LECTURE NOTES

BERNOULLI POLYNOMIALS AND APPLICATIONS

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Abstract. In this lecture notes we try to familiarize the audience with the theory of Bernoulli polynomials; we study their properties, and we give, with proofs and references, some of the most relevant results related to them. Several applications to these polynomials are presented, including a unified approach to the asymptotic expansion of the error term in many numerical quadrature formulæ, and many new and sharp inequalities, that bound some trigonometric sums.

Contents

2010 *Mathematics Subject Classification.* 11B68, 11L03, 30D10, 32B05, 43E05, 42A16, 65D32.
Key words and phrases. Fourier series, analytic functions, power series expansion, Bernoulli polynomials, Bernoulli numbers, harmonic numbers, asymptotic expansion, numerical quadrature, Riemann sum, trapezoidal rule, Simpson's rule, Gauss quadrature rule, Romberg's rule, sum of cosecants, sum of cotangents.
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1. Introduction

There are many ways to introduce Bernoulli polynomials and numbers. We opted for the algebraic approach relying on the difference operator. But first, let us introduce some notation.

Let the real vector space of polynomials with real coefficients be denoted by $\mathbb{R}[X]$. For a nonnegative integer n, let $\mathbb{R}_n[X]$ be the subspace of $\mathbb{R}[X]$ consisting of polynomials of degree smaller or equal to n.

If P is a polynomial from $\mathbb{R}[X]$, we define $\Delta P \stackrel{\text{def}}{=} P(X+1) - P(X)$, and we denote by Δ the linear operator, defined on $\mathbb{R}[X]$, by $P \mapsto \Delta P$.

Lemma 1.1. *The linear operator Φ defined by*

$$\Phi : \mathbb{R}[X] \longrightarrow \mathbb{R}[X] \times \mathbb{R}, P \mapsto \left(\Delta P, \int_0^1 P(t)\,dt\right) \tag{1.1}$$

is bijective.

Proof. Consider $P \in \ker\Phi$, then $P \in \ker\Delta$ and $\int_0^1 P(t)\,dt = 0$. Now, if we consider $Q(X) = P(X) - P(0)$, then clearly we have

$$Q(X+1) = P(X+1) - P(0) = P(X) - P(0) = Q(X)$$

This implies by induction that $Q(n) = 0$ for every nonnegative integer n, so $Q = 0$, since it has infinitely many zeros. Thus, $P(X) = P(0)$, but we have also $\int_0^1 P(t)\,dt = 0$, so $P(0) = 0$, and consequently $P = 0$. This proves that Φ is injective.

Clearly, for a nonnegative integer n we have $\deg\Delta(X^{n+1}) = n$. Thus

$$P \in \mathbb{R}_{n+1}[X] \implies \Delta P \in \mathbb{R}_n[X]$$

Therefore,

$$\forall n \in \mathbb{N},\quad \Phi(\mathbb{R}_{n+1}[X]) \subset \mathbb{R}_n[X] \times \mathbb{R}.$$

But the fact that Φ is injective implies that

$$\dim\Phi(\mathbb{R}_{n+1}[X]) = \dim\mathbb{R}_{n+1}[X] = 1 + \dim\mathbb{R}_n[X] = \dim(\mathbb{R}_n[X] \times \mathbb{R}),$$

and consequently

$$\forall n \in \mathbb{N},\quad \Phi(\mathbb{R}_{n+1}[X]) = \mathbb{R}_n[X] \times \mathbb{R}$$

This, proves that Φ is surjective, and the lemma follows. □

Let us consider the basis $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$ of $\mathbb{R}[X] \times \mathbb{R}$ defined by $e_0 = (0, 1)$ and $e_n = (nX^{n-1}, 0)$ for $n \in \mathbb{N}^*$. We can define the Bernoulli polynomials, In terms of this basis and of the isomorphism Φ of Lemma 1.1 as follows:

Definition 1.2. *The sequence of* ***Bernoulli polynomials*** $(B_n)_{n\in\mathbb{N}}$ *is defined by*

$$B_n = \Phi^{-1}(e_n) \qquad \text{for } n \geq 0.$$

According to Lemma 1.1, this definition takes a more practical form as follows :

Corollary 1.3. *The sequence of Bernoulli polynomials $(B_n)_{n\in\mathbb{N}}$ is* ***uniquely*** *defined by the conditions:*

$$\begin{aligned} &① \quad B_0(X) = 1.\\ &② \quad \forall n \in \mathbb{N}^*, \quad B_n(X+1) - B_n(X) = nX^{n-1}.\\ &③ \quad \forall n \in \mathbb{N}^*, \quad \int_0^1 B_n(t)\,dt = 0. \end{aligned} \tag{1.2}$$

For instance, it is straightforward to see that

$$B_1(X) = X - \frac{1}{2}, \qquad \text{and} \qquad B_2(X) = X^2 - X + \frac{1}{6}.$$

2. Properties of Bernoulli polynomials

In the next proposition, we summarize some simple properties of Bernoulli polynomials :

Proposition 2.1. *The sequence of Bernoulli polynomials $(B_n)_{n\in\mathbb{N}}$ satisfies the following properties:*

i. For every positive integer n we have $B'_n(X) = nB_{n-1}(X)$.
ii. For every positive integer n we have $B_n(1 - X) = (-1)^n B_n(X)$.
iii. For every nonnegative integer n and every positive integer p we have

$$\frac{1}{p}\sum_{k=0}^{p-1} B_n\left(\frac{X+k}{p}\right) = \frac{1}{p^n}B_n(X).$$

Proof. (i) Consider the sequence of polynomials $(Q_n)_{n\in\mathbb{N}}$ defined by $Q_n = \frac{1}{n+1}B'_{n+1}$. It is straightforward to see that $Q_0(X) = 1$ and for $n \geq 1$:

$$\Delta Q_n = \frac{1}{n+1}(\Delta B_{n+1})' = \frac{1}{n+1}((n+1)X^n)' = nX^{n-1}$$

and

$$\int_0^1 Q_n(t)\,dt = \frac{1}{n+1}\int_0^1 B'_{n+1}(t)\,dt = \frac{\Delta B_{n+1}(0)}{n+1} = 0.$$

This proves that the sequence of $(Q_n)_{n\in\mathbb{N}}$ satisfies the conditions ①, ② and ③ of Corollary 1.3, and (i) follows because of the unicity assertion.

(ii) Consider again the sequence $(Q_n)_{n\in\mathbb{N}}$ defined by $Q_n(X) = (-1)^n B_n(1 - X)$. Clearly $Q_0(X) = 1$ and for $n \geq 1$:

$$\begin{aligned}\Delta Q_n &= (-1)^n(B_n(-X) - B_n(1 - X)) = (-1)^{n-1}\Delta B_n(-X)\\ &= (-1)^{n-1}n(-X)^{n-1} = nX^{n-1}.\end{aligned}$$

Moreover, for $n \geq 1$, $\int_0^1 Q_n(t)\,dt = (-1)^n \int_0^1 B_n(1 - t)\,dt = 0$. This proves that the sequence of $(Q_n)_{n\in\mathbb{N}}$ satisfies the conditions ①, ② and ③ of Corollary 1.3, and (ii) follows from the unicity assertion.

(iii) Similarly, consider the sequence of polynomials $(Q_n)_{n\in\mathbb{N}}$ defined by

$$Q_n(X) = p^{n-1}\sum_{k=0}^{p-1} B_n\left(\frac{X+k}{p}\right)$$

Clearly $Q_0(X) = 1$ and for $n \geq 1$:

$$\Delta Q_n = p^{n-1}\sum_{k=0}^{p-1} B_n\left(\frac{X+k+1}{p}\right) - p^{n-1}\sum_{k=0}^{p-1} B_n\left(\frac{X+k}{p}\right)$$

$$= p^{n-1}\left(\sum_{k=1} B_n\left(\frac{X+k}{p}\right) - \sum_{k=0} B_n\left(\frac{X+k}{p}\right)\right)$$

$$= p^{n-1}\left(B_n\left(\frac{X+p}{p}\right) - B_n\left(\frac{X}{p}\right)\right) = p^{n-1}\Delta B_n\left(\frac{1}{p}X\right) = nX^{n-1}.$$

Moreover, for $n \geq 1$,

$$\int_0^1 Q_n(t)dt = p^{n-1}\sum_{k=0}^{p-1}\int_0^1 B_n\left(\frac{t+k}{p}\right) dt$$

$$= p^{n-1}\sum_{k=0}^{p-1}\int_{k/p}^{(k+1)/p} B_n(t)\, dt = p^{n-1}\int_0^1 B_n(t)\, dt = 0.$$

This proves that the sequence of $(Q_n)_{n\in\mathbb{N}}$ satisfies the conditions ①, ② and ③ of Corollary 1.3, and (*iii*) follows by unicity. The proof of Proposition 2.1 is complete. □