A new class of generalized Bernoulli polynomials and Euler polynomials

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Abstract

The main purpose of this paper is to introduce and investigate a new class of generalized Bernoulli polynomials and Euler polynomials based on the q-integers. The q-analogues of well-known formulas are derived. The q-analogue of the Srivastava–Pintér addition theorem is obtained. We give new identities involving q-Bernstein polynomials.

1 Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1} \left(1 - q^j a\right), \quad n \in \mathbb{N}, \quad (a;q)_\infty = \prod_{j=0}^{\infty} \left(1 - q^j a\right), \quad |q| < 1, \quad a \in \mathbb{C}.$$

The q-numbers and q-numbers factorial is defined by

$$[a]_q = \frac{1 - q^a}{1 - q} \quad (q \neq 1); \quad [0]_q! = 1; \quad [n]_q! = [1]_q [2]_q \ldots [n]_q \qquad n \in \mathbb{N}, \quad a \in \mathbb{C}$$

respectively. The q-polynomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q;q)_n}{(q;q)_{n-k} (q;q)_k}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{1}{2}k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0.$$

The q-binomial formula is known as

$$(1-a)_q^n = (a;q)_n = \prod_{j=0}^{n-1} \left(1 - q^j a\right) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k.$$

In the standard approach to the q-calculus two exponential functions are used:

$$e_q(z)=\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!}=\prod_{k=0}^{\infty}\frac{1}{(1-(1-q)\,q^k z)},\quad 0<|q|<1,\ |z|<\frac{1}{|1-q|},$$
$$E_q(z)=\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}z^n}{[n]_q!}=\prod_{k=0}^{\infty}\left(1+(1-q)\,q^k z\right),\qquad 0<|q|<1,\ z\in\mathbb{C}.$$

From this form we easily see that $e_q(z)\,E_q(-z)=1$. Moreover,

$$D_q e_q(z)=e_q(z),\qquad D_q E_q(z)=E_q(qz),$$

where D_q is defined by

$$D_q f(z):=\frac{f(qz)-f(z)}{qz-z},\qquad 0<|q|<1,\ 0\neq z\in\mathbb{C}.$$

The above q-standard notation can be found in [1].

Over 70 years ago, Carlitz extended the classical Bernoulli and Euler numbes and polynomials and introduced the q-Bernoulli and the q-Euler numbers and polynomials (see [2], [3] and [4]). There are numerous recent investigations on this subject by, among many other authors, Cenki et al. ([12], [13], [14]), Choi et al. ([15] and [16]), Kim et al. ([17]-[24]), Ozden and Simsek [25], Ryoo et al. [28], Simsek ([29], [30] and [31]), and Luo and Srivastava [11], Srivastava et al. [32].

We first give here the definitions of the q-Bernoulli and the q-Euler polynomials of higher order as follows.

Definition 1 *Let $q,\alpha\in\mathbb{C},\ 0<|q|<1$. The q-Bernoulli numbers $\mathfrak{B}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,y)$ in x,y of order α are defined by means of the generating function functions:*

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!},\qquad |t|<2\pi,$$
$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(tx)\,E_q(ty)=\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\,\frac{t^n}{[n]_q!},\qquad |t|<2\pi.$$

Definition 2 *Let $q,\alpha\in\mathbb{C},\ 0<|q|<1$. The q-Euler numbers $\mathfrak{E}_{n,q}^{(\alpha)}$ and polynomials $\mathfrak{E}_{n,q}^{(\alpha)}(x,y)$ in x,y of order α are defined by means of the generating functions:*

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}\frac{t^n}{[n]_q!},\qquad |t|<\pi,$$
$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha}e_q(tx)\,E_q(ty)=\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)\,\frac{t^n}{[n]_q!},\qquad |t|<\pi.$$

It is obvious that

$$\mathfrak{B}_{n,q}^{(\alpha)}=\mathfrak{B}_{n,q}^{(\alpha)}(0,0),\quad \lim_{q\to1^-}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=B_n^{(\alpha)}(x+y),\quad \lim_{q\to1^-}\mathfrak{B}_{n,q}^{(\alpha)}=B_n^{(\alpha)},$$
$$\mathfrak{E}_{n,q}^{(\alpha)}=\mathfrak{E}_{n,q}^{(\alpha)}(0,0),\quad \lim_{q\to1^-}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)=E_n^{(\alpha)}(x+y),\quad \lim_{q\to1^-}\mathfrak{E}_{n,q}^{(\alpha)}=E_n^{(\alpha)},$$
$$\lim_{q\to1^-}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)=B_n^{(\alpha)}(x),\quad \lim_{q\to1^-}\mathfrak{B}_{n,q}^{(\alpha)}(0,y)=B_n^{(\alpha)}(y),$$
$$\lim_{q\to1^-}\mathfrak{E}_{n,q}^{(\alpha)}(x,0)=E_n^{(\alpha)}(x),\quad \lim_{q\to1^-}\mathfrak{E}_{n,q}^{(\alpha)}(0,y)=E_n^{(\alpha)}(y).$$

Here $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ denote the classical Bernoulli and Euler polynomials of order α which are defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\,\frac{t^n}{[n]_q!} \qquad \text{and} \qquad \left(\frac{2}{e^t+1}\right)^{\alpha} e^{tx} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\,\frac{t^n}{[n]_q!}.$$

In fact Definitions 1 and 2 define two different type $\mathfrak{B}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{B}_{n,q}^{(\alpha)}(0,y)$ of the q-Bernoulli polynomials and two different type $\mathfrak{E}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{E}_{n,q}^{(\alpha)}(0,y)$ of the q-Euler polynomials. Both polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{B}_{n,q}^{(\alpha)}(0,y)$ ($\mathfrak{E}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{E}_{n,q}^{(\alpha)}(0,y)$) coincide with the classical highe order Bernoulli polynomilas (Euler polynomilas) in the limiting case $q \to 1^-$.

For the q-Bernoulli numbers $\mathfrak{B}_{n,q}$, the q-Euler numbers $\mathfrak{E}_{n,q}$ of order n, we have

$$\mathfrak{B}_{n,q} = \mathfrak{B}_{n,q}(0,0) = \mathfrak{B}_{n,q}^{(1)}(0,0), \qquad \mathfrak{E}_{n,q} = \mathfrak{E}_{n,q}(0,0) = \mathfrak{E}_{n,q}^{(1)}(0,0),$$

respectively. Note that the q-Bernoulli numbers $\mathfrak{B}_{n,q}$ are defined and studied in [26].

The aim of the present paper is to obtain some results for the above defined q-Bernoulli and q-Euler polynomials. In this paper the q-analogues of well-known results, for example, Srivastava and Pintér [10], Cheon [5], etc., will be given. Also the formulas involving the q-Stirling numbers of the second kind, q-Bernoulli polynomials and Phillips q-Bernstein polynomials are derived.

2 Preliminaries and Lemmas

In this section we shall provide some basic formulas for the q-Bernoulli and q-Euler polynomials in order to obtain the main results of this paper in the next section. The following result is q-analogue of the addition theorem for the classical Bernoulli and Euler polynomials.

Lemma 3 (Addition Theorems) *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(x+y)_q^{n-k}, \quad \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(x+y)_q^{n-k},$$

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{(\alpha)}(x,0)\,y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(0,y)\,x^{n-k}, \tag{1}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{(\alpha)}(x,0)\,y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(0,y)\,x^{n-k}. \tag{2}$$

In particular, setting $x=0$ and $y=0$ in (1) and (2), we get the following formulas for q-Bernoulli and q-Euler polynomials, respectively.

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)} x^{n-k}, \quad \mathfrak{B}_{n,q}^{(\alpha)}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{(\alpha)} y^{n-k}, \tag{3}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)} x^{n-k}, \quad \mathfrak{E}_{n,q}^{(\alpha)}(0,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{(\alpha)} y^{n-k}. \tag{4}$$

Setting $y=1$ and $x=1$ in (1) and (2), we get

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{B}_{k,q}^{(\alpha)}(x,0), \quad \mathfrak{B}_{n,q}^{(\alpha)}(1,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(\alpha)}(0,y), \tag{5}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(x,1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} \mathfrak{E}_{k,q}^{(\alpha)}(x,0), \quad \mathfrak{E}_{n,q}^{(\alpha)}(1,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{(\alpha)}(0,y). \tag{6}$$

Clearly (5) and (6) are q-analogues of

$$B_n^{(\alpha)}(x+1)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)B_k^{(\alpha)}(x), \quad E_n^{(\alpha)}(x+1)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)E_k^{(\alpha)}(x),$$

respectively.

Lemma 4 *We have*

$$\begin{aligned} D_{q,x}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)&=[n]_q\,\mathfrak{B}_{n-1,q}^{(\alpha)}(x,y), \quad D_{q,y}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)=[n]_q\,\mathfrak{B}_{n-1,q}^{(\alpha)}(x,qy),\\ D_{q,x}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)&=[n]_q\,\mathfrak{E}_{n-1,q}^{(\alpha)}(x,y), \quad D_{q,y}\mathfrak{E}_{n,q}^{(\alpha)}(x,y)=[n]_q\,\mathfrak{E}_{n-1,q}^{(\alpha)}(x,qy).\end{aligned}$$

Lemma 5 (Difference Equations) *We have*

$$\mathfrak{B}_{n,q}^{(\alpha)}(1,y)-\mathfrak{B}_{n,q}^{(\alpha)}(0,y)=[n]_q\,\mathfrak{B}_{n-1,q}^{(\alpha-1)}(0,y), \tag{7}$$

$$\mathfrak{E}_{n,q}^{(\alpha)}(1,y)+\mathfrak{E}_{n,q}^{(\alpha)}(0,y)=2\mathfrak{E}_{n,q}^{(\alpha-1)}(0,y), \tag{8}$$

$$\begin{aligned}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)-\mathfrak{B}_{n,q}^{(\alpha)}(x,-1)&=[n]_q\,\mathfrak{B}_{n-1,q}^{(\alpha-1)}(x,-1),\\ \mathfrak{E}_{n,q}^{(\alpha)}(x,0)+\mathfrak{E}_{n,q}^{(\alpha)}(x,-1)&=2\mathfrak{E}_{n,q}^{(\alpha-1)}(x,-1).\end{aligned}$$

From (7) and (3), (8) and (4) we obtain the following formulas.

Lemma 6 *We have*

$$\mathfrak{B}_{n-1,q}^{(\alpha-1)}(0,y)=\frac{1}{[n+1]_q}\sum_{k=0}^{n}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\mathfrak{B}_{k,q}^{(\alpha)}(0,y), \tag{9}$$

$$\mathfrak{E}_{n,q}^{(\alpha-1)}(0,y)=\frac{1}{2}\left[\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{k,q}^{(\alpha)}(0,y)+\mathfrak{E}_{n,q}^{(\alpha)}(0,y)\right]. \tag{10}$$

Putting $\alpha=1$ in (9) and (10), and noting that

$$\mathfrak{B}_{n,q}^{(0)}(0,y)=\mathfrak{E}_{n,q}^{(0)}(0,y)=q^{n(n-1)/2}y^n,$$

we arrive at the following expansions:

$$\begin{aligned} y^n&=\frac{1}{q^{n(n-1)/2}[n+1]_q}\sum_{k=0}^{n}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\mathfrak{B}_{k,q}(0,y),\\ y^n&=\frac{1}{2q^{n(n-1)/2}}\left[\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{E}_{k,q}(0,y)+\mathfrak{E}_{n,q}(0,y)\right],\end{aligned}$$

which are q-analoques of the following familiar expansions

$$y^n=\frac{1}{n+1}\sum_{k=0}^{n}\left(\begin{array}{c}n+1\\k\end{array}\right)B_k(y), \quad y^n=\frac{1}{2}\left[\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)E_k(y)+E_n(y)\right], \tag{11}$$

respectively.

Lemma 7 (Recurrence Relationships) *The polynomials $\mathfrak{B}_{n,q}^{(\alpha)}(x,0)$ and $\mathfrak{E}_{n,q}^{(\alpha)}(x,0)$ satisfy the following difference relationships:*

$$\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q m^j\mathfrak{B}_{j,q}^{(\alpha)}(x,0)-\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q m^j\mathfrak{B}_{j,q}^{(\alpha)}(x,-1)=[k]_q\sum_{j=0}^{k-1}\left[\begin{array}{c}k-1\\j\end{array}\right]_q m^{j+1}\mathfrak{B}_{j,q}^{(\alpha-1)}(x,-1), \tag{12}$$

$$\mathfrak{B}_{k,q}^{(\alpha)}\left(\frac{1}{m},y\right)-\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{B}_{j,q}^{(\alpha)}(0,y)=[k]_q\sum_{j=0}^{k-1}\left[\begin{array}{c}k-1\\j\end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j-1}\mathfrak{B}_{j,q}^{(\alpha-1)}(0,y), \tag{13}$$

$$\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q m^j\mathfrak{E}_{j,q}^{(\alpha)}(x,0)+\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q m^j\mathfrak{E}_{j,q}^{(\alpha)}(x,-1)=2\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q m^j\mathfrak{E}_{j,q}^{(\alpha-1)}(x,-1), \tag{14}$$

$$\mathfrak{E}_{k,q}^{(\alpha)}\left(\frac{1}{m},y\right)+\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{E}_{j,q}^{(\alpha)}(0,y)=2\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{E}_{j,q}^{(\alpha-1)}(0,y). \tag{15}$$

3 Explicit relationship between the q-Bernoulli and q-Euler polynomials

In this section we shall investigate some explicit relationships between the q-Bernoulli and q-Euler polynomials. Here some q-analogues of known results will be given. We also obtain new formulas and their some special cases below. These formulas are some extensions of the formulas of Srivastava and Á. Pintér, Cheon and others.

We present natural q-extensions of th main results of the papers [10], [8], see Theorems 8 and 13.

Theorem 8 *For $n\in\mathbb{N}_0$, the following relationship*

$$\begin{aligned}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)&=\frac{1}{2m^n}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\left[m^k\mathfrak{B}_{k,q}^{(\alpha)}(x,0)+\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q m^j\mathfrak{B}_{j,q}^{(\alpha)}(x,-1)\right.\\&\left.+[k]_q\sum_{j=0}^{k-1}\left[\begin{array}{c}k-1\\j\end{array}\right]_q m^{j+1}\mathfrak{B}_{j,q}^{(\alpha-1)}(x,-1)\right]\mathfrak{E}_{n-k,q}(0,my),\\ \mathfrak{B}_{n,q}^{(\alpha)}(x,y)&=\frac{1}{2m^n}\sum_{k=0}^{n}\left[\begin{array}{c}n\\j\end{array}\right]_q m^k\left[\mathfrak{B}_{k,q}^{(\alpha)}(0,y)+\sum_{j=0}^{k}\left[\begin{array}{c}k\\j\end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{B}_{j,q}^{(\alpha)}(0,y)\right.\\&\left.+[k]_q\sum_{j=0}^{k-1}\left[\begin{array}{c}k-1\\j\end{array}\right]_q\left(\frac{1}{m}-1\right)_q^{k-1-j}\mathfrak{B}_{j,q}^{(\alpha-1)}(0,y)\right]\mathfrak{E}_{n-k,q}(mx,0)\end{aligned}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Proof. Using the following identity

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(tx)\,E_q(ty)=\frac{2}{e_q\left(\frac{t}{m}\right)+1}\cdot E_q\left(\frac{t}{m}my\right)\cdot\frac{e_q\left(\frac{t}{m}\right)+1}{2}\cdot\left(\frac{t}{e_q(t)-1}\right)^{\alpha}e_q(tx)$$

we have

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}$$
$$=: I_1+I_2.$$

It is clear that

$$I_2=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}(0,my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q m^{k-n}\mathfrak{B}_{k,q}^{(\alpha)}(x,0)\,\mathfrak{E}_{n-k,q}(0,my)\frac{t^n}{[n]_q!}.$$

On the other hand

$$I_1=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q m^{-n}\mathfrak{E}_{j,q}(0,my)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathfrak{B}_{k,q}^{(\alpha)}(x,0)\sum_{j=0}^{n-k}\begin{bmatrix} n-k\\ j\end{bmatrix}_q m^{k-n}\mathfrak{E}_{j,q}(0,my)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}m^{-n}\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q\mathfrak{E}_{j,q}(0,my)\sum_{k=0}^{n-j}\begin{bmatrix} n-j\\ k\end{bmatrix}_q m^k\mathfrak{B}_{k,q}^{(\alpha)}(x,0)\frac{t^n}{[n]_q!}.$$

Therefore

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q m^{k-n}\left[\mathfrak{B}_{k,q}^{(\alpha)}(x,0)+m^{-k}\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{B}_{j,q}^{(\alpha)}(x,0)\right]\mathfrak{E}_{n-k,q}(0,my)\frac{t^n}{[n]_q!}.$$

It remains to use the formula (12). ■

Next we discuss some special cases of Theorem 8.

Corollary 9 *For $n\in\mathbb{N}_0$, $m\in\mathbb{N}$ the following relationship*

$$\mathfrak{B}_{n,q}(x,y)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\Bigg[m^k\mathfrak{B}_{k,q}(x,0)+\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q m^j\mathfrak{B}_{j,q}(x,-1)$$
$$+[k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1\\ j\end{bmatrix}_q m^{j+1}(x-1)_q^j\Bigg]\mathfrak{E}_{n-k,q}(0,my),$$

$$\mathfrak{B}_{n,q}(x,y)=\frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix}_q m^k\Bigg[m^k\mathfrak{B}_{k,q}(0,y)+\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q\left(\frac{1}{m}-1\right)_q^{k-j}\mathfrak{B}_{j,q}(0,y)$$
$$+[k]_q\sum_{j=0}^{k-1}\begin{bmatrix} k-1\\ j\end{bmatrix}_q q^{\frac{1}{2}j(j-1)}\left(\frac{1}{m}-1\right)_q^{k-1-j}y^j\Bigg]\mathfrak{E}_{n-k,q}(mx,0)$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Corollary 10 *[8] For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship holds true.*

$$B_n(x+y) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(B_k(y) + \frac{k}{2} y^{k-1} \right) E_{n-k}(x),$$

$$B_n(x+y) = \frac{1}{2m^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left[m^k B_k(x) + m^k B_k\left(x - 1 + \frac{1}{m} \right) + km\left(1 + m(x-1)\right)^{k-1} \right] E_{n-k,q}(my).$$

Corollary 11 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(\mathfrak{B}_{k,q}(0,y) + q^{\frac{1}{2}(k-1)(k-2)} \frac{[k]_q}{2} y^{k-1} \right) \mathfrak{E}_{n-k,q}(x,0). \tag{16}$$

Corollary 12 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{B}_{n,q}(x,0) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q}(x,0) + \left(\mathfrak{B}_{1,q} + \frac{1}{2} \right) \mathfrak{E}_{n-1,q}(x,0), \tag{17}$$

$$\mathfrak{B}_{n,q}(0,y) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \mathfrak{B}_{k,q} \mathfrak{E}_{n-k,q}(0,y) + \left(\mathfrak{B}_{1,q} + \frac{1}{2} \right) \mathfrak{E}_{n-1,q}(0,y). \tag{18}$$

The formulas (16)-(18) are q-extension of the Cheon's main result [5]. Notice that $\mathfrak{B}_{1,q} = -\frac{1}{[2]_q}$, see [26], and the extra term becomes zeo for $q \to 1^-$.

Theorem 13 *For $n \in \mathbb{N}_0$, the following relationship*

$$\begin{aligned} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \frac{1}{m^{n-1}[k+1]_q} & \left[2\sum_{j=0}^{k+1} \left[\begin{array}{c} k+1 \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k+1-j} \mathfrak{E}_{j,q}^{(\alpha-1)}(0,y) \right. \\ & \left. - \sum_{j=0}^{k+1} \left[\begin{array}{c} k+1 \\ j \end{array} \right]_q \left(\frac{1}{m} - 1 \right)_q^{k+1-j} \mathfrak{E}_{j,q}^{(\alpha)}(0,y) - \mathfrak{E}_{k+1,q}^{(\alpha)}(0,y) \right] \mathfrak{B}_{n-k,q}(mx,0), \end{aligned}$$

$$\begin{aligned} \mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \frac{1}{m^n[k+1]_q} & \left[2\sum_{j=0}^{k+1} \left[\begin{array}{c} k+1 \\ j \end{array} \right]_q m^j \mathfrak{E}_{j,q}^{(\alpha-1)}(x,-1) \right. \\ & \left. - \sum_{j=0}^{k+1} \left[\begin{array}{c} k+1 \\ j \end{array} \right]_q m^j \mathfrak{E}_{j,q}^{(\alpha)}(x,-1) - m^{k+1} \mathfrak{E}_{k+1,q}^{(\alpha)}(x,0) \right] \mathfrak{B}_{n-k,q}(0,my) \end{aligned}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Proof. The proof is based on the following identities

$$\left(\frac{2}{e_q(t)+1} \right)^{\alpha} e_q(tx) E_q(ty) = \left(\frac{2}{e_q(t)+1} \right)^{\alpha} E_q(ty) \cdot \frac{e_q\left(\frac{t}{m}\right) - 1}{t} \cdot \frac{t}{e_q\left(\frac{t}{m}\right) - 1} e_q\left(\frac{t}{m} mx \right),$$

$$\left(\frac{2}{e_q(t)+1} \right)^{\alpha} e_q(tx) E_q(ty) = \left(\frac{2}{e_q(t)+1} \right)^{\alpha} e_q(tx) \cdot \frac{e_q\left(\frac{t}{m}\right) - 1}{t} \cdot \frac{t}{e_q\left(\frac{t}{m}\right) - 1} E_q\left(\frac{t}{m} my \right)$$

and similar to that of Theorem 8. ■

Next we discuss some special cases of Theorem 13.

Corollary 14 *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship*

$$\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{m^{-n}}{[k+1]_q} \left[2\sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j (x-1)_q^j \right.$$
$$\left. -\sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix}_q m^j \mathfrak{E}_{j,q}(x,-1) - m^{k+1}\mathfrak{E}_{k+1,q}(x,0) \right] \mathfrak{B}_{n-k,q}(0,my)$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

Corollary 15 *[8] For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship holds true.*

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left(y^{k+1} - E_{k+1}(y)\right) B_{n-k}(x),$$
$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \frac{m^{k-n+1}}{k+1} \left[2\left(x + \frac{1-m}{m}\right)^{k+1} - E_{k+1}\left(x + \frac{1-m}{m}\right) - E_{k+1}(x) \right] B_{n-k}(my).$$

Corollary 16 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \left(q^{\frac{1}{2}k(k+1)} y^{k+1} - \mathfrak{E}_{k+1,q}(0,y)\right) \mathfrak{B}_{n-k,q}(x,0).$$

Corollary 17 *For $n \in \mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{E}_{n,q}(x,0) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{2}{[k+1]_q} \mathfrak{E}_{k+1,q} \mathfrak{B}_{n-k,q}(x,0),$$
$$\mathfrak{E}_{n,q}(0,y) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \mathfrak{E}_{k+1,q} \mathfrak{B}_{n-k,q}(0,y).$$

These formulas are q-analogues of the formula of Srivastava and Á. Pintér [10].

4 q-Stirling Numbers and q-Bernoulli Polynomials

In this section, we aim to derive several formulas involving the q-Bernoulli polynomials, the q-Euler polynomials of order α, the q-Stirling numbers of the second kind and q-Bernstein polynomials.

Theorem 18 *Each of the following relationships holds true for the Stirling numbers $S_2(n,k)$ of the second kind:*

$$\mathfrak{B}_{n,q}^{(\alpha)}(x,y) = \sum_{j=0}^{n} \binom{mx}{j} j! \sum_{k=0}^{n-j} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{j-n} \mathfrak{B}_{k,q}^{(\alpha)}(0,y) S_2(n-k,j),$$
$$\mathfrak{E}_{n,q}^{(\alpha)}(x,y) = \sum_{j=0}^{n} \binom{mx}{j} j! \sum_{k=0}^{n-j} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{j-n} \mathfrak{E}_{k,q}^{(\alpha)}(0,y) S_2(n-k,j).$$

The familiar q-Stirling numbers $S(n,k)$ of the second kind are defined by

$$\frac{(e_q(t)-1)^k}{[k]_q!} = \sum_{m=0}^{\infty} S_{2,q}(m,k) \frac{t^m}{[m]_q!},$$

where $k \in \mathbb{N}$. Next we give relationship between q-Bernstein basis defined by Phillips [27] and q-Bernoulli polynomials

$$b_{n,k}(q;x) := x^k (1-x)_q^{n-k}.$$

Theorem 19 *We have*

$$b_{n,k}(q;x) = x^k \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array}\right]_q S_{2,q}(m,k)\, \mathfrak{B}_{n-m,q}^{(k)}(1,-x). \tag{19}$$

Proof. The proof follows from the following identities.

$$\begin{aligned}\frac{x^k t^k}{[k]_q!} e_q(t) E_q(-xt) &= \frac{x^k t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(1-x)_q^n t^n}{[n]_q!} = \sum_{n=k}^{\infty} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{x^k (1-x)_q^{n-k} t^n}{[n]_q!} \\ &= \sum_{n=k}^{\infty} b_{n,k}(q;x) \frac{t^n}{[n]_q!}.\end{aligned}$$

and

$$\begin{aligned}\frac{x^k t^k}{[k]_q!} e_q(t) E_q(-xt) &= \frac{x^k (e_q(t)-1)^k}{[k]_q!} \frac{t^k}{(e_q(t)-1)^k} e_q(t) E_q(-xt) \\ &= x^k \sum_{m=0}^{\infty} S_{2,q}(m,k) \frac{t^m}{[m]_q!} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(k)}(1,-x) \frac{t^n}{[n]_q!} \\ &= x^k \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array}\right]_q S_{2,q}(m,k)\, \mathfrak{B}_{n-m,q}^{(k)}(1,-x) \right) \frac{t^n}{[n]_q!}.\end{aligned}$$

■

Finally, in their limit case when $q \to 1^-$, these last result (19) would reduce to the following formula for the classical Bernoulli polynomials $B_n^{(k)}(x)$ and the Bernstein basis $b_{n,k}(x) = x^k (1-x)^{n-k}$:

$$b_{n,k}(x) = x^k \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array}\right]_q S_2(m,k)\, \mathfrak{B}_{n-m}^{(k)}(1-x).$$

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