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Some Formulas for a Family of Numbers Analogous to the Higher-Order Bernoulli Numbers

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Abstract

In this paper the authors establish several formulas and results for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$, which are analogous to the higher-order Bernoulli numbers. Some applications of these families of D numbers are also presented.

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1 Introduction

The main purpose of this paper is to prove several formulas and results for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$, which are (in a sense) analogous to the higher-order Bernoulli numbers. We also discuss some applications of the various results which are presented here for these D numbers. With a view to making our presentation as much self-contained as possible, and for the convenience of (and ready reference by) the interested reader, we have closely followed and chosen to freely reproduce here some basic definitions and preliminary results from such recent publications as (for example) [7] (and indeed also from the book [9]). We thus begin this paper by introducing the Bernoulli polynomials $B_n^{(k)}(x)$ of order k and degree n, which may be defined (for any integer k) by means of the following generating function (see, for example, [2, 4, 8, 9, 13, 14, 15]):

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} \qquad (|t| < 2\pi;\ k \in \mathbb{Z}), \tag{1}$$

where $\mathbb{Z}$ denotes the set of integers. In partucular, the numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k and the numbers $B_n^{(1)} = B_n$ are referred to as the ordinary Bernoulli numbers. By using the generating function (1), we can get

$$\frac{\mathrm{d}}{\mathrm{d}x}B_n^{(k)}(x) = nB_{n-1}^{(k)}(x),$$

$$B_n^{(k+1)}(x) = \frac{k-n}{k}B_n^{(k)}(x) + (x-k)\frac{n}{k}B_{n-1}^{(k)}(x)$$

and

$$B_n^{(k+1)}(x+1) = \frac{nx}{k}B_{n-1}^{(k)}(x) - \frac{n-k}{k}B_n^{(k)}(x),$$

where

$$n \in \mathbb{N} \qquad (\mathbb{N} := \{1,2,3,\cdots\} = \mathbb{N}_0 \setminus \{0\}).$$

Specifically, the numbers $B_n^{(n)}$ are called the Nörlund numbers (see [1, 3, 4]). A generating function for the numbers $B_n^{(n-k)}$ is given by (see [9])

$$\frac{1}{1+t}\left(\frac{t}{\log(1+t)}\right)^{k+1} = \sum_{n=0}^{\infty} B_n^{(n-k)}\frac{t^n}{n!}.$$

For many interesting applications of the numbers $B_n^{(n)}$ and $B_n^{(n-1)}$, one may refer to [9] (see also [7]).

The so-called D numbers $D_{2n}^{(k)}$ of the first kind are defined by means of the following generating function (see [5, 6, 7, 8, 9, 10]):

$$(t\csc t)^k = \sum_{n=0}^{\infty}(-1)^n D_{2n}^{(k)}\frac{t^{2n}}{(2n)!} \qquad (|t| < \pi). \tag{2}$$

Indeed, by using (1) and (2), and by observing that

$$\csc t = \frac{2i}{e^{it} - e^{-it}},$$

we easily find that

$$\sum_{n=0}^{\infty} (-1)^n D_{2n}^{(k)} \frac{t^{2n}}{(2n)!} = \left(\frac{2it}{e^{it} - e^{-it}} \right)^k = \left(\frac{2it}{e^{2it} - 1} \right)^k e^{kit} = \sum_{n=0}^{\infty} B_n^{(k)} \left(\frac{k}{2} \right) \frac{(2it)^n}{n!}.$$

Therefore, we have

$$D_{2n}^{(k)} = 4^n B_{2n}^{(k)} \left(\frac{k}{2} \right). \tag{3}$$

Upon setting $k = 1$ and $k = 2$ in this last equation (3), if we note that

$$B_{2n}^{(1)} \left(\frac{1}{2} \right) = (2^{1-2n} - 1) B_{2n} \qquad \text{and} \qquad B_{2n}^{(2)}(1) = (1 - 2n) B_{2n},$$

we have

$$D_{2n}^{(1)} = (2 - 2^{2n}) B_{2n} \qquad \text{and} \qquad D_{2n}^{(2)} = 4^n (1 - 2n) B_{2n}.$$

On the other hand, if we put $k = -1$ in (2) and note that

$$\sin t = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!} \qquad \text{and} \qquad \cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!},$$

then it is easily seen that

$$D_{2n}^{(-1)} = \frac{1}{2n+1} \qquad \text{and} \qquad D_{2n}^{(-2)} = \frac{4^n}{(n+1)(2n+1)}.$$

The D numbers $D_{2n}^{(k)}$ satisfy the following recurrence relation (see [5]):

$$D_{2n}^{(k)} = \frac{(2n-k+2)(2n-k+1)}{(k-2)(k-1)} D_{2n}^{(k-2)} - \frac{2n(2n-1)(k-2)}{k-1} D_{2n-2}^{(k-2)}. \tag{4}$$

In light of (4), we can immediately deduce the following known results (see [9]):

$$D_{2n}^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n} \qquad \text{and} \qquad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1} (n!)^2$$

and

$$D_{2n}^{(2n+3)} = \frac{(-1)^n (2n)!}{2 \cdot 4^{2n}} \binom{2n+2}{n+1} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} \right).$$

Recently, Liu [7] derived the following exponential generating function for $D_{2n}^{(2n-k)}$:

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1+t^2}} \left(\frac{t}{\log(t+\sqrt{1+t^2})} \right)^{k+1}. \tag{5}$$

Some of the important applications of the numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ include (for example) each of the following known results:

$$\int_0^{\frac{\pi}{2}} \frac{\sin t}{t} \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \tag{6}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin t}{t} \, \mathrm{d}t = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n!)^2} \tag{7}$$

and

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \, D_{2n}^{(2n-1)}}{(2n-1)(2n)!}. \tag{8}$$

The D numbers $d_{2n}^{(k)}$ of the second kind may be defined by means of the following generating function:

$$\left(\frac{t}{\log(t+\sqrt{1+t^2})} \right)^k = \sum_{n=0}^{\infty} d_{2n}^{(k)} t^{2n}. \tag{9}$$

The numbers $d_n = d_n^{(1)}$ are referred to as the ordinary D numbers of the second kind. Some consequences of the generating function (9) are given below:

$$d_0 = 1, \quad d_2 = \frac{1}{6}, \quad d_4 = -\frac{17}{360}, \quad d_6 = \frac{367}{15120}, \quad d_8 = -\frac{195013}{27216000} \quad \text{and} \quad d_{10} = \frac{1295803}{252806400}.$$

Indeed, by using the generating function (9), we also find that

$$\sum_{n=1}^{\infty} 2n d_{2n}^{(k)} t^{2n-1} = k \left(\frac{t}{\log(t+\sqrt{1+t^2})} \right)^{k-1} \frac{\log(t+\sqrt{1+t^2}) - \frac{t}{\sqrt{1+t^2}}}{\left[\log(t+\sqrt{1+t^2})\right]^2},$$

that is, that

$$\sum_{n=1}^{\infty} 2n d_{2n}^{(k)} t^{2n} = k \sum_{n=0}^{\infty} d_{2n}^{(k)} t^{2n} - k \sum_{n=0}^{\infty} D_{2n}^{(2n-k)} \frac{t^{2n}}{(2n)!}. \tag{10}$$

Consequently, we have (see [7, Theorem 4])

$$(2n)! d_{2n}^{(k)} = \frac{k}{k-2n} D_{2n}^{(2n-k)}. \tag{11}$$

By applying (11), we obtain

$$(2n)!d_{2n}^{(2n+1)} = 1 \qquad \text{and} \qquad (2n)!d_{2n}^{(2n+2)} = \frac{4^n}{2n+1},$$

and

$$(2n)!d_{2n}^{(2n-1)} = (2n-1)(2^{2n}-2)B_{2n} \qquad \text{and} \qquad (2n)!d_{2n}^{(2n-2)} = 4^n(n-1)(2n-1)B_{2n}.$$

We turn now to the *central factorial numbers* $\mathfrak{t}(n,k)$ of the first kind, which are usually defined by (see [11])

$$x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\cdots\left(x+\frac{n}{2}-n+1\right) = \sum_{k=0}^{n} \mathfrak{t}(n,k)x^k \tag{12}$$

or, equivalently, by means of the following generating function:

$$\left[2\log\left(\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right)\right]^k = k!\sum_{n=k}^{\infty} \mathfrak{t}(n,k)\frac{x^n}{n!}. \tag{13}$$

By appealing to (12) or (13), we can show that

$$\mathfrak{t}(n,k) = \mathfrak{t}(n-2,k-2) - \frac{1}{4}(n-2)^2\,\mathfrak{t}(n-2,k) \tag{14}$$

and that

$$\mathfrak{t}(n,0) = \delta_{n,0} \quad (n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}) \qquad \text{and} \qquad \mathfrak{t}(n,n) = 1 \quad (n \in \mathbb{N})$$

and

$$\mathfrak{t}(n,k) = 0 \quad (n+k \quad \text{odd}) \quad \text{and} \quad \mathfrak{t}(n,k) = 0 \quad (k > n \quad \text{or} \quad k < 0),$$

where $\delta_{m,n}$ denotes the Kronecker symbol.

Next, by making use of (12), we obtain

$$(x^2-1^2)(x^2-3^2)\cdots[x^2-(2n-1)^2] = \sum_{k=0}^{n} 4^{n-k}\mathfrak{t}(2n+1,2k+1)x^{2k} \tag{15}$$

and

$$x^2(x^2-1^2)(x^2-2^2)\cdots[x^2-(n-1)^2] = \sum_{k=0}^{n} \mathfrak{t}(2n,2k)x^{2k}. \tag{16}$$

By applying (15) and (16), we find for $n \in \mathbb{N}_0$ that

$$\mathfrak{t}(2n+1,1) = \left(-\frac{1}{4}\right)^n \cdot 1^2 \cdot 3^2 \cdots (2n-1)^2 \quad \text{and} \quad \mathfrak{t}(2n+2,2) = (-1)^n(n!)^2$$

and

$$\mathfrak{t}(2n+2,4) = (-1)^{n+1}(n!)^2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right) \qquad (n \in \mathbb{N}).$$

By using (4) and (14), we have

$$D_{2n-2k}^{(2n+1)} = \frac{4^{n-k}}{\binom{2n}{2k}}\,\mathfrak{t}(2n+1,2k+1) \qquad (n \geqq k \geqq 0) \tag{17}$$

and

$$D_{2n-2k}^{(2n)} = \frac{4^{n-k}}{\binom{2n-1}{2k-1}}\,\mathfrak{t}(2n,2k) \qquad (n \geqq k \geqq 1). \tag{18}$$

In Sections 2 and 3 of this paper, we shall state and prove several formulas and results for the D numbers $D_{2n}^{(k)}$ and $d_{2n}^{(k)}$, respectively. Then, in Section 4, we discuss some applications of these families of D numbers.

2 Formulas and Results Involving the Numbers $D_{2n}^{(k)}$

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$D_{2n}^{(2n)} = \int_0^1 (x^2-1^2)(x^2-3^2)\cdots[x^2-(2n-1)^2]\mathrm{d}x. \tag{19}$$

Proof. By applying (15) and (17) and noting that (see [5])

$$D_{2k}^{(-1)} = \frac{1}{2k+1} \qquad \text{and} \qquad D_{2n}^{(k)} = \sum_{j=0}^{n}\binom{2n}{2j}D_{2n-2j}^{(k-l)}D_{2j}^{(l)},$$

we get

$$\begin{aligned}
D_{2n}^{(2n)} &= \sum_{k=0}^{n}\binom{2n}{2k}D_{2n-2k}^{(2n+1)}D_{2k}^{(-1)} = \sum_{k=0}^{n}\binom{2n}{2k}D_{2n-2k}^{(2n+1)}D_{2k}^{(-1)}\int_0^1(2k+1)x^{2k}\,\mathrm{d}x \\
&= \int_0^1\sum_{k=0}^{n}\binom{2n}{2k}D_{2n-2k}^{(2n+1)}x^{2k}\,\mathrm{d}x = \int_0^1\sum_{k=0}^{n}4^{n-k}\mathfrak{t}(2n+1,2k+1)x^{2k}\,\mathrm{d}x \\
&= \int_0^1(x^2-1^2)(x^2-3^2)\cdots[x^2-(2n-1)^2]\mathrm{d}x,
\end{aligned}$$

which completes the proof of the assertion (19) of Theorem 1. □

Theorem 2. *Let $n \in \mathbb{N}$. Then*

$$D_{2n}^{(2n-1)} = (1-2n) \sum_{k=0}^{n} \binom{2n}{2k} D_{2n-2k}^{(2n)}. \tag{20}$$

Proof. By using (5), (18) and (13), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2n}{2k} D_{2n-2k}^{(2n)} \frac{t^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{n4^{n-k}}{k} \, \mathfrak{t}(2n,2k) \frac{t^{2n}}{(2n)!} \\
&= \frac{t}{\sqrt{1+t^2} \log\left(t+\sqrt{1+t^2}\right)} + \sum_{k=1}^{\infty} \frac{t}{2k} \sum_{n=k}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ 4^{n-k} \, \mathfrak{t}(2n,2k) \, \frac{t^{2n}}{(2n)!} \right\} \\
&= \frac{t}{\sqrt{1+t^2} \, \log\left(t+\sqrt{1+t^2}\right)} + \sum_{k=1}^{\infty} \frac{t}{(2k)\cdot(2k)!} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \left[\log\left(t+\sqrt{1+t^2}\right) \right]^{2k} \right\} \\
&= \frac{t}{\sqrt{1+t^2} \, \log\left(t+\sqrt{1+t^2}\right)} \sum_{k=0}^{\infty} \frac{\left[\log\left(t+\sqrt{1+t^2}\right)\right]^{2k}}{(2k)!} \\
&= \frac{t}{\log\left(t+\sqrt{1+t^2}\right)} = \sum_{n=0}^{\infty} d_{2n} t^{2n},
\end{aligned}$$

which, in view of (11), yields

$$\sum_{k=0}^{n} \binom{2n}{2k} D_{2n-2k}^{(2n)} = (2n)! d_{2n} = \frac{1}{1-2n} D_{2n}^{(2n-1)}.$$

This completes the proof of the assertion (20) of Theorem 2. □

Theorem 3. *Let $n \in \mathbb{N}$. Then*

$$D_{2n}^{(2n-1)} = (1-2n) \int_0^1 x^2(x^2-2^2) \cdots [x^2-(2n-2)^2] \mathrm{d}x. \tag{21}$$

Proof. By using (16), (18) and (20), we have

$$
\begin{aligned}
&\int_0^1 x^2(x^2-2^2)\cdots[x^2-(2n-2)^2]\mathrm{d}x \\
&\quad = 2\int_0^{\frac{1}{2}} 4x^2(4x^2-2^2)\cdots[4x^2-(2n-2)^2]\mathrm{d}x \\
&\quad = 2^{2n+1}\int_0^{\frac{1}{2}}\sum_{k=0}^{n}\mathfrak{t}(2n,2k)x^{2k}\,\mathrm{d}x = \sum_{k=0}^{n}\frac{4^{n-k}}{2k+1}\mathfrak{t}(2n,2k) \\
&\quad = \sum_{k=0}^{n}\frac{\binom{2n-1}{2k-1}}{2k+1}D_{2n-2k}^{(2n)} = \frac{1}{n}\sum_{k=0}^{n}\frac{k}{2k+1}\binom{2n}{2k}D_{2n-2k}^{(2n)} \\
&\quad = \frac{1}{2n}\sum_{k=0}^{n}\binom{2n}{2k}D_{2n-2k}^{(2n)} - \frac{1}{2n}\sum_{k=0}^{n}\frac{1}{2k+1}\binom{2n}{2k}D_{2n-2k}^{(2n)} \\
&\quad = \frac{1}{2n(1-2n)}D_{2n}^{(2n-1)} - \frac{1}{2n}\sum_{k=0}^{n}\binom{2n}{2k}D_{2n-2k}^{(2n)}D_{2k}^{(-1)} \\
&\quad = \frac{1}{2n(1-2n)}D_{2n}^{(2n-1)} - \frac{1}{2n}D_{2n}^{(2n-1)} = \frac{1}{1-2n}D_{2n}^{(2n-1)},
\end{aligned}
$$

which completes the proof of the assertion (21) of Theorem 3. □

3 Formulas and Results Involving the Numbers $d_{2n}^{(k)}$

Theorem 4. *Let $n,k\in\mathbb{N}$. Then*

$$
k(k+1)d_{2n}^{(k+2)} = (2n-k)(2n-k-1)d_{2n}^{(k)} + (2n-k-2)^2 d_{2n-2}^{(k)}. \tag{22}
$$

Proof. By making use of (9), we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty}(2n-k)(2n-k-1)d_{2n}^{(k)}t^{2n-k-2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^k \\
= \frac{k(k+1)}{1+t^2}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+2} &+ \left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+1}\frac{kt}{(1+t^2)^{3/2}}.
\end{aligned} \tag{23}
$$

On the other hand, we also have

$$
\begin{aligned}
\sum_{n=1}^{\infty}(2n-k-2)d_{2n-2}^{(k)}t^{2n-k-3} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^k \\
&= -\frac{k}{\sqrt{1+t^2}}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+1}.
\end{aligned}
$$

Therefore, we get

$$\sum_{n=1}^{\infty}(2n-k-2)^2 d_{2n-2}^{(k)} t^{2n-k-3} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{kt}{\sqrt{1+t^2}}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+1}$$
$$= \frac{k(k+1)t}{1+t^2}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+2} - \left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+1}\frac{k}{(1+t^2)^{3/2}},$$

that is,

$$\sum_{n=1}^{\infty}(2n-k-2)^2 d_{2n-2}^{(k)} t^{2n-k-2} = \frac{k(k+1)t^2}{1+t^2}\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+2}$$
$$- \left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+1}\frac{kt}{(1+t^2)^{3/2}}. \tag{24}$$

Now, by using (23) and (24), we obtain

$$\sum_{n=0}^{\infty}(2n-k)(2n-k-1)d_{2n}^{(k)}t^{2n-k-2} + \sum_{n=1}^{\infty}(2n-k-2)^2 d_{2n-2}^{(k)} t^{2n-k-2}$$
$$= k(k+1)\left(\frac{1}{\log\left(t+\sqrt{1+t^2}\right)}\right)^{k+2} = k(k+1)\sum_{n=0}^{\infty} d_{2n}^{(k+2)} t^{2n-k-2}. \tag{25}$$

Finally, by comparing the coefficients of t^{2n-k-2} on both sides of (25), we are led easily to (22). This completes the proof of Theorem 4. $\square$

Remark 5. Upon setting $k = 2n-2$ in Theorem 4, if we make use of (11), we immediately obtain the following result:

$$(2n)!d_{2n}^{(2n)} = 4^n B_{2n} \qquad (n \in \mathbb{N}_0). \tag{26}$$

A generalization of the above result (26) and other analogous results can be found in the recent work by Liu [7, Corollary 1 and Theorem 4].

Theorem 6. *Let $n, k \in \mathbb{N}$ and $n \geqq k+1$. Then*

$$(2k)!(2n-2k-1)!d_{2n}^{(2k+1)} = \sum_{j=0}^{k}(2n-1-2j)!\sigma(n,k,j)d_{2n-2j}, \tag{27}$$

where

$$\sigma(n,k,j) = \sum_{\substack{v_1,\cdots,v_{k-j+1}\in\mathbb{N}_0 \\ (v_1+\cdots+v_{k-j+1}=j)}} (2n-2j-1)^{2v_1}(2n-2j-3)^{2v_2}\cdots(2n-2k-1)^{2v_{k-j+1}}.$$

Proof. We prove the assertion (27) of Theorem 6 by using the principle of mathematical induction. Indeed, when $k = 1$, (27) is true by virtue of (22). Suppose now that (27) is true for some natural number $k \in \mathbb{N} \setminus \{1\}$. Then, by the superposition of (22), we have

$$\begin{aligned}
&(2k+1)(2k+2)d_{2n}^{(2k+3)} \\
&\quad = (2n-2k-1)(2n-2k-2)d_{2n}^{(2k+1)} + (2n-2k-3)^2 d_{2n-2}^{(2k+1)} \\
&\quad = \frac{(2n-2k-1)(2n-2k-2)}{(2k)!(2n-2k-1)!} \sum_{j=0}^{k} (2n-1-2j)!\sigma(n,k,j)d_{2n-2j} \\
&\qquad + \frac{(2n-2k-3)^2}{(2k)!(2n-2k-3)!} \sum_{j=0}^{k} (2n-3-2j)!\sigma(n-1,k,j)d_{2n-2-2j} \\
&\quad = \frac{1}{(2k)!(2n-2k-3)!} \sum_{j=0}^{k} (2n-1-2j)!\sigma(n,k,j)d_{2n-2j} \\
&\qquad + \frac{(2n-2k-3)^2}{(2k)!(2n-2k-3)!} \sum_{j=1}^{k+1} (2n-2-2j)!\sigma(n-1,k,j-1)d_{2n-2j}.
\end{aligned} \tag{28}$$

In light of this last result (28), and by noting that

$$\sigma(n,k+1,0) = \sigma(n,k,0),$$

$$\sigma(n,k+1,k+1) = (2n-2k-3)^2 \sigma(n-1,k,k)$$

and

$$\sigma(n,k+1,j) = \sigma(n,k,j) + (2n-2k-3)^2 \sigma(n-1,k,j-1),$$

we find that

$$
\begin{aligned}
(2k+2)!&(2n-2k-3)!d_{2n}^{(2k+3)}\\
&=\sum_{j=0}^{k}(2n-1-2j)!\sigma(n,k,j)d_{2n-2j}\\
&\quad+(2n-2k-3)^2\sum_{j=1}^{k+1}(2n-1-2j)!\sigma(n-1,k,j-1)d_{2n-2j}\\
&=(2n-1)!\sigma(n,k,0)d_{2n}+(2n-2k-3)^2(2n-3-2k)!\sigma(n-1,k,k)d_{2n-2k-2}\\
&\quad+\sum_{j=1}^{k}(2n-1-2j)!\left[\sigma(n,k,j)+(2n-2k-3)^2\sigma(n-1,k,j-1)\right]d_{2n-2j}\\
&=(2n-1)!\sigma(n,k+1,0)d_{2n}+(2n-3-2k)!\sigma(n,k+1,k+1)d_{2n-2k-2}\\
&\quad+\sum_{j=1}^{k}(2n-1-2j)!\sigma(n,k+1,j)d_{2n-2j}\\
&=\sum_{j=0}^{k+1}(2n-1-2j)!\sigma(n,k+1,j)d_{2n-2j},
\end{aligned}
$$

which shows that (27) is also true for the natural number $k+1$. Thus, by the principle of mathematical induction, (27) holds true for all $k \in \mathbb{N}$. This completes the proof of Theorem 6. □

Remark 7. Upon setting $k = 1, 2, 3$ in (27), we can immediately deduce

$$2!d_{2n}^{(3)} = (2n-1)(2n-2)d_{2n} + (2n-3)^2 d_{2n-2},$$

$$
\begin{aligned}
4!d_{2n}^{(5)} &= (2n-1)(2n-2)(2n-3)(2n-4)d_{2n}\\
&\quad+(2n-3)(2n-4)\left[(2n-3)^2+(2n-5)^2\right]d_{2n-2}+(2n-5)^4 d_{2n-4}
\end{aligned}
$$

and

$$
\begin{aligned}
6!d_{2n}^{(7)} &= (2n-1)(2n-2)(2n-3)(2n-4)(2n-5)(2n-6)d_{2n}\\
&\quad+(2n-3)(2n-4)(2n-5)(2n-6)\left[(2n-3)^2+(2n-5)^2\right.\\
&\quad\left.+(2n-7)^2\right]d_{2n-2}+(2n-5)(2n-6)\left[(2n-5)^4\right.\\
&\quad\left.+(2n-5)^2(2n-7)^2+(2n-7)^4\right]d_{2n-4}+(2n-7)^6 d_{2n-6}.
\end{aligned}
$$

Theorem 8. *Let $n, k \in \mathbb{N}$ and $n \geqq k$. Then*

$$(2k-1)!(2n-2k)!d_{2n}^{(2k)} = \sum_{j=0}^{k-1}(2n-2-2j)!\tau(n,k,j)d_{2n-2j}^{(2)}, \tag{29}$$

where

$$\tau(n,k,j) = \sum_{\substack{v_1,\cdots,v_{k-j}\in\mathbb{N}_0 \\ (v_1+\cdots+v_{k-j}=j)}} (2n-2j-2)^{2v_1}(2n-2j-4)^{2v_2}\cdots(2n-2k)^{2v_{k-j}}.$$

Proof. We prove the assertion (29) of Theorem 8 by using the principle of mathematical induction. In fact, when $k=1$ and $k=2$, (29) is true by (22). Suppose now that (29) is true for some natural number $k \in \mathbb{N} \setminus \{1\}$. Then, by the superposition of (22), we have

$$\begin{aligned}
&2k(2k+1)d_{2n}^{(2k+2)} \\
&\quad = (2n-2k)(2n-2k-1)d_{2n}^{(2k)} + (2n-2k-2)^2 d_{2n-2}^{(2k)} \\
&\quad = \frac{(2n-2k)(2n-2k-1)}{(2k-1)!(2n-2k)!}\sum_{j=0}^{k-1}(2n-2-2j)!\tau(n,k,j)d_{2n-2j}^{(2)} \\
&\qquad + \frac{(2n-2k-2)^2}{(2k-1)!(2n-2k-2)!}\sum_{j=0}^{k-1}(2n-4-2j)!\tau(n-1,k,j)d_{2n-2-2j}^{(2)} \\
&\quad = \frac{1}{(2k-1)!(2n-2k-2)!}\sum_{j=0}^{k-1}(2n-2-2j)!\tau(n,k,j)d_{2n-2j}^{(2)} \\
&\qquad + \frac{(2n-2k-2)^2}{(2k-1)!(2n-2k-2)!}\sum_{j=1}^{k}(2n-2-2j)!\tau(n-1,k,j-1)d_{2n-2j}^{(2)}.
\end{aligned} \tag{30}$$

By using (30), and noting that

$$\tau(n,k+1,0) = \tau(n,k,0),$$

$$\tau(n,k+1,k) = (2n-2k-2)^2\tau(n-1,k,k-1)$$

and

$$\tau(n,k+1,j) = \tau(n,k,j) + (2n-2k-2)^2\tau(n-1,k,j-1),$$

we find that

$$
\begin{aligned}
&(2k+1)!(2n-2k-2)!d_{2n}^{(2k+2)} \\
&\quad = \sum_{j=0}^{k-1}(2n-2-2j)!\tau(n,k,j)d_{2n-2j}^{(2)} \\
&\qquad + (2n-2k-2)^2\sum_{j=1}^{k}(2n-2-2j)!\tau(n-1,k,j-1)d_{2n-2j}^{(2)} \\
&\quad = (2n-2)!\tau(n,k,0)d_{2n}^{(2)} + (2n-2k-2)^2(2n-2-2k)!\tau(n-1,k,k-1)d_{2n-2k}^{(2)} \\
&\qquad + \sum_{j=1}^{k-1}(2n-2-2j)!\left[\tau(n,k,j) + (2n-2k-2)^2\tau(n-1,k,j-1)\right]d_{2n-2j}^{(2)} \\
&\quad = (2n-2)!\tau(n,k+1,0)d_{2n}^{(2)} + (2n-2-2k)!\tau(n,k+1,k)d_{2n-2k}^{(2)} \\
&\qquad + \sum_{j=1}^{k-1}(2n-2-2j)!\left[\tau(n,k,j) + (2n-2k-2)^2\tau(n-1,k,j-1)\right]d_{2n-2j}^{(2)} \\
&\quad = \sum_{j=0}^{k}(2n-2-2j)!\tau(n,k+1,j)d_{2n-2j}^{(2)},
\end{aligned}
$$

which shows that (29) holds true also for the natural number $k+1$. This evidently completes the proof of Theorem 8 by the principle of mathematical induction on $k \in \mathbb{N}$. □

Remark 9. By setting $k = 2, 3, 4$ in (29), we can immediately deduce

$$3!d_{2n}^{(4)} = (2n-2)(2n-3)d_{2n}^{(2)} + (2n-4)^2 d_{2n-2}^{(2)},$$

$$
\begin{aligned}
5!d_{2n}^{(6)} &= (2n-2)(2n-3)(2n-4)(2n-5)d_{2n}^{(2)} \\
&\quad + (2n-4)(2n-5)[(2n-4)^2 + (2n-6)^2]d_{2n-2}^{(2)} + (2n-6)^4 d_{2n-4}^{(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
7!d_{2n}^{(8)} &= (2n-2)(2n-3)(2n-4)(2n-5)(2n-6)(2n-7)d_{2n}^{(2)} \\
&\quad + (2n-4)(2n-5)(2n-6)(2n-7)[(2n-4)^2 + (2n-6)^2 + (2n-8)^2]d_{2n-2}^{(2)} \\
&\quad + (2n-6)(2n-7)[(2n-6)^4 + (2n-6)^2(2n-8)^2 + (2n-8)^4]d_{2n-4}^{(2)} \\
&\quad + (2n-8)^6 d_{2n-6}^{(2)}.
\end{aligned}
$$

4 A Set of Applications

We first give the following application of the results presented in the preceding sections.

Theorem 10. *Let $k \in \mathbb{Z}$. Then*

$$\int_0^{\frac{\pi}{2}} \left(\frac{\sin t}{t}\right)^{k+1} \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n-k)}}{(2n+1)!}. \tag{31}$$

Proof. By using (5), we find that

$$\sum_{n=0}^{\infty} (-1)^n D_{2n}^{(2n-k)} \frac{x^{2n}}{(2n)!} = \frac{1}{\sqrt{1-x^2}} \left(\frac{ix}{\log(ix+\sqrt{1-x^2})}\right)^{k+1}.$$

Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n-k)}}{(2n+1)!} &= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\frac{ix}{\log(ix+\sqrt{1-x^2})}\right)^{k+1} \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta} \left(\frac{i\sin\theta}{\log(i\sin\theta+\cos\theta)}\right)^{k+1} d(\sin\theta) = \int_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{\theta}\right)^{k+1} d\theta,\end{aligned}$$

which obviously completes the proof of Theorem 10. □

Remark 11. Setting $k=0$ in (31), we immediately obtain (6). Moreover, if we set $k=-1$ and $k=-2$ in (31) and note that

$$D_{2n}^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \binom{2n}{n} \qquad \text{and} \qquad D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1} (n!)^2,$$

we get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)4^n} \binom{2n}{n} = \frac{\pi}{2}$$

and

$$\int_0^{\frac{\pi}{2}} \frac{t}{\sin t} \,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2 \binom{2n}{n}},$$

The sum on the right-hand side of this last result can be expressed as a Clausenian hypergeometric series as follows:

$${}_3F_2\left(\frac{1}{2}, 1, 1; \frac{3}{2}, \frac{3}{2}; 1\right).$$

Moreover, by using a computer algebra system, the integral on the left-hand side can be found to be twice the Catalan constant $\mathbf{G}$ which is defined by (see, for details, [14, p. 43 et seq.])

$$\mathbf{G} := \frac{1}{2} \int_0^1 \mathbf{K}(\kappa) \mathrm{d}\kappa = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \doteq 0.91596\,55941\,77219\,015 \cdots,$$

where $\mathbf{K}(\kappa)$ is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(\kappa) := \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}t}{\sqrt{1-\kappa^2 \sin^2 t}} \qquad (|\kappa| < 1).$$

In fact, the above observation can be verified *analytically* by setting $\tau = \tan\left(\frac{t}{2}\right)$ and $\mathrm{d}\tau = \frac{1}{2}\sec^2\left(\frac{t}{2}\right)\mathrm{d}t$ in the following known result [13, p. 110, Equation 2.3(37)]:

$$\int_0^1 \frac{\arctan \tau}{\tau}\,\mathrm{d}\tau = \mathbf{G}.$$

Theorem 12. *Let $k \in \mathbb{Z}$. Then*

$$\int_0^{\frac{\pi}{2}} \left(\frac{\sin t}{t}\right)^k \cos t\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} k D_{2n}^{(2n-k)}}{(2n-k)(2n+1)!}. \tag{32}$$

Proof. By using (9) and (11), we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} k D_{2n}^{(2n-k)}}{(2n-k)(2n)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n d_{2n}^{(k)} x^{2n} = \left(\frac{ix}{\log(ix+\sqrt{1-x^2})}\right)^k.$$

Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^{n+1} k D_{2n}^{(2n-k)}}{(2n-k)(2n+1)!} &= \int_0^1 \left(\frac{ix}{\log(ix+\sqrt{1-x^2})}\right)^k \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{i\sin t}{\log(i\sin t+\cos t)}\right)^k \mathrm{d}(\sin t) = \int_0^{\frac{\pi}{2}} \left(\frac{\sin t}{t}\right)^k \cos t\,\mathrm{d}t,\end{aligned}$$

which evidently completes the proof of Theorem 12. □

Remark 13. By setting $k = 1$ and $k = -1$ in (32), we have

$$\int_0^{\pi} \frac{\sin t}{t}\,\mathrm{d}t = 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n+1)!}$$

and

$$\int_0^{\frac{\pi}{2}} t\cot t\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{1}{4^n\,(2n+1)^2}\binom{2n}{n}$$

or, equivalently,

$$\int_0^1 \frac{\arcsin \tau}{\tau}\,\mathrm{d}\tau = \sum_{n=0}^{\infty} \frac{1}{4^n\,(2n+1)^2}\binom{2n}{n}$$

by setting $\sin t = \tau$ and $\cos t\,\mathrm{d}t = \mathrm{d}\tau$. The sum on the right-hand side of this last result can be expressed in a closed form via a Clausenian hypergeometric series as follows:

$$\int_0^1 \frac{\arcsin \tau}{\tau}\,\mathrm{d}\tau = \sum_{n=0}^{\infty} \frac{1}{4^n\,(2n+1)^2}\binom{2n}{n} = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; 1\right) = \frac{\pi}{2}\,\ln 2.$$

Theorem 14. *Let $k \in \mathbb{Z}$. Then*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}k}{(2n-k)(2n)!} D_{2n}^{(2n-k)} = \left(\frac{2}{\pi}\right)^k, \tag{33}$$

provided that each member of (33) exists.

Proof. By applying (9) and (11), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}k}{(2n-k)(2n)!} D_{2n}^{(2n-k)} x^{2n} &= \sum_{n=0}^{\infty} (-1)^n d_{2n}^{(k)} x^{2n} \\ &= \left(\frac{ix}{\log(ix+\sqrt{1-x^2})}\right)^k = \left(\frac{1}{\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)4^n}\binom{2n}{n} x^{2n}}\right)^k.\end{aligned}$$

Therefore, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}k}{(2n-k)(2n)!} D_{2n}^{(2n-k)} = \left(\frac{1}{\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)4^n}\binom{2n}{n}}\right)^k = \left(\frac{2}{\pi}\right)^k.$$

This completes the proof of Theorem 14. $\square$

Remark 15. Upon setting $k = 1$ in (33), we immediately obtain (8). If, on the other hand, we put $k = -2$ in (33) and note that

$$D_{2n}^{(2n+2)} = \frac{(-1)^n 4^n}{2n+1}(n!)^2,$$

then we find that

$$\sum_{n=0}^{\infty} \frac{4^n (n!)^2}{(2n+2)!} = \frac{\pi^2}{8}.$$

This last identity follows also from a known power-series expansion for the function $(\arcsin x)^2$.

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