On a class of generalized q-Bernoulli and q-Euler polynomials

Nazim I Mahmudov, M Eini Keleshteri

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Abstract

The main purpose of this paper is to introduce and investigate a new class of generalized q-Bernoulli and q-Euler polynomials. The q-analogues of well-known formulas are derived. A generalization of the Srivastava-Pintér addition theorem is obtained.

1 Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, N_0 denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{\quad}{1-q} \quad (q \neq 1); \qquad [0]_q! = 1; \qquad [n]_q! = [1]_q [2]_q \cdots [n]_q, \quad n \in \mathrm{N}, a \in \mathrm{C},$$

respectively. The q-polynomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\quad}{(q;q)_{n-k}(q;q)_k}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x + y)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n \in N_0.$$

The q-binomial formula is known as

$$(1 - a)_q^n = \prod_{j=0}^{n-1} \left(1 - q^j a\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k.$$

In the standard approach to the q-calculus, two exponential functions are used:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, |z| < \frac{1}{|1-q|},$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q)q^k z\right), \quad 0 < |q| < 1, z \in C.$$

From this form, we easily see that $e_q(z)E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \qquad D_q E_q(z) = E_q(qz),$$

where D_q is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, 0 \neq z \in C.$$

The above q-standard notation can be found in [1].

Carlitz firstly extended the classical Bernoulli and Euler numbers and polynomials, introducing them as q-Bernoulli and q-Euler numbers and polynomials [2−4]. There are numerous recent investigations on this subject by, among many other authors, Cenki *et al.* [5−7], Choi *et al.* [8] and [9], Kim *et al.* [10−13], Ozden and Simsek [14], Ryoo *et al.* [15], Simsek [16, 17] and [18], and Luo and Srivastava [19], Srivastava *et al.* [20], Mahmudov [21, 22].

Recently, Natalini and Bernardini [23], Bretti *et al.* [24], Kurt [25, 26], Tremblay *et al.* [27, 28] studied the properties of the following generalized Bernoulli and Euler polynomials:

$$\left(\frac{t^m}{e^t - \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^\alpha e^{tx} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x)\frac{t^n}{n!},$$

$$\left(\frac{2^m}{e^t + \sum_{k=0}^{m-1} \frac{t^k}{k!}}\right)^\alpha e^{tx} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}, \quad \alpha \in C, 1^\alpha := 1.$$

(1)

Motivated by the generalizations in (1) of the classical Bernoulli and Euler polynomials, we introduce and investigate here the so-called generalized two-dimensional q-Bernoulli and q-Euler polynomials, which are defined as follows.

Definition 1 Let $q, \alpha \in C$, $m \in N$, $0 < |q| < 1$. The generalized two-dimensional q-Bernoulli polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{t}{e_q(t) - T_{m-1,q}(t)} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!},$$

where $T_{m-1,q}(t) = \sum_{k=0}^{m-1} \frac{t^k}{[k]_q!}$.

Definition 2 Let $q, \alpha \in C$, $0 < |q| < 1$, $m \in N$. The generalized two-dimensional q-Euler polynomials $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating functions

$$\left(\frac{2}{e_q(t) + T_{m-1,q}(t)} \right)^\alpha e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) \frac{t^n}{[n]_q!}.$$

It is obvious that

$$\lim_{q \to 1^-} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = B_n^{[m-1,\alpha]}(x+y),$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]} = \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,0), \qquad \lim_{q \to 1^-} \mathfrak{B}_{n,q}^{[m-1,\alpha]} = B_n^{[m-1,\alpha]},$$

$$\lim_{q \to 1^-} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = E_n^{[m-1,\alpha]}(x+y),$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]} = \mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,0), \qquad \lim_{q \to 1^-} \mathfrak{E}_{n,q}^{[m-1,\alpha]} = E_n^{[m-1,\alpha]},$$

$$\lim_{q \to 1^-} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0) = B_n^{[m-1,\alpha]}(x), \qquad \lim_{q \to 1^-} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) = B_n^{[m-1,\alpha]}(y),$$

$$\lim_{q \to 1^-} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0) = E_n^{[m-1,\alpha]}(x), \qquad \lim_{q \to 1^-} \mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y) = E_n^{[m-1,\alpha]}(y).$$

Here $B_n^{[m-1,\alpha]}(x)$ and $E_n^{[m-1,\alpha]}(x)$ denote the generalized Bernoulli and Euler polynomials defined in (1). Notice that $B_n^{[m-1,\alpha]}(x)$ was introduced by Natalini [23], and $E_n^{[m-1,\alpha]}(x)$ was introduced by Kurt [25].

In fact Definitions 1 and 2 define two different types $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Bernoulli polynomials and two different types $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)$ of the generalized q-Euler polynomials. Both polynomials $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)$ ($\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0)$ and $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y)$) coincide with the classical higher-order generalized Bernoulli polynomials (Euler polynomials) in the limiting case $q \to 1^-$.

2 Preliminaries and lemmas

In this section we provide some basic formulas for the generalized q-Bernoulli and q-Euler polynomials to obtain the main results of this paper in the next section. The following result is a q-analogue of the addition theorem for the classical Bernoulli and Euler polynomials.

Lemma 3 *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k},$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x+y)_q^{n-k},$$

(2)

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)y^{n-k}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)x^{n-k},$$

(3)

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x,0)y^{n-k}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(0,y)x^{n-k}.$$

(4)

In particular, setting $x = 0$ and $y = 0$ in (3) and (4), we get the following formulae for the generalized q-Bernoulli and q-Euler polynomials, respectively,

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]} x^{n-k},$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{[m-1,\alpha]} y^{n-k},$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]} x^{n-k},$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{[m-1,\alpha]} y^{n-k}.$$

Setting $y = 1$ and $x = 1$ in (3) and (4), we get, respectively,

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0),$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y),$$

(5)

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{(\alpha)}(x,0),$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(1,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(0,y).$$

(6)

Clearly, (5) and (6) are the generalization of q-analogues of

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x), \qquad E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x),$$

respectively.

Lemma 4 *The generalized q-Bernoulli and q-Euler polynomials satisfy the following relations*:

$$\mathfrak{B}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\mathfrak{B}_{k,q}^{[m-1,\beta]}(0,y),$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha+\beta]}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,\alpha]}(x,0)\mathfrak{E}_{k,q}^{[m-1,\beta]}(0,y).$$

Lemma 5 *We have*

$$D_{q,x}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q\mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(x,y), \qquad D_{q,y}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q\mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(x,qy),$$

$$D_{q,x}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q\mathfrak{E}_{n-1,q}^{[m-1,\alpha]}(x,y), \qquad D_{q,y}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y) = [n]_q\mathfrak{E}_{n-1,q}^{[m-1,\alpha]}(x,qy).$$

Lemma 6 *The generalized q-Bernoulli and q-Euler polynomials satisfy the following relations*:

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) = \frac{\overline{}}{[n-m]_q!}\mathfrak{B}_{n-m,q}^{[m-1,\alpha-1]}(0,y), \quad n \geq m,$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(1,y) + \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{E}_{n,q}^{[m-1,\alpha]}(0,y) = 2^m \mathfrak{E}_{n,q}^{[m-1,\alpha-1]}(0,y),$$

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0) - \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,-1) = \frac{\overline{}}{[n-m]_q!}\mathfrak{B}_{n-m,q}^{[m-1,\alpha-1]}(x,-1), \quad n \geq m,$$

$$\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,0) + \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,-1) = 2^m \mathfrak{E}_{n,q}^{[m-1,\alpha-1]}(x,-1).$$

(7)

Proof We prove only (7). The proof is based on the following equality:

$$\sum_{n=0}^{\infty}\left(\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)\right)\frac{\overline{}}{[n]_q!}$$

$$= \left(\frac{\overline{}}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} e_q(t)E_q(ty) - T_{m-1,q}(t)\left(\frac{\overline{}}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} E_q(ty)$$

$$= \left(\frac{\overline{}}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} E_q(ty)\left(e_q(t) - T_{m-1,q}(t)\right)$$

$$= t^m \left(\frac{\overline{}}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha-1} E_q(ty) = \sum_{n=0}^{\infty}\frac{\overline{}}{[n]_q!}\mathfrak{B}_{n,q}^{[m-1,\alpha-1]}(0,y)\frac{\overline{}}{[n+m]_q!}.$$

Here we used the following relation:

$$T_{m-1,q}(t)\left(\frac{\overline{}}{e_q(t) - T_{m-1,q}(t)}\right)^{\alpha} E_q(ty)$$

$$= \sum_{n=0}^{m-1}\frac{\overline{}}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{\overline{}}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\left(\frac{\overline{}}{[n]_q!} + \frac{\overline{}}{[n]_q!} + \frac{\overline{}}{[n]_q![2]_q!} + \cdots + \frac{\overline{}}{[n]_q![m-1]_q!}\right)$$

$$= \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{\overline{}}{[n]_q!} + \sum_{n=0}^{\infty}[n]_q\mathfrak{B}_{n-1,q}^{[m-1,\alpha]}(0,y)\frac{\overline{}}{[n]_q!}$$

$$+ \sum_{n=0}^{\infty}\frac{\overline{}}{[2]_q!}\mathfrak{B}_{n-2,q}^{[m-1,\alpha]}(0,y)\frac{\overline{}}{[n]_q!}$$

$$+ \cdots + \sum_{n=0}^{\infty}\frac{\overline{}}{[m-1]_q!}\mathfrak{B}_{n-m+1,q}^{[m-1,\alpha]}(0,y)\frac{\overline{}}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)\frac{\overline{}}{[n]_q!}.$$

□

Corollary 7 *Taking $q \to 1^-$, we have*

$$\mathfrak{B}_n^{[m-1,\alpha]}(y+1) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k}^{[m-1,\alpha]}(y) = \frac{\overline{}}{[n-m]_q!} \mathfrak{B}_{n-m}^{[m-1,\alpha-1]}(y), \quad n \geq m,$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(y+1) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_n^{[m-1,\alpha]}(y) = 2^m \mathfrak{E}_n^{[m-1,\alpha-1]}(y).$$

Lemma 8 *The generalized q-Bernoulli polynomials satisfy the following relations*:

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y)$$

$$= [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\mathfrak{B}_{n-1-k,q}^{[0,-1]}.$$

(8)

Proof

Indeed,

$$\sum_{n=0}^{\infty} \left(\mathfrak{B}_{n,q}^{[m-1,\alpha]}(1,y) - \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,y) \right) \frac{}{[n]_q!}$$

$$= \left(\frac{}{e_q(t)-T_{m-1,q}(t)} \right)^{\alpha} e_q(t)E_q(ty) - T_{m-1,q}(t)\left(\frac{}{e_q(t)-T_{m-1,q}(t)} \right)^{\alpha} E_q(ty)$$

$$= \left(\frac{}{e_q(t)-T_{m-1,q}(t)} \right)^{\alpha} E_q(ty)\frac{}{t}\, t$$

$$= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\frac{}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[0,-1]}\frac{}{[n]_q!}$$

$$= \sum_{n=1}^{\infty} [n]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\mathfrak{B}_{n-1-k,q}^{[0,-1]}\frac{}{[n]_q!}.$$

□

Remark 9 Notice taking limit in (8) as $q \to 1^-$, we get

$$\mathfrak{B}_n^{[m-1,\alpha]}(y+1) - \sum_{k=0}^{\min(n,m-1)} \binom{n}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(y) = n\sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{B}_k^{[m-1,\alpha]}(y)\mathfrak{B}_{n-1-k}^{[0,-1]}.$$

It is a correct form of formula (2.7) from [27] for $\lambda = 1$.

Lemma 10 *We have*

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\cdots}{[k+m]_q!} \mathfrak{B}_{n-k,q}^{[m-1,1]}(x,0), \qquad y^n = \frac{\cdots}{q^{\frac{1}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\cdots}{[k+m]_q!} \mathfrak{B}_{n-k,q}^{[m-1,1]}(0,y),$$

$$x^n = \frac{\cdots}{2^m} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,1]}(x,0) \right),$$

$$y^n = \frac{\cdots}{2^m q^{\frac{1}{2}}} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{[m-1,1]}(0,y) + \sum_{k=0}^{\min(n,m-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{n,q}^{[m-1,1]}(0,y) \right).$$

From Lemma 10 we obtain the list of generalized q-Bernoulli polynomials as follows

$$\mathfrak{B}_{0,q}^{[m-1,1]}(x,0) = [m]_q!, \qquad \mathfrak{B}_{0,q}^{[m-1,1]}(0,y) = [m]_q!,$$

$$\mathfrak{B}_{1,q}^{[m-1,1]}(x,0) = [m]_q! \left(x - \frac{\cdots}{[m+1]_q} \right), \qquad \mathfrak{B}_{1,q}^{[m-1,1]}(0,y) = [m]_q! \left(y - \frac{\cdots}{[m+1]_q} \right),$$

$$\mathfrak{B}_{2,q}^{[m-1,1]}(x,0) = x^2 - \frac{\cdots}{[m+1]_q} x + \frac{\cdots \cdots}{[m+1]_q^2 [m+2]_q},$$

$$\mathfrak{B}_{2,q}^{[m-1,1]}(0,y) = qy^2 - \frac{\cdots}{[m+1]_q} y + \frac{\cdots \cdots}{[m+1]_q^2 [m+2]_q}.$$

3 Explicit relationship between the q-Bernoulli and q-Euler polynomials

In this section, we give some generalizations of the Srivastava-Pintér addition theorem. We also obtain new formulae and their some special cases below.

We present natural q-extensions of the main results of the papers [29, 30].

Theorem 11 *The relationships*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)$$

$$= \frac{\cdots}{2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left[\frac{\cdots}{l^{n-k}} \mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0) + \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{\cdots}{l^{k-j}} \mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0) \right] \mathfrak{E}_{n-k,q}(0,ly),$$

(9)

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \frac{\cdots}{2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\cdots}{l^{n-k}} \left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) + \mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{\cdots}{l}, y \right) \right] \mathfrak{E}_{n-k,q}(lx,0)$$

(10)

hold true between the generalized q-Bernoulli polynomials and q-Euler polynomials.

Proof First we prove (9). Using the identity

$$\left(\frac{1}{e_q(t) - \sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx)E_q(ty)$$

$$= \frac{1}{e_q(\frac{t}{l})+1} \cdot E_q\left(\frac{1}{l}ly\right) \cdot \frac{2}{2} \cdot \left(\frac{1}{e_q(t) - \sum_{i=0}^{m-1}\frac{t^i}{[i]_q!}}\right)^{\alpha} e_q(tx),$$

we have

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{t^n}{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,ly)\frac{t^n}{l^n[n]_q!} \sum_{k=0}^{\infty}\frac{t^k}{l^k[k]_q!} \sum_{j=0}^{\infty}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\frac{t^j}{[j]_q!}$$

$$+ \frac{1}{2}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!} \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}$$

$$= : I_1 + I_2.$$

It is clear that

$$I_2 = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\mathfrak{E}_{n-k,q}(0,ly)\frac{t^n}{[n]_q!}.$$

On the other hand,

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(0,ly)\frac{t^k}{l^k[k]_q!}\sum_{j=}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q\mathfrak{E}_{j,q}(0,ly)\frac{t^k}{l^k[k]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\j\end{bmatrix}_q\frac{1}{l^{n-k}}\mathfrak{E}_{j,q}(0,ly)\frac{t^n}{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\mathfrak{E}_{j,q}(0,ly)\sum_{k=0}^{n-j}\begin{bmatrix}n-j\\k\end{bmatrix}_q\frac{1}{l^{n-k}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0)\frac{t^n}{[n]_q!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\overline{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left[\frac{1}{l^{n-k}}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(x,0) + \sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q\frac{1}{l^{k-j}}\mathfrak{B}_{j,q}^{[m-1,\alpha]}(x,0)\right]$$

$$\times \mathfrak{E}_{n-k,q}(0,ly)\overline{[n]_q!}.$$

Next we prove (10). Using the identity

$$\left(\overline{\frac{1}{e_q(t) - \sum_{i=0}^{m-1}\frac{1}{[i]_q!}}}\right)^{\alpha} e_q(tx)E_q(ty)$$

$$= \frac{1}{e_q(\frac{t}{l})+1} \cdot e_q\left(\frac{t}{l}lx\right) \cdot \frac{}{2} \cdot \left(\overline{\frac{1}{e_q(t) - \sum_{i=0}^{m-1}\frac{1}{[i]_q!}}}\right)^{\alpha} E_q(ty),$$

we have

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\overline{[n]_q!} = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(lx,0)\overline{\frac{1}{l^n[n]_q!}}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}\left(\frac{x}{l},y\right)\overline{[n]_q!}$$

$$+ \frac{1}{2}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\overline{\frac{1}{l^k[k]_q!}}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\overline{[n]_q!}$$

$$=: I_1 + I_2.$$

It is clear that

$$I_2 = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(0,y)\overline{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\overline{\frac{1}{l^k[k]_q!}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y)\mathfrak{E}_{n-k,q}(lx,0)\overline{[n]_q!}.$$

On the other hand,

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}\left(\frac{x}{l},y\right)\overline{[n]_q!}\sum_{k=0}^{\infty}\mathfrak{E}_{k,q}(lx,0)\overline{\frac{1}{m^k[k]_q!}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q l^{k-n}\mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{x}{l},y\right)\mathfrak{E}_{n-k,q}(lx,0)\overline{[n]_q!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\overline{[n]_q!}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q l^{k-n}\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) + \mathfrak{B}_{k,q}^{[m-1,\alpha]}\left(\frac{1}{l},y\right)\right]\mathfrak{E}_{n-k,q}(lx,0)\overline{[n]_q!}.$$

□

Next we discuss some special cases of Theorem 11.

Theorem 12 *The relationship*

$$\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left[\mathfrak{B}_{k,q}^{[m-1,\alpha]}(0,y) + \sum_{k=0}^{\min(n,m-1)}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{n-k,q}^{[m-1,\alpha]}(0,\ldots\right.$$

$$+[k]_q\sum_{j=0}^{k-1}\begin{bmatrix}k-1\\j\end{bmatrix}_q\mathfrak{B}_{j,q}^{[m-1,\alpha]}(0,y)\mathfrak{B}_{k-1-j,q}^{[0,-1]}\right]\mathfrak{E}_{n-k,q}(x,0)$$

holds true between the generalized q-Bernoulli polynomials and the q-Euler polynomials.

Remark 13 Taking $q \to 1^-$ in Theorem 12, we obtain the Srivastava-Pintér addition theorem for the generalized Bernoulli and Euler polynomials.

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\mathfrak{B}_k^{[m-1,\alpha]}(y) + \sum_{k=0}^{\min(n,m-1)}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y)\right.$$

$$+k\sum_{j=0}^{k-1}\binom{k-1}{j}\mathfrak{B}_j^{[m-1,\alpha]}(y)\mathfrak{B}_{k-1-j}^{[0,-1]}\right]\mathfrak{E}_{n-k}(x).$$

(11)

Notice that the Srivastava-Pintér addition theorem for the generalized Apostol-Bernoulli polynomials and the Apostol-Euler polynomials was given in [27]. The formula (11) is a correct version of Theorem 3 [27] for $\lambda = 1$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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