}, \\
i\tan_q t &= \frac{e_q(it) - e_q(-it)}{e_q(it) + e_q(-it)}, & \cot_q t &= i\frac{e_q(it) + e_q(-it)}{e_q(it) - e_q(-it)}.
\end{aligned}
$$

Now by choosing $z = 2it$ in $\widehat{\mathfrak{B}}(z)$, we get

$$\widehat{\mathfrak{B}}(2it) = \frac{2it}{\mathcal{E}_q(2it) - 1} = \frac{te_q(-it)}{\sin_q t} = \sum_{n=0}^{\infty} \mathfrak{b}_{n,q} \frac{(2it)^n}{[n]_q!}.$$

It follows that

$$
\begin{aligned}
\widehat{\mathfrak{B}}(2it) &= \frac{te_q(-it)}{\sin_q t} = \frac{t}{\sin_q t}(\cos_q t - i\sin_q t) = t\cot_q t - it \\
&= \mathfrak{b}_{0,q} + 2it\mathfrak{b}_{1,q} + \sum_{n=2}^{\infty} \mathfrak{b}_{n,q} \frac{(2it)^n}{[n]_q!} \\
&= 1 - it + \sum_{n=2}^{\infty} \mathfrak{b}_{n,q} \frac{(2it)^n}{[n]_q!}.
\end{aligned}
$$

Since $t\cot_q t$ is even in the above sum odd coefficients $\mathfrak{b}_{2k+1,q}$, $k = 1, 2, ...$are zero we get

$$t\cot_q t = 1 + \sum_{n=2}^{\infty} \mathfrak{b}_{n,q} \frac{(2it)^n}{[n]_q!} = 1 + \sum_{n=1}^{\infty} \mathfrak{b}_{n,q} \frac{(2it)^{2n}}{[2n]_q!}.$$

By choosing $z = 2it$ in $\widehat{\mathfrak{G}}(z)$, we get

$$\widehat{\mathfrak{G}}(2it) = \frac{4it}{\mathcal{E}_q(2it) + 1} = \frac{2ite_q(-it)}{\cos_q t} = \sum_{n=0}^{\infty} \mathfrak{g}_{n,q} \frac{(2it)^n}{[n]_q!}.$$

$$\widehat{\mathfrak{G}}(2it)=\frac{4it}{\mathcal{E}_q(2it)+1}=\frac{2ite_q(-it)}{\cos_q t}=\frac{2it}{\cos_q t}(\cos_q t-i\sin_q t)=2it+2t\tan_q t$$
$$=\mathfrak{g}_{0,q}+2it\mathfrak{g}_{1,q}+\sum_{n=2}^{\infty}\mathfrak{g}_{n,q}\frac{(2it)^n}{[n]_q!}$$
$$=2it+\sum_{n=2}^{\infty}\mathfrak{g}_{n,q}\frac{(2it)^n}{[n]_q!}.$$

It follows that

$$2t\tan_q t=\sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(2it)^{2n}}{[2n]_q!},\qquad \tan_q t=\sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(-1)^n(2t)^{2n-1}}{[2n]_q!}$$
$$\tanh_q t=-i\tan_q(it)=-i\sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(-1)^n(2it)^{2n-1}}{[2n]_q!}$$
$$=-\sum_{n=1}^{\infty}\mathfrak{g}_{2n,q}\frac{(2t)^{2n-1}}{[2n]_q!}=-\sum_{n=1}^{\infty}\mathfrak{g}_{2n+2,q}\frac{(2t)^{2n+1}}{[2n+2]_q!}.$$

Thus

$$\tanh_q t=-i\tan_q(it)=\frac{e_q(t)-e_q(-t)}{e_q(t)+e_q(-t)}=\frac{\mathcal{E}_q(2t)-1}{\mathcal{E}_q(2t)+1}$$
$$=\sum_{n=1}^{\infty}\mathfrak{T}_{2n+1,q}\frac{(-1)^k t^{2n+1}}{[2n+1]_q!},$$

and

$$\mathfrak{T}_{2n+1,q}=\mathfrak{g}_{2n+2,q}\frac{(-1)^{k-1}2^{2n+1}}{[2n+2]_q}.$$

■

The following result is q-analogue of the addition theorem for the classical Bernoulli, Euler and Genocchi polynomials.

Lemma 8 (Addition Theorems) *For all $x,y\in\mathbb{C}$ we have*

$$\begin{aligned}
\mathfrak{B}_{n,q}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{b}_{k,q}(x\oplus_q y)^{n-k}, &\qquad \mathfrak{B}_{n,q}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x)y^{n-k},\\
\mathfrak{E}_{n,q}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{e}_{k,q}(x\oplus_q y)^{n-k}, &\qquad \mathfrak{E}_{n,q}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{E}_{k,q}(x)y^{n-k},\\
\mathfrak{G}_{n,q}(x,y)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{g}_{k,q}(x\oplus_q y)^{n-k}, &\qquad \mathfrak{G}_{n,q}(x,y)&=\sum_{k=0}^{n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{G}_{k,q}(x)y^{n-k}.
\end{aligned}\tag{6}$$

Proof. We prove only the first formula. It is a consequence of the following identity

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}=\frac{t}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)\mathcal{E}_q(ty)=\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(x\oplus_q y)^n\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{b}_{k,q}(x\oplus_q y)^{n-k}\frac{t^n}{[n]_q!}.$$

■

In particular, setting $y=0$ in (6), we get the following formulas for q-Bernoulli, q-Euler and q-Genocchi polynomials, respectively.

$$\mathfrak{B}_{n,q}(x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{b}_{k,q}x^{n-k},\quad \mathfrak{E}_{n,q}(x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{e}_{k,q}x^{n-k}, \tag{7}$$

$$\mathfrak{G}_{n,q}(x)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{g}_{k,q}x^{n-k}. \tag{8}$$

Setting $y=1$ in (6), we get

$$\mathfrak{B}_{n,q}(x,1)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x),\quad \mathfrak{E}_{n,q}(x,1)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{E}_{k,q}(x), \tag{9}$$

$$\mathfrak{G}_{n,q}(x,1)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{G}_{k,q}(x). \tag{10}$$

Clearly (9) and (10) are q-analogues of

$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x),\ E_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}E_k(x),\ G_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}G_k(x),$$

respectively.

Lemma 9 *The odd coefficient of the q-Bernoulli numbers except the first one are zero, that means* $\mathfrak{b}_{n,q}=0$ *where* $n=2r+1,\ (r\in\mathbb{N})$.
Proof. *It follows from the fact that the function*

$$f(t)=\sum_{n=0}^{\infty}\mathfrak{b}_{n,q}\frac{t^n}{[n]_q!}-\mathfrak{b}_{1,q}t=\frac{t}{\mathcal{E}_q(t)-1}+\frac{t}{2}=\frac{t}{2}\left(\frac{\mathcal{E}_q(t)+1}{\mathcal{E}_q(t)-1}\right)$$

is even, and the coefficient of t^n *in the Taylor expansion about zero of any even function vanish for all odd* n*. note that this could not happen in the last q-analogue of these numbers, because in the case of improved exponential function* $\mathcal{E}_q(-t)=(\mathcal{E}_q(t))^{-1}$. ■

By using (**??**) and q-derivative approaching to the next lemma.

Lemma 10 *We have*

$$D_{q,x}\mathfrak{B}_{n,q}(x)=[n]_q\frac{\mathfrak{B}_{n-1,q}(x)+\mathfrak{B}_{n-1,q}(qx)}{2},\quad D_{q,x}\mathfrak{E}_{n,q}(x)=[n]_q\frac{\mathfrak{E}_{n-1,q}(x)+\mathfrak{E}_{n-1,q}(qx)}{2},$$
$$D_{q,x}\mathfrak{G}_{n,q}(x)=[n]_q\frac{\mathfrak{G}_{n-1,q}(x)+\mathfrak{G}_{n-1,q}(qx)}{2}.$$

Lemma 11 (Difference Equations) *We have*

$$\mathfrak{B}_{n,q}(x,1)-\mathfrak{B}_{n,q}(x)=\frac{(-1;q)_{n-1}}{2^{n-1}}[n]_q x^{n-1},\quad n\geq 1, \tag{11}$$

$$\mathfrak{E}_{n,q}(x,1)+\mathfrak{E}_{n,q}(x)=2\frac{(-1;q)_n}{2^n}x^n,\quad n\geq 0, \tag{12}$$

$$\mathfrak{G}_{n,q}(x,1)+\mathfrak{G}_{n,q}(x)=2\frac{(-1;q)_{n-1}}{2^{n-1}}[n]_q x^{n-1},\quad n\geq 1. \tag{13}$$

Proof. We prove the identity for the q-Bernoulli polynomials. From the identity

$$\frac{t\mathcal{E}_q(t)}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)=t\mathcal{E}_q(tx)+\frac{t}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx),$$

it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\frac{(-1,q)_n}{2^n}x^n\frac{t^{n+1}}{[n]_q!}+\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}.$$

■

From (11) and (7), (12) and (8) we obtain the following formulas.

Lemma 12 *We have*

$$x^n=\frac{2^n}{(-1;q)_n[n]_q}\sum_{k=0}^{n}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q \frac{(-1;q)_{n+1-k}}{2^{n+1-k}}\mathfrak{B}_{k,q}(x) \quad (14)$$

$$x^n=\frac{2^{n-1}}{(-1;q)_n}\left(\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1;q)_{n-k}}{2^{n-k}}\mathfrak{E}_{k,q}(x)+\mathfrak{E}_{n,q}(x)\right), \quad (15)$$

$$x^n=\frac{2^{n-1}}{(-1;q)_n[n+1]_q}\left(\sum_{k=0}^{n+1}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q \frac{(-1;q)_{n+1-k}}{2^{n+1-k}}\mathfrak{G}_{k,q}(x)+\mathfrak{G}_{n+1,q}(x)\right). \quad (16)$$

The above formulas are q-analoques of the following familiar expansions

$$x^n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x), \quad x^n=\frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)\right], \quad (17)$$

$$x^n=\frac{1}{2(n+1)}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}E_k(x)+E_{n+1}(x)\right],$$

respectively.

Lemma 13 *The following identities hold true.*

$$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x,y)-\mathfrak{B}_{n,q}(x,y)=[n]_q(x\oplus_q y)^{n-1},$$

$$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{E}_{k,q}(x,y)+\mathfrak{E}_{n,q}(x,y)=2(x\oplus_q y)^{n},$$

$$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{G}_{k,q}(x,y)+\mathfrak{G}_{n,q}(x,y)=2[n]_q(x\oplus_q y)^{n-1}.$$

Proof. We the identity for the q-Bernoulli polynomials. From the identity

$$\frac{t\mathcal{E}_q(t)}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)\mathcal{E}_q(ty)=t\mathcal{E}_q(tx)\mathcal{E}_q(ty)+\frac{t}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)\mathcal{E}_q(ty),$$

it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_{n-k}}{2^{n-k}}\mathfrak{B}_{k,q}(x,y)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{(-1,q)_k(-1,q)_{n-k}}{2^n}x^k y^{n-k}\frac{t^{n+1}}{[n]_q!}+\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}.$$

■

3 Some new formulas

The classical Cayley transformation $z \rightarrow \mathrm{Cay}(z, a) := \frac{1+az}{1-az}$ motivates us to approaching to the formula for $\mathcal{E}_q(qt)$, In addition by substitute it in the generating formula we have:

$$\widehat{\mathfrak{B}}_q(qt)\,\widehat{\mathfrak{B}}_q(t) = \left(\widehat{\mathfrak{B}}_q(qt) - q\widehat{\mathfrak{B}}_q(t)\,(1+(1-q)\frac{t}{2})\right)\frac{1}{1-q} \times \frac{2}{\mathcal{E}_q(t)+1}$$

The right hand side can be presented by improved q-Euler numbers .Now the equating coefficients of t^n we get the following identity.In the case that $n=0$, we find the first improved q-Euler number which is exactly 1.

Proposition 14 *For all $n \geq 1$,*

$$\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{B}_{k,q}\mathfrak{B}_{n-k,q}q^k = -q\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q \mathfrak{B}_{k,q}\mathfrak{E}_{n-k,q}[k-1]_q - \frac{q}{2}\sum_{k=0}^{n-1}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_q \mathfrak{B}_{k,q}\mathfrak{E}_{n-k-1,q}[n]_q$$

Let take a q-derivative from generating function, after simplifying the equation and by knowing the quotient rule for quantum derivative , also using that

$$\mathcal{E}_q(qt) = \frac{1-(1-q)\frac{t}{2}}{1+(1-q)\frac{t}{2}}\mathcal{E}_q(t)\,, D_q(\mathcal{E}_q(t)) = \frac{\mathcal{E}_q(qt)+\mathcal{E}_q(t)}{2},$$

we have:

$$\widehat{B}_q(qt)\,\widehat{B}_q(t) = \frac{2+(1-q)t}{2\mathcal{E}_q(t)\,(q-1)}\left(q\widehat{B}_q(t) - \widehat{B}_q(qt)\right)$$

It is clear that $\mathcal{E}_q^{-1}(t) = \mathcal{E}_q(-t)$. Now the equating coefficient of t^n we lead to the following identity.

Proposition 15 *For all $n \geq 1$,*

$$\sum_{k=0}^{2n}\left[\begin{array}{c} 2n \\ k \end{array}\right]_q \mathfrak{B}_{k,q}\mathfrak{B}_{2n-k,q}q^k = -q\sum_{k=0}^{2n}\left\{\begin{array}{c} 2n \\ k \end{array}\right\}_q \mathfrak{B}_{k,q}[k-1]_q(-1)^k + \frac{q(1-q)}{2}\sum_{k=0}^{2n-1}\left\{\begin{array}{c} 2n-1 \\ k \end{array}\right\}_q \mathfrak{B}_{k,q}[k-1]_q(-1)^k$$

$$\sum_{k=0}^{2n+1}\left[\begin{array}{c} 2n+1 \\ k \end{array}\right]_q \mathfrak{B}_{k,q}\mathfrak{B}_{2n-k+1,q}q^k = q\sum_{k=0}^{2n+1}\left\{\begin{array}{c} 2n+1 \\ k \end{array}\right\}_q \mathfrak{B}_{k,q}[k-1]_q(-1)^k - \frac{q(1-q)}{2}\sum_{k=0}^{2n}\left\{\begin{array}{c} 2n \\ k \end{array}\right\}_q \mathfrak{B}_{k,q}[k-1]_q(-1)^k$$

We may also derive a differential equation for $\widehat{B}_q(t)$ .If we differentiate both sides of generating function with respect to t, after a little calculation we find that

$$\frac{\partial}{\partial t}\widehat{B}_q(t) = \widehat{B}_q(t)\left(\frac{1}{t} - \frac{(1-q)\mathcal{E}_q(t)}{\mathcal{E}_q(t)-1}\left(\sum_{k=0}^{\infty}\frac{4q^k}{4-(1-q)^2q^{2k}}\right)\right)$$

If we differentiate with respect to q, we instead obtain

$$\frac{\partial}{\partial q}\widehat{B}_q(t) = -\widehat{B}_q^2(t)\,\mathcal{E}_q(t)\sum_{k=0}^{\infty}\frac{4t(kq^{k-1}-(k+1)q^k)}{4-(1-q)^2q^{2k}}$$

Again using generating function and combining this with the t derivative we get the partial differential equation

Proposition 16

$$\frac{\partial}{\partial t}\widehat{B}_q(t)-\frac{\partial}{\partial q}\widehat{B}_q(t)=\frac{\widehat{B}_q(t)}{t}+\frac{\widehat{B}_q^2(t)\,\mathcal{E}_q(t)}{t}\sum_{k=0}^{\infty}\frac{4t(kq^{k-1}-(k+1)q^k)-q^k(1-q)}{4-(1-q)^2q^{2k}}$$

4 Explicit relationship between the q-Bernoulli and q-Euler polynomials

In this section we will give some explicit relationships between the q-Bernoulli and q-Euler polynomials. Here some q-analogues of known results will be given. We also obtain new formulas and their some special cases below. These formulas are some extensions of the formulas of Srivastava and Pintér, Cheon and others.

We present natural q-extensions of the main results in the papers [?] and [13], see Theorems 17 and 19.

Theorem 17 *For $n\in\mathbb{N}_0$, the following relationships hold true:*

$$\begin{aligned}\mathfrak{B}_{n,q}(x,y)&=\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q m^{k-n}\left[\mathfrak{B}_{k,q}(x)+\sum_{j=0}^{k}\left\{\begin{array}{c}k\\j\end{array}\right\}_q\frac{\mathfrak{B}_{j,q}(x)}{m^{k-j}}\right]\mathfrak{E}_{n-k,q}(my)\\&=\frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q m^{k-n}\left[\mathfrak{B}_{k,q}(x)+\mathfrak{B}_{k,q}\left(x,\frac{1}{m}\right)\right]\mathfrak{E}_{n-k,q}(my).\end{aligned}$$

Proof. Using the following identity

$$\frac{t}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)\,\mathcal{E}_q(ty)=\frac{t}{\mathcal{E}_q(t)-1}\mathcal{E}_q(tx)\cdot\frac{\mathcal{E}_q\left(\frac{t}{m}\right)+1}{2}\cdot\frac{2}{\mathcal{E}_q\left(\frac{t}{m}\right)+1}\mathcal{E}_q\left(\frac{t}{m}my\right)$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{(-1,q)_n}{m^n2^n}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\\&\quad+\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\\&\quad=:I_1+I_2.\end{aligned}$$

It is clear that

$$I_2=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right]_q m^{k-n}\mathfrak{B}_{k,q}(x)\,\mathfrak{E}_{n-k,q}(my)\frac{t^n}{[n]_q!}.$$

On the other hand

$$\begin{aligned}I_1&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left\{\begin{array}{c}n\\j\end{array}\right\}_q\mathfrak{B}_{j,q}(x)\frac{t^n}{m^{n-j}[n]_q!}\\&=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{n-k,q}(my)\sum_{j=0}^{k}\left\{\begin{array}{c}k\\j\end{array}\right\}_q\frac{\mathfrak{B}_{j,q}(x)}{m^{n-k}m^{k-j}}\frac{t^n}{[n]_q!}.\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q m^{k-n}\left[\mathfrak{B}_{k,q}(x)+\sum_{j=0}^{k}\left\{\begin{array}{c}k\\j\end{array}\right\}_q\frac{\mathfrak{B}_{j,q}(x)}{m^{k-j}}\right]\mathfrak{E}_{n-k,q}(my)\frac{t^n}{[n]_q!}.$$

It remains to equate coefficient of t^n. ■

Next we discuss some special cases of Theorem 17.

Corollary 18 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(\mathfrak{B}_{k,q}(x) + \frac{(-1;q)_{k-1}}{2^k} [k]_q x^{k-1} \right) \mathfrak{E}_{n-k,q}(y). \tag{18}$$

The formula (18) ia a q-extension of the Cheon's main result [5].

Theorem 19 *For $n \in \mathbb{N}_0$, the following relationships*

$$\mathfrak{E}_{n,q}(x,y) = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \frac{1}{m^{n+1-k}} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_q \left(\sum_{j=0}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\}_q \frac{\mathfrak{E}_{j,q}(x)}{m^{k-j}} - \mathfrak{E}_{k,q}(y) \right) \mathfrak{B}_{n+1-k,q}(mx)$$

hold true between the q-Bernoulli polynomials and q-Euler polynomials.

Proof. The proof is based on the following identity

$$\frac{2}{\mathcal{E}_q(t)+1} \mathcal{E}_q(tx) \mathcal{E}_q(ty) = \frac{2}{\mathcal{E}_q(t)+1} \mathcal{E}_q(ty) \cdot \frac{\mathcal{E}_q\left(\frac{t}{m}\right) - 1}{t} \cdot \frac{t}{\mathcal{E}_q\left(\frac{t}{m}\right) - 1} \mathcal{E}_q\left(\frac{t}{m} mx\right).$$

Indeed

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(y) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \frac{t^{n-1}}{m^n \{n\}_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \\ &- \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(y) \frac{t^{n-1}}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \\ &=: I_1 - I_2. \end{aligned}$$

It follows that

$$\begin{aligned} I_2 &= \frac{1}{t} \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}(y) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} = \frac{1}{t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{1}{m^{n-k}} \mathfrak{E}_{k,q}(y) \mathfrak{B}_{n-k,q}(mx) \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \left[\begin{array}{c} n+1 \\ k \end{array} \right]_q \frac{1}{m^{n+1-k}} \mathfrak{E}_{k,q}(y) \mathfrak{B}_{n+1-k,q}(mx) \frac{t^n}{[n]_q!}, \end{aligned}$$

and

$$\begin{aligned} I_1 &= \frac{1}{t} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(mx) \frac{t^n}{m^n [n]_q!} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_q \frac{\mathfrak{E}_{k,q}(y)}{m^{n-k}} \frac{t^n}{[n]_q!} \\ &= \frac{1}{t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{n-k,q}(mx) \sum_{j=0}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\}_q \frac{\mathfrak{E}_{j,q}(y)}{m^{n-k} m^{k-j}} \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ j \end{array} \right]_q \mathfrak{B}_{n-j,q}(mx) \sum_{k=0}^{j} \left\{ \begin{array}{c} j \\ k \end{array} \right\}_q \frac{\mathfrak{E}_{k,q}(x)}{m^{n-k}} \frac{t^{n-1}}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \frac{1}{[n+1]_q} \sum_{j=0}^{n+1} \left[\begin{array}{c} n+1 \\ j \end{array} \right]_q \mathfrak{B}_{n+1-j,q}(mx) \sum_{k=0}^{j} \left\{ \begin{array}{c} j \\ k \end{array} \right\}_q \frac{\mathfrak{E}_{k,q}(x)}{m^{n+1-k}} \frac{t^n}{[n]_q!}. \end{aligned}$$

■

Next we give an interesting relationship between the q-Genocchi polynomials and the q-Bernoulli polynomials.

Theorem 20 *For $n \in \mathbb{N}_0$, the following relationship*

$$\mathfrak{G}_{n,q}(x,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n-k}}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q\left(\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}}\mathfrak{G}_{j,q}(x)-\mathfrak{G}_{k,q}(x)\right)\mathfrak{B}_{n+1-k,q}(my),$$

$$\mathfrak{B}_{n,q}(x,y)=\frac{1}{2[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n-k}}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q\left(\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}}\mathfrak{B}_{j,q}(x)+\mathfrak{B}_{k,q}(x)\right)\mathfrak{G}_{n+1-k,q}(my)$$

holds true between the q-Genocchi and the q-Bernoulli polynomials.

Proof. Using the following identity

$$\begin{aligned}
&\frac{2t}{\mathcal{E}_q(t)+1}\mathcal{E}_q(tx)\,\mathcal{E}_q(ty)\\
&=\frac{2t}{\mathcal{E}_q(t)+1}\mathcal{E}_q(tx)\cdot\left(\mathcal{E}_q\left(\frac{t}{m}\right)-1\right)\frac{m}{t}\cdot\frac{\frac{t}{m}}{\mathcal{E}_q\left(\frac{t}{m}\right)-1}\cdot\mathcal{E}_q\left(\frac{t}{m}my\right)
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(-1,q)_n}{m^n2^n}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\
&-\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\
&=\frac{m}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\frac{(-1,q)_{n-k}}{m^{n-k}2^{n-k}}\mathfrak{G}_{k,q}(x)-\mathfrak{G}_{n,q}(x)\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\
&=\frac{m}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{m^{n-k}}\left[\begin{array}{c} n \\ k \end{array}\right]_q\left(\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}}\mathfrak{G}_{j,q}(x)-\mathfrak{G}_{k,q}(x)\right)\mathfrak{B}_{n-k,q}(my)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\sum_{k=0}^{n+1}\frac{1}{m^{n-k}}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q\left(\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\frac{(-1,q)_{k-j}}{m^{k-j}2^{k-j}}\mathfrak{G}_{j,q}(x)-\mathfrak{G}_{k,q}(x)\right)\mathfrak{B}_{n+1-k,q}(my)\frac{t^n}{[n]_q!}.
\end{aligned}$$

The second identity can be proved in a like manner. ■

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