The Fibonacci Quarterly 2005 (43,3): 208-212

SOME IDENTITIES INVOLVING BERNOULLI NUMBERS

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(Submitted September 2002-Final Revision February 2003)

1. INTRODUCTION AND RESULTS

Let x be a complex number with $|x| < 2\pi$. The Bernoulli numbers $B_n (n = 0, 1, 2, \cdots)$ are defined by the coefficients in the expansion of (see [1], [3] and [4])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi. \tag{1}$$

By (1), we have $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \cdots$, and $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$, we have (see [2])

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m. \tag{2}$$

The main purpose of this paper is to prove some new identities involving Bernoulli numbers. That is, we shall prove the following main conclusion.

Theorem 1: Let $n \geq 1, k \geq 0$ be any integers, then

(a)
$$\sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2 - 2^{2j}}{(2k+1)^{2j}} B_{2j} = \frac{(2n+1)2^{2n+1}}{(2k+1)^{2n+1}} \sum_{i=0}^{k} i^{2n}. \tag{3}$$

(b)
$$\sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2 - 2^{2j}}{(2k+2)^{2j}} B_{2j} = \frac{2n+1}{2^{2n}(k+1)^{2n+1}} \sum_{i=0}^{k} (2i+1)^{2n}. \tag{4}$$

Taking $k = 0, 1, 2$ in Theorem 1, we may immediately deduce the following Corollary 1:

Corollary 1: Let $n \geq 1$ be any integers, then

(a)
$$\sum_{j=0}^{n} \binom{2n+1}{2j} \left(2 - 2^{2j}\right) B_{2j} = 0, \tag{5}$$

$$\sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2 - 2^{2j}}{3^{2j}} B_{2j} = \frac{(2n+1)2^{2n+1}}{3^{2n+1}}, \tag{6}$$

$$\sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2 - 2^{2j}}{5^{2j}} B_{2j} = \frac{2(2n+1)(4^{2n} + 2^{2n})}{5^{2n+1}}. \tag{7}$$

(b)
$$\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{2-2^{2j}}{2^{2j}}B_{2j}=\frac{2n+1}{2^{2n}}, \tag{8}$$

$$\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{2-2^{2j}}{4^{2j}}B_{2j}=\frac{(2n+1)(1+3^{2n})}{2^{4n+1}}, \tag{9}$$

$$\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{2-2^{2j}}{6^{2j}}B_{2j}=\frac{(2n+1))(1+3^{2n}+5^{2n})}{2^{2n}3^{2n+1}}. \tag{10}$$

Theorem 2: Let $n\geq 0, k\geq 0$ be any integers, then

(a)
$$\sum_{j=0}^{n}\binom{2n}{2j}\sum_{i=0}^{k}(2i+1)^{2n-2j}(2-2^{2j})(2k+2)^{2j}B_{2j}=(k+1)(2-2^{2n})B_{2n}, \tag{11}$$

(b)
$$\sum_{j=0}^{n}\binom{2n}{2j}\sum_{i=1}^{k}(2i)^{2n-2j}(2-2^{2j})(2k+1)^{2j}B_{2j}$$
$$=(2k+1)(1-2^{2n-1})(1-(2k+1)^{2n-1})B_{2n}. \tag{12}$$

Taking $k=0,1,2$ in Theorem 2(a) and $k=1,2,3$ in Theorem 2(b), we may immediately deduce the following Corollary 2:

Corollary 2: Let $n\geq 0$ be any integer, then

(a)
$$\sum_{j=0}^{n}\binom{2n}{2j}\left(2-2^{2j}\right)2^{2j}B_{2j}=(2-2^{2n})B_{2n}, \tag{13}$$

$$\sum_{j=0}^{n}\binom{2n}{2j}(1+3^{2n-2j})(2-2^{2j})4^{2j}B_{2j}=2(2-2^{2n})B_{2n}, \tag{14}$$

$$\sum_{j=0}^{n}\binom{2n}{2j}(1+3^{2n-2j}+5^{2n-2j})(2-2^{2j})6^{2j}B_{2j}=3(2-2^{2n})B_{2n}. \tag{15}$$

(b)
$$\sum_{j=0}^{n}\binom{2n}{2j}2^{2n-2j}\left(2-2^{2j}\right)3^{2j}B_{2j}=3(1-2^{2n-1})(1-3^{2n-1})B_{2n}, \tag{16}$$

$$\sum_{j=0}^{n}\binom{2n}{2j}(2^{2n-2j}+4^{2n-2j})(2-2^{2j})5^{2j}B_{2j}=5(1-2^{2n-1})(1-5^{2n-1})B_{2n}, \tag{17}$$

$$\sum_{j=0}^{n}\binom{2n}{2j}(2^{2n-2j}+4^{2n-2j}+6^{2n-2j})(2-2^{2j})7^{2j}B_{2j}=7(1-2^{2n-1})(1-7^{2n-1})B_{2n}. \tag{18}$$

2. SOME LEMMAS

Lemma 1: (see [3, p. 260])

$$\frac{1}{\sin x}=\frac{1}{x}+\sum_{n=1}^{\infty}(-1)^n(2-2^{2n})\frac{B_{2n}}{(2n)!}x^{2n-1},\ 0<|x|<\pi. \tag{19}$$

Proof:

$$\frac{x}{\sin x} = \frac{2ix}{e^{ix} - e^{-ix}} = \frac{2ix}{e^{ix} - 1} - \frac{2ix}{e^{2ix} - 1}$$

$$= 2\sum_{n=0}^{\infty} \frac{B_n (ix)^n}{n!} - \sum_{n=0}^{\infty} \frac{B_n (2ix)^n}{n!}, \ |x| < \pi.$$

Take the real part of this expansion. □

Lemma 2:

(a)
$$\sum_{n=0}^{\infty} (\sin nx) t^n = \frac{t \sin x}{1 - 2t\cos x + t^2}, \ |t| < 1 \tag{20}$$

(b)
$$\sum_{n=0}^{\infty} (\cos nx) t^n = \frac{1 - t\cos x}{1 - 2t\cos x + t^2}, \ |t| < 1. \tag{21}$$

Proof:

$$\sum_{n=0}^{\infty} (\cos nx + i \sin nx) t^n = \sum_{n=0}^{\infty} (e^{ix} t)^n = \frac{1}{1 - e^{ix} t} = \frac{1}{1 - t\cos x - it \sin x}$$

$$= \frac{1 - t\cos x}{1 - 2t\cos x + t^2} + \frac{it \sin x}{1 - 2t \cos x + t^2}, \ |t| < 1.$$

Take the real and imaginary parts. □

Lemma 3:

$$\sum_{n=0}^{\infty} \sin(n+1)x t^n = \frac{\sin x}{1 - 2t\cos x + t^2}, \ |t| < 1. \tag{22}$$

Proof:

$$\sum_{n=0}^{\infty} \sin(n+1)x t^n = \operatorname{Im}\left(\sum_{n=0}^{\infty} e^{i(n+1)x} t^n\right) = \operatorname{Im}\left(e^{ix} \sum_{n=0}^{\infty} (e^{ix} t)^n\right)$$

$$= \frac{t \sin x \cos x + \sin x (1 - t\cos x)}{1 - 2t\cos x + t^2} = \frac{\sin x}{1 - 2t\cos x + t^2}. \quad \square$$

Lemma 4:

$$\sum_{j=0}^{m} \cos(m - 2j)x = \frac{\sin(m+1)x}{\sin x}. \tag{23}$$

Proof:

$$\sum_{j=0}^{m} e^{i(m-2j)x} = e^{imx}\sum_{j=0}^{m}(e^{-2ix})^j = e^{imx}\frac{1-e^{(-2ix)(m+1)}}{1-e^{-2ix}}$$
$$= e^{i(m+1)x}\frac{1-e^{(-2ix)(m+1)}}{e^{ix}-e^{-ix}} = \frac{(e^{i(m+1)x}-e^{-i(m+1)x})/2i}{(e^{ix}-e^{-ix})/2i}$$
$$= \frac{\sin(m+1)x}{\sin x}.$$

Take the real part of this equation. □

3. PROOF OF THE THEOREMS

Proof of Theorem 1: By Lemmas 1 and 4, since $\cos(m-2i)x = \sum_{n=0}^{\infty}(-1)^n (m-2i)^{2n}\frac{x^{2n}}{(2n)!}$ and $\sin(m+1)x = \sum_{n=0}^{\infty}(-1)^n(m+1)^{2n+1}\frac{x^{2n+1}}{(2n+1)!}$, we have

$$\sum_{i=0}^{m}\sum_{n=0}^{\infty}(-1)^n(m-2i)^{2n}\frac{x^{2n}}{(2n)!}$$
$$= \left(\frac{1}{x}+\sum_{n=1}^{\infty}(-1)^n(2-2^{2n})\frac{B_{2n}}{(2n)!}x^{2n-1}\right)\left(\sum_{n=0}^{\infty}(-1)^n(m+1)^{2n+1}\frac{x^{2n+1}}{(2n+1)!}\right)$$
$$= \left(\sum_{n=0}^{\infty}(-1)^n(2-2^{2n})\frac{B_{2n}}{(2n)!}x^{2n}\right)\left(\sum_{n=0}^{\infty}(-1)^n(m+1)^{2n+1}\frac{x^{2n}}{(2n+1)!}\right)$$
$$= \sum_{n=0}^{\infty}(-1)^n(m+1)^{2n+1}\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{2-2^{2j}}{(m+1)^{2j}}B_{2j}\frac{x^{2n}}{(2n+1)!}. \tag{26}$$

Comparing the coefficient of x^{2n} on both sides of (26), we get

$$\frac{(-1)^n}{(2n)!}\sum_{i=0}^{m}(m-2i)^{2n} = \frac{(-1)^n(m+1)^{2n+1}}{(2n+1)!}\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{2-2^{2j}}{(m+1)^{2j}}B_{2j}, \quad \text{i.e.}$$
$$\sum_{j=0}^{n}\binom{2n+1}{2j}\frac{2-2^{2j}}{(m+1)^{2j}}B_{2j} = \frac{2n+1}{(m+1)^{2n+1}}\sum_{i=0}^{m}(m-2i)^{2n}. \tag{27}$$

Set $m = 2k$ in (27) we immediately obtain (3). Set $m = 2k+1$ in (27), we immediately obtain (4). □

Proof of Theorem 2: By Lemma 4, we have

$$\frac{1}{\sin x} = \frac{1}{\sin(m+1)x}\sum_{i=0}^{m}\cos(m-2i)x. \tag{28}$$

By

$$\cos(m-2i)x = \sum_{n=0}^{\infty}(-1)^n(m-2i)^{2n}\frac{x^{2n}}{(2n)!},\ \sin(m+1)x = \sum_{n=0}^{\infty}(-1)^n(m+1)^{2n+1}\frac{x^{2n+1}}{(2n+1)!},$$

(28) and Lemma 1, we have

$$\frac{1}{x}+\sum_{n=1}^{\infty}(-1)^n(2-2^{2n})\frac{B_{2n}}{(2n)!}x^{2n-1} = \left(\frac{1}{(m+1)x}+\sum_{n=1}^{\infty}(-1)^n(2-2^{2n})(m+1)^{2n-1}\frac{B_{2n}}{(2n)!}x^{2n-1}\right)$$
$$\sum_{i=0}^{m}\sum_{n=0}^{\infty}(-1)^n(m-2i)^{2n}\frac{x^{2n}}{(2n)!},\ \text{i.e.}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n(2-2^{2n})B_{2n}\frac{x^{2n}}{(2n)!}\\ &=\left(\sum_{n=0}^{\infty}(-1)^n(2-2^{2n})(m+1)^{2n-1}\frac{B_{2n}}{(2n)!}x^{2n}\right)\sum_{i=0}^{m}\sum_{n=0}^{\infty}(-1)^n(m-2i)^{2n}\frac{x^{2n}}{(2n)!}\\ &=\sum_{n=0}^{\infty}(-1)^n\sum_{j=0}^{n}\binom{2n}{2j}\sum_{i=0}^{m}(m-2i)^{2n-2j}(2-2^{2j})(m+1)^{2j-1}B_{2j}\frac{x^{2n}}{(2n)!},\end{aligned} \tag{29}$$

and comparing the coefficient of x^{2n} on both sides of (29), we get

$$\sum_{j=0}^{n}\binom{2n}{2j}\sum_{i=0}^{m}(m-2i)^{2n-2j}(2-2^{2j})(m+1)^{2j-1}B_{2j} = (2-2^{2n})B_{2n}. \tag{30}$$

Set $m = 2k+1$ in (30) we immediately obtain (11). Set $m = 2k$ in (30) and bring the term with $i = k$ to the other side of the equation, to immediately obtain (12). □

ACKNOWLEDGMENT

The authors would like to thank the anonymous referee for valuable comments, including the proofs of Lemmas 1, 2, 3 and 4, which improved the presentation of original paper.

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AMS Classification Numbers: 11B68

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