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SOME REMARKS ON A q-ANALOGUE OF BERNOULLI NUMBERS

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Abstract. Using the p-adic q-integral due to T. Kim [4], we define a number $B_n^*(q)$ and a polynomial $B_n^*(x;q)$ which are p-adic q-analogue of the ordinary Bernoulli number and Bernoulli polynomial, respectively. We investigate some properties of these. Also, we give slightly different construction of Tsumura's p-adic function $\ell_p(u,s,\chi)$ [14] using the p-adic q-integral in [4].

1. Introduction

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ be the p-adic valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-1}$. If $q \in \mathbb{C}_p$, one normally assumes $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. We use the notation

$$[x] = [x;q] = \frac{1-q^x}{1-q}. \tag{1.1}$$

Hence, $\lim_{q\to 1}[x;q] = x$ for any x with $|x|_p \le 1$. Let $UD(\mathbb{Z}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. It is well-known that the I_0-integral of $f \in UD(\mathbb{Z}_p)$ exists and is given by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x) \frac{1}{p^N}, \tag{1.2}$$

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where μ_0 is the ordinary p-adic distribution defined by $\mu_0(x+p^N\mathbb{Z}_p) = \frac{1}{p^N}$. The q-analogue of μ_0, denote by μ_q, defined by T. Kim [4] as follows: Let d be a fixed integer and p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \tag{1.3}$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ with $0 \leq a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N;q]} \tag{1.4}$$

is known as a distribution on X. In the case of $d = 1$, this distribution yields an I_q-integral for $f \in UD(\mathbb{Z}_p)$

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\frac{q^x}{[p^N]}. \tag{1.5}$$

In particular, the relation between the I_0-integral and I_q-integral is given by

$$\int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \frac{\log q}{q-1}\int_{\mathbb{Z}_p} q^{-x}f(x)d\mu_q(x) \quad \text{for } f \in UD(\mathbb{Z}_p). \tag{1.5$'$}$$

We recall variant Bernoulli numbers given by below in the symbolic form: For $n \geq 0$

- $B_0 = 1, \qquad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$

  (Ordinary Bernoulli numbers)
- $\beta_0(q) = 1, \qquad q(q\beta(q)+1)^n - \beta_n(q) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$

  (Carlitz's q-Bernoulli numbers (see [1]))
- $B_0(q) = \dfrac{q-1}{\log q}, \qquad (qB(q)+1)^n - B_n(q) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$

  (Tsumura's q-Bernoulli numbers (see [15]))
- $\mathcal{B}_0(q) = 0, \qquad q(\mathcal{B}(q)+1)^n - \mathcal{B}_n(q) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$

  (Kim's q-Bernoulli numbers (see [2], [9])).

It is known that variant Bernoulli numbers are connected with the I_0- and I_q-integral as follows: For $n \geq 0$

$$I_0(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_0(x) = B_n \text{ (Witt's formula);} \tag{1.6}$$

$$I_q([x]^n) = \int_{\mathbb{Z}_p} [x]^n d\mu_q(x) = \beta_n(q) \text{ (see [4]);} \tag{1.7}$$

$$I_q(q^{-x}[x]^n) = \int_{\mathbb{Z}_p} q^{-x}[x]^n d\mu_q(x) = B_n(q) \text{ (see [3]);} \tag{1.8}$$

$$I_q(q^{-x}x^n) \overset{(1.5')}{=} \int_{\mathbb{Z}_p} q^{-x}x^n d\mu_q(x) = \frac{q-1}{\log q} B_n. \tag{1.9}$$

N. Koblitz [11] constructed the p-adic q-L-series which interpolated Carlitz's q-Bernoulli numbers $\beta_n(q)$. J. Satoh [13] constructed the complex q-L-series which interpolated Carlitz's q-Bernoulli numbers $\beta_n(q)$. T. Kim [4] proved that Carlitz's q-Bernoulli numbers $\beta_n(q)$ can be represented as an integral by the q-analogue μ_q of the ordinary p-adic invariant measure. In the complex case, H. Tsumura [15] studied a q-analogue of the Dirichlet L-series which interpolated q-Bernoulli numbers $B_n(q)$. In the p-adic case, T. Kim [3] constructed the p-adic q-L-function using the congruence on q-Bernoulli numbers $B_n(q)$.

In this paper, we consider a uniformly (strictly) differentiable function $f(x) = x^n$ $(n \geq 0)$ in the I_q-integral given by (1.5) and put

$$B_n^*(q) = \int_{\mathbb{Z}_p} x^n d\mu_q(x); \quad B_n^*(x;q) = \int_{\mathbb{Z}_p} (x+t)^n d\mu_q(t).$$

The purpose of this paper is to investigate the properties of a number $B_n^*(q)$ and a polynomial $B_n^*(x;q)$. Also, we give slightly different construction of Tsumura's p-adic function $\ell_p(u,s,\chi)$ [14] using the p-adic q-integral in [4].

2. Another p-adic q-Bernoulli number $B_n^*(q)$ and its basic properties

Set $f(x) = x^n \in UD(\mathbb{Z}_p)$ for $n \geq 0$ in the equation (1.5).

Now, for any integer $n \geq 0$ we define a number $B_n^*(q)$ and a polynomial $B_n^*(x;q)$ in the variable $x \in \mathbb{C}_p$ with $|x|_p \leq 1$, respectively, by

$$B_n^*(q) \overset{\text{def}}{=} \int_{\mathbb{Z}_p} x^n d\mu_q(x); \quad B_n^*(x;q) \overset{\text{def}}{=} \int_{\mathbb{Z}_p} (x+t)^n d\mu_q(t). \tag{2.1}$$

The generating function, denote by $G_q^*(t)$, of $B_n^*(q)$ is given by

$$G_q^*(t) = \frac{q-1}{\log q}\left(\frac{\log q + t}{qe^t - 1}\right) = \sum_{n=0}^{\infty} B_n^*(q)\frac{t^n}{n!}. \tag{2.2}$$

Indeed, for $f(x) = q^x e^{xt} \in UD(\mathbb{Z}_p)$ using the equation $I_0(f_1) = I_0(f) + f'(0)$, where $f_1(x) = f(x+1)$ for all $x \in \mathbb{Z}_p$, we have

$$I_0\left(q^x e^{xt}\right) = \frac{\log q + t}{qe^t - 1}. \tag{2.3}$$

From the formula (1.5′), we obtain

$$\begin{aligned} G_q^*(t) &= \frac{q-1}{\log q}\left(\frac{\log q + t}{qe^t - 1}\right) = \frac{q-1}{\log q} I_0\left(q^x e^{xt}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{q-1}{\log q} I_0\left(x^n q^x\right)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty} I_q(x^n)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} B_n^*(q)\frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

We can easily prove the following.

Proposition 2.1. *For $n \geq 0$ and $x \in \mathbb{Z}_p$, we have*

(1) $B_0^*(q) = 1$, $q(B^*(q)+1)^n - B_n^*(q) = \begin{cases} \frac{q-1}{\log q} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$;
(2) $\lim_{q \to 1} B_n^*(q) = B_n$;
(3) $B_n^*(x;q) = (B^*(q)+x)^n$ *and* $\lim_{q\to 1} B_n^*(x;q) = B_n(x)$, *where $B_n(x)$ is the ordinary Bernoulli polynomial.*

From Proposition 2.1 we may say that a number $B_n^*(q)$ and a polynomial $B_n^*(x;q)$ are another p-adic q-analogue of ordinary Bernoulli number and Bernoulli polynomial, respectively.

Proposition 2.2. *For $m,\ n \geq 0$ and $x \in \mathbb{Z}_p$, we have*

(1) $I_0(x^m q^{nx}) = \frac{n\log q}{q-1} I_{q^n}(x^m) = \frac{\log q}{q-1} B_m^*(q^n)$;
(2) $I_q(x^m q^{nx}) = \sum_{s=0}^{\infty} \frac{(\log q^n)^s}{s!} B_{m+s}^*(q)$.

LEMMA 2.3. *For $n \geq 0$*

$$\int_{\mathbb{Z}_p} x^n d\mu_q(x) = \int_X x^n d\mu_q(x).$$

Proof. Note that $d\mu_q(x) = \frac{(q-1)q^x}{\log q}\, d\mu_0(x)$ for $|1-q|_p < p^{-1/(p-1)}$ (see [4], [10]). It is well known that $\int_{\mathbb{Z}_p} x^n\, d\mu_0(x) = \int_X x^n\, d\mu_0(x)$ for $n \geq 0$ (cf. [4, Lemma 1]) and $q^x = \sum_{s=0}^{\infty} \frac{x^s (\log q)^s}{s!}$ for $|1-q|_p < p^{-1/(p-1)}$. The result now follows easily. □

LEMMA 2.4. *For any positive integer d and $k \geq 0$*

$$B_k^*(x;q) = d^k [d]^{-1} \sum_{i=0}^{d-1} q^i B_k^* \left(\frac{x+i}{d}; q^d \right).$$

Proof. By Lemma 2.3 we have

$$\begin{aligned}
B_k^*(x;q) &= \lim_{N\to\infty} \frac{1}{[dp^N]} \sum_{n=0}^{dp^N-1} q^n (x+n)^k \\
&= \lim_{N\to\infty} \frac{1}{[d]} \frac{1}{[p^N;q^d]} \sum_{i=0}^{d-1} \sum_{n=0}^{p^N-1} q^{i+dn} (x+i+dn)^k \\
&= [d]^{-1} \sum_{i=0}^{d-1} d^k q^i \lim_{N\to\infty} \frac{1}{[p^N;q^d]} \sum_{n=0}^{p^N-1} (q^d)^n \left(\frac{x+i}{d} + n \right)^k \\
&= [d]^{-1} \sum_{i=0}^{d-1} d^k q^i \int_{\mathbb{Z}_p} \left(\frac{x+i}{d} + t \right)^k d\mu_{q^d}(t) \\
&= d^k [d]^{-1} \sum_{i=0}^{d-1} q^i B_k^* \left(\frac{x+i}{d}; q^d \right).
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *For $k \geq 0$, let $\mu_k^* = \mu_{k;q}^*$ be define by*

$$\mu_k^*(a + dp^N\mathbb{Z}_p) \stackrel{\text{def}}{=} (dp^N)^k[dp^N]^{-1}q^a B_k^*\left(\frac{a}{dp^N}; q^{dp^N}\right),$$

where N and d are positive integers. Then μ_k^* is a distribution on X.

Proof. Since $\mu_{0;q}^* = \mu_q$ which is a distribution on X (see [4]), for any positive integer k we show that μ_k^* is a distribution on X. For that, it is suffices to check that

$$\begin{aligned}
&\sum_{i=0}^{p-1} \mu_k^*(a + idp^N + dp^{N+1}\mathbb{Z}_p) \\
&= (dp^{N+1})^k q^a [dp^{N+1}]^{-1} \sum_{i=0}^{p-1} q^{idp^N} B_k^*\left(\frac{a + idp^N}{dp^{N+1}}; q^{dp^{N+1}}\right) \\
&= (dp^N)^k q^a [dp^N]^{-1} \left\{ \frac{p^k}{[p; q^{dp^N}]} \sum_{i=0}^{p-1} (q^{dp^N})^i B_k^*\left(\frac{\frac{a}{dp^N} + i}{p}; (q^{dp^N})^p\right) \right\} \\
&= (dp^N)^k [dp^N]^{-1} q^a B_k^*\left(\frac{a}{dp^N}; q^{dp^N}\right) \text{ (using Lemma 2.4)} \\
&= \mu_k^*(a + dp^N\mathbb{Z}_p).
\end{aligned}$$

This completes the proof. □

3. A generalized q-Bernoulli numbers $B_{k,\chi}^*(q)$ and related properties

Let χ be a Dirichlet character with conductor d, where d is a positive integer. For $k \geq 0$ we define the k-th generalized q-Bernoulli number belonging to the character χ by

$$B_{k,\chi}^*(q) \stackrel{\text{def}}{=} \int_X \chi(x) x^k d\mu_q(x). \tag{3.1}$$

Using a similar method used in the proof of Lemma 2.4, we may perform the integral in the right hand side of (3.1) to get

$$B_{k,\chi}^*(q) = d^k[d]^{-1} \sum_{a=0}^{d-1} q^a \chi(a) B_k^*\left(\frac{a}{d}; q^d\right). \tag{3.2}$$

Proposition 3.1. *For $k \geq 0$, we have*

(1) $\int_X \chi(x) d\mu_k^*(x) = B_{k,\chi}^*(q)$;
(2) $\int_{pX} \chi(x) d\mu_k^*(x) = \chi(p)p^k[p]^{-1}B_{k,\chi}^*(q^p)$;
(3) $\int_X \chi(x) d\mu_{k;q^c}^*(\frac{1}{c}x) = \chi(c)B_{k,\chi}^*(q^c)$;
(4) $\int_{pX} \chi(x) d\mu_{k;q^c}^*(\frac{1}{c}x) = \chi(c)\chi(p)p^k[p]^{-1}B_{k,\chi}^*(q^{pc})$.

Proof. Using the definition of $\mu_{k;q}^*$ given by Theorem 2.5 and the formula (3.2), the proofs are clear. □

Corollary 3.2. *For $k \geq 0$*

$$\int_X \chi(x)x^k \, d\mu_q(x) = \int_X \chi(x) \, d\mu_k^*(x).$$

Proof. The definition of $B_{k,\chi}^*(q)$ and Proposition 3.1(1) imply

$$\int_X \chi(x)x^k \, d\mu_q(x) = \int_X \chi(x) \, d\mu_k^*(x).$$

This completes the proof. □

We set

$$p^* = \begin{cases} p & \text{if } p > 2 \\ 4 & \text{if } p = 2. \end{cases} \tag{3.3}$$

Let $\bar{d} = [d, p^*]$ be the least common multiple of conductor d of χ and p^*. By using the I_q-integral, we have the Witt's type formula in the p-adic cyclotomic field $\mathbb{Q}_p(\chi)$ as follows:

$$B_{k,\chi}^*(q) = \lim_{N\to\infty} \sum_{x=1}^{\bar{d}p^N} \chi(x)x^k \frac{q^x}{[\bar{d}p^N]}, \quad k \geq 0. \tag{3.4}$$

For any rational integers s and t, let $\chi^s = \chi^{s,k;q}$ be an operator on $f(q)$ as follows:

$$\chi^s f(q) \stackrel{\text{def}}{=} s^k[s]^{-1}\chi(s)f(q^s); \tag{3.5}$$

$$\chi^s\chi^t \overset{\text{def}}{=} [s]^{-1}[s;q^t]\left(\chi^{s,k;q^t}\circ\chi^{t,k;q}\right). \tag{3.6}$$

Now we choose a rational integer number c such that $(c,\bar{d})=1$ and $c\neq\pm1$, and we put

$$\mu_k^c = \frac{1}{k}\frac{\log q}{q-1}\left(\mu^*_{k;q}(U) - c^{k+1}[c]^{-1}\mu^*_{k;q^c}\left(\frac{1}{c}U\right)\right),\ k\geq 1, \tag{3.7}$$

where $U\subset X$ is compact open set. Then μ_k^c must be a distribution on X (μ_k^c is not a measure on X) and, using Proposition 3.1 and the definition of the operator χ^p, χ^c and $\chi^p\chi^c$ given by (3.5) and (3.6), this distribution yields an integral on $X^*=X-pX$ as follows:

$$\begin{aligned}
&\int_{X^*}\chi(x)d\mu_k^c(x)\\
&=\int_X\chi(x)d\mu_k^c(x)-\int_{pX}\chi(x)d\mu_k^c(x)\\
&=\frac{1}{k}\frac{\log q}{q-1}\left(\int_X\chi(x)d\mu^*_{k;q}(x)-c^{k+1}[c]^{-1}\int_X\chi(x)d\mu^*_{k;q^c}(\frac{1}{c}x)\right)\\
&\qquad -\frac{1}{k}\frac{\log q}{q-1}\left(\int_{pX}\chi(x)d\mu^*_{k;q}(x)-c^{k+1}[c]^{-1}\int_{pX}\chi(x)d\mu^*_{k;q^c}(\frac{1}{c}x)\right)\\
&=\frac{1}{k}\frac{\log q}{q-1}\Big\{\left(B^*_{k,\chi}(q)-p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^p)\right)\\
&\qquad\qquad -\chi(c)c^{k+1}[c]^{-1}\left(B^*_{k,\chi}(q^c)-p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^{pc})\right)\Big\}\\
&=\frac{\log q}{q-1}(1-\chi^p)(1-c\chi^c)\frac{B^*_{k,\chi}(q)}{k}.
\end{aligned}$$

Hence we obtain

$$\int_{X^*}\chi(x)d\mu_k^c(x)=\frac{\log q}{q-1}(1-\chi^p)(1-c\chi^c)\frac{B^*_{k,\chi}(q)}{k}. \tag{3.8}$$

Theorem 3.3. *For $k\geq 1$*

$$\int_{X^*}\chi(x)d\mu_k^c(x)=\lim_{N\to\infty}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(cx)(cx)^{k-1}\left[-\frac{cx}{\bar{d}p^N}\right]_g q^{cx},$$

where $\sum^*$ means to take sums over the rational integers prime to p in the given range, c is a rational integer number such that $(c,\bar{d})=1$ and $c\neq\pm1$, and $[\cdot]_g$ is Gauss's symbol.

Proof. From (3.8) we must show that

$$\lim_{N\to\infty}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(cx)(cx)^{k-1}\left[-\frac{cx}{\bar{d}p^N}\right]_g q^{cx}=\frac{\log q}{q-1}(1-\chi^p)(1-c\chi^c)\frac{B^*_{k,\chi}(q)}{k}.$$

We can rewrite $B^*_{k,\chi}(q),\ k\geq 1$, given by (3.4) as

$$\begin{aligned}
&B^*_{k,\chi}(q)\\
&=\lim_{N\to\infty}\frac{1}{[\bar{d}p^N]}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(x)x^kq^x+\lim_{N\to\infty}\frac{1}{[\bar{d}p^{N-1};q^p]}\frac{1}{[p]}\sum_{y=1}^{\bar{d}p^{N-1}}\chi(py)(py)^kq^{py}\\
&=\lim_{N\to\infty}\frac{1}{[\bar{d}p^N]}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(x)x^kq^x+p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^p).
\end{aligned}$$

That is,

$$\text{(A)}\qquad \lim_{N\to\infty}\frac{1}{[\bar{d}p^N]}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(x)x^kq^x=B^*_{k,\chi}(q)-p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^p).$$

We choose a rational integer number c such that $(c,\bar{d})=1$ and $c\neq\pm1$. Let x and x_N be the rational integers such that $1\leq x,x_N\leq\bar{d}p^N$ and $(x,p)=(x_N,p)=1$, and determine a rational integer number $r_N(x)$ by $x_N=cx+r_N(x)\bar{d}p^N$, i.e.,

$$\text{(B)}\qquad r_N(x)=-\frac{cx}{\bar{d}p^N}+\frac{x_N}{\bar{d}p^N}=\left[-\frac{cx}{\bar{d}p^N}\right]_g,$$

where $[\cdot]_g$ is Gauss's symbol. Then we have

$$\begin{aligned}
&\frac{1}{[\bar{d}p^N]}\sum_{x_N=1}^{\bar{d}p^N}{}^{*}\chi(x_N)(x_N)^kq^{x_N}\\
&=\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(cx)\Big\{(cx)^k+k(cx)^{k-1}\big(r_N(x)\bar{d}p^N\big)+\cdots+\big(r_N(x)\bar{d}p^N\big)^k\Big\}\frac{q^{x_N}}{[\bar{d}p^N]}\\
&\equiv\frac{1}{[\bar{d}p^N]}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(cx)(cx)^kq^{x_N}+k\frac{\bar{d}p^N}{[\bar{d}p^N]}\sum_{x=1}^{\bar{d}p^N}{}^{*}\chi(cx)(cx)^{k-1}r_N(x)q^{x_N}\\
&\qquad(\operatorname{mod}\ (\bar{d}p^N)^2[\bar{d}p^N]^{-1}).
\end{aligned}$$

Since

$$\lim_{N\to\infty} q^{r_N(x)\bar{d}p^N} = 1 \text{ for } |1-q|_p < p^{-1/(p-1)}; \quad \lim_{N\to\infty} \frac{\bar{d}p^N}{[\bar{d}p^N]} = \frac{q-1}{\log q},$$

using the formula (A) we find that

$$\begin{aligned}
& B^*_{k,\chi}(q) - p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^p) \\
= & \lim_{N\to\infty} \frac{1}{[\bar{d}p^N]} \sideset{}{^*}\sum_{x_N=1}^{\bar{d}p^N} \chi(x_N)(x_N)^k q^{x_N} \\
= & \lim_{N\to\infty} \frac{1}{[\bar{d}p^N]} \sideset{}{^*}\sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^k q^{cx} \\
& + k\frac{q-1}{\log q} \lim_{N\to\infty} \sideset{}{^*}\sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^{k-1} r_N(x) q^{cx} \\
= & \, \chi(c)c^{k+1}[c]^{-1} \lim_{N\to\infty} \frac{1}{[\bar{d}p^N; q^c]} \sideset{}{^*}\sum_{x=1}^{\bar{d}p^N} \chi(x)x^k (q^c)^x \\
& + k\frac{q-1}{\log q} \lim_{N\to\infty} \sideset{}{^*}\sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^{k-1} r_N(x) q^{cx} \\
= & \, \chi(c)c^{k+1}[c]^{-1} \left\{ B^*_{k,\chi}(q^c) - p^k[p]^{-1}\chi(p)B_{k,\chi}(q^{pc}) \right\} \\
& + k\frac{q-1}{\log q} \lim_{N\to\infty} \sideset{}{^*}\sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^{k-1} r_N(x) q^{cx},
\end{aligned}$$

that is, using (B) and the definition of the operators χ^p, χ^c and $\chi^p\chi^c$ given by (3.5) and (3.6) we have

$$\begin{aligned}
& \lim_{N\to\infty} \sideset{}{^*}\sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^{k-1} \left[-\frac{cx}{\bar{d}p^N}\right]_g q^{cx} \\
= & \frac{1}{k}\frac{\log q}{q-1} \Big\{ \left(B^*_{k,\chi}(q) - p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^p)\right) \\
& \qquad - \chi(c)c^{k+1}[c]^{-1}\left(B^*_{k,\chi}(q^c) - p^k[p]^{-1}\chi(p)B^*_{k,\chi}(q^{pc})\right) \Big\} \\
= & \frac{\log q}{q-1}(1-\chi^p)(1-c\chi^c)\frac{B^*_{k,\chi}(q)}{k}.
\end{aligned}$$

This completes the proof. $\square$

Now, we will consider a q-analogue of Nasybullin's lemma (see [8, Theorem 1]; we follow the notation of [8]).

Let $B_n^*(x;q)$ be the nth q-Bernoulli polynomials in (2.2). The nth q-Bernoulli functions $P_n(x)$ are define by $P_n(x) = P(x;q) = B_n^*(x;q)$ for $0 \leq x < 1$. They are periodic with period 1 and agree with the q-Bernoulli polynomials $B_n^*(x;q)$ in the interval $0 \leq x < 1$.

By Lemma 2.4 we have

$$d^n[d]^{-1}\sum_{i=0}^{d-1} q^i B_n^*\left(\frac{x+i}{d};q^d\right) = B_n^*(x;q).$$

Hence for any real number x

$$d^n[d]^{-1}\sum_{i=0}^{d-1} q^i P_n\left(\frac{x+i}{d};q^d\right) = P_n(x;q).$$

From the above that the function $P_n(x;q)$ satisfies the property of q-Nasybullin's lemma with constants $A = d^{-n}[d]$, $B = 0$. Then $\rho \neq 0$ is equal to $d^{-n}[d]$, as $\rho^2 = A\rho + B\rho$ reduces simply to $\rho^2 = d^{-n}[d]\rho$. Thus we define the function $\mu_n = \mu_{n;q}$ on $a + \bar{d}p^N\mathbb{Z}_p$ by

$$\mu_n(a+\bar{d}p^N\mathbb{Z}_p) := (\bar{d}p^N)^n[\bar{d}p^N]^{-1}q^a P_n\left(\frac{a}{\bar{d}p^N};q^{\bar{d}p^N}\right).$$

This can be extended to a measure on $\varprojlim_N(\mathbb{Z}/\bar{d}p^N\mathbb{Z})$ for $N \geq 0$.

Let χ be a primitive Dirichlet character with conductor $\bar{d}$. Then the generalized q-Bernoulli number in (3.4) is defined by

$$B_{k,\chi}^*(q) = \lim_{N\to\infty}\frac{1}{[\bar{d}p^N]}\sum_{n=0}^{\bar{d}p^N-1}\chi(n)n^k q^n = \frac{\bar{d}^k}{[\bar{d}]}\sum_{a=0}^{\bar{d}-1} q^a\chi(a)B_k^*\left(\frac{a}{\bar{d}};q^d\right).$$

Let

$$\begin{aligned}
L(\mu_n,\chi) &= \lim_{N\to\infty}{\sum_{a=0}^{\bar{d}p^N-1}}^{*}\chi(a)\mu_n(a+\bar{d}p^N\mathbb{Z}_p)\\
&= \lim_{N\to\infty}\sum_{\substack{a \pmod{\bar{d}p^N}\\ (a,p)=1}}\chi(a)\mu_n(a+\bar{d}p^N\mathbb{Z}_p),
\end{aligned}$$

where $\sum^*$ means to take sums over the rational integers prime to p in the given range. Then since the character χ is constant on $a + \bar{d}\mathbb{Z}_p$,

$$\begin{aligned} L(\mu_n, \chi) = & \lim_{N\to\infty} \sum_{a \pmod{\bar{d}p^N}} \chi(a)\mu_n(a + \bar{d}p^N\mathbb{Z}_p) \\ & - \lim_{N\to\infty} \sum_{\substack{a \pmod{\bar{d}p^N} \\ p|a}} \chi(a)\mu_n(a + \bar{d}p^N\mathbb{Z}_p) \\ & = B^*_{n,\chi}(q) - p^n[p]^{-1}\chi(p)B^*_{n,\chi}(q^p), \end{aligned}$$

where $B^*_{n,\chi}(q)$ is the nth q-Bernoulli number containing χ. Thus we obtain

$$L\left(\mu_n, \chi\omega^{-n}\right) = B^*_{n,\chi\omega^{-n}}(q) - p^n[p]^{-1}\chi\omega^{-n}(p)B^*_{n,\chi\omega^{-n}}(q^p)$$

where $n \geq 1$ and ω is the Teichmüller character mod p^*.

4. I_q-integral and Tsumura's p-adic function

Let $z \in \mathbb{C}_p$ be such that $z^{dp^N} \neq 1$ for all N. In [10], N. Koblitz defined

$$E_z(a + dp^N\mathbb{Z}_p) = \frac{z^a}{1 - z^{dp^N}}. \tag{4.1}$$

He obtained

Proposition 4.1 ([10]). *E_z is a distribution on X. Let $D_1 = \{x \in \mathbb{C}_p \mid |x-1|_p < 1\}$, and let $\overline{D}_1 = \mathbb{C}_p \backslash D_1$ be the complement of the open unit disc around 1. Then E_z is a measure if and only if $z \in \overline{D}_1$.*

Note that if $q \in \overline{D}_1$ and $\mathrm{ord}_p(1-q) \neq -\infty$, then $\mu_q(a + dp^N\mathbb{Z}_p) = (1-q)E_q(a + dp^N\mathbb{Z}_p)$. Thus $\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]}$ in $q \in \overline{D}_1$ and $\mathrm{ord}_p(1-q) \neq -\infty$ is the similar measure as Koblitz measure.

Hereafter, we assume that $q \in \overline{D}_1$ and $\mathrm{ord}_p(1-q) \neq -\infty$.

Now, for $t \in \mathbb{C}_p$ with $\mathrm{ord}_p t > \frac{1}{p-1}$, we define a number $H^*_m(q)$ by

$$\frac{q-1}{qe^t - 1} = \sum_{m=0}^{\infty} H^*_m(q)\frac{t^m}{m!}. \tag{4.2}$$

Note that $H_m^*(q^{-1}) = H_m(q)$ where the number $H_m(q)$ defined by $\frac{1-q}{e^t-q} = \sum_{m=0}^{\infty} \frac{H_m(q)}{m!} t^m$ is called the m-th Euler number belonging to q, which lies in an algebraic closure of $\mathbb{Q}_p$.

We can express the numbers $H_m^*(q)$ as an integral over $\mathbb{Z}_p$, for $d = 1, X = \mathbb{Z}_p$, by using the measure μ_q, that is,

$$\int_{\mathbb{Z}_p} x^m d\mu_q(x) = H_m^*(q) \quad \text{for } m \geq 0. \tag{4.3}$$

Indeed, we find that

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) &= \lim_{N\to\infty} \frac{1-q}{1-q^{p^N}} \sum_{a=0}^{p^N-1} e^{at} q^a \\ &= \frac{1-q}{1-qe^t} \lim_{N\to\infty} \frac{1-e^{tp^N} q^{p^N}}{1-q^{p^N}} = \frac{q-1}{qe^t-1}, \end{aligned}$$

since e^{tp^N} approaches 1 as $N \to \infty$, the limit is 1. Let $t \in \mathbb{C}_p$ with $\text{ord}_p t > \frac{1}{p-1}$. Then we obtain

$$\sum_{m=0}^{\infty} H_m^*(q) \frac{t^m}{m!} = \int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} x^m d\mu_q(x) \frac{t^m}{m!}. \tag{4.4}$$

Hence, comparing the above formulas, $\int_{\mathbb{Z}_p} x^m d\mu_q(x) = H_m^*(q)$ for $m \geq 0$.

Note that if $q \in D_1$ then $\int_{\mathbb{Z}_p} x^m d\mu_q(x) = B_m^*(q)$ (see Section 2).

Let ω denote the Teichmüller character mod p^*. For $x \in X^*$, we set $\langle x \rangle = x/\omega(x)$. For $s \in \mathbb{Z}_p$, we define

$$\ell_{p,q}(s) \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{1-q}{1-q^{p^N}} \sum_{m=0}^{p^N-1} \frac{q^m}{m^s}. \tag{4.5}$$

Then we obtain $\ell_{p,q}(-k) = \lim_{N\to\infty} \frac{1-q}{1-q^{p^N}} \sum_{m=0}^{p^N-1} q^m m^k = H_k^*(q)$ for $k \geq 0$.

Let χ be a primitive Dirichlet character with conductor d. For $k \geq 0$, the generalized numbers $H_{k,\chi}^*(q)$ is defined by

$$H_{k,\chi}^*(q) = \int_X \chi(x) x^k d\mu_q(x). \tag{4.6}$$

For $s \in \mathbb{Z}_p$, we define the function $\ell_{p,q}$ by

$$\ell_{p,q}(s,\chi) = \int_{X^*} \langle x \rangle^{-s} \chi(x) d\mu_q(x) \tag{4.7}$$

which is slightly different from the one in [14]. The value of this function at non-positive integers are given by

Proposition 4.2. *For any $k \geq 0$, we have*

$$\ell_{p,q}(-k, \chi\omega^k) = H^*_{k,\chi}(q) - p^k[p]^{-1}\chi(p)H^*_{k,\chi}(q^p).$$

Proof. Since $\mu_q(pU) = \mu_{q^p}(U)$ for $U \subset X$, $\int_{pX} \chi(x)x^k d\mu_q(x) = [p]^{-1}\int_X \chi(px)(px)^k d\mu_{q^p}(x) = p^k[p]^{-1}\chi(p)H^*_{k,\chi}(q^p)$. The proof now follows directly. $\square$

For $\alpha, \beta \in \mathbb{C}_p$ and any function $f(q)$, we set

$$(\alpha + \beta p^k) \circledast f(q) := \alpha f(q) + \beta p^k f(q^p). \tag{4.8}$$

We have the following Kummer congruences.

Corollary 4.3. *If $k \equiv k' \pmod{(p-1)p^N}$, then*

$$(1 - \chi(p)p^k) \circledast \frac{H^*_{k,\chi}(q)}{1-q} \equiv (1 - \chi(p)p^{k'}) \circledast \frac{H^*_{k',\chi}(q)}{1-q} \pmod{p^N}.$$

Proof. Note that (see [10, Proposition 2])

$$\left|\frac{\mu_q(a + dp^N\mathbb{Z}_p)}{1-q}\right|_p = \left|\frac{q^a}{(1-q)[dp^N]}\right|_p = \left|\frac{q^a}{1-q^{dp^N}}\right|_p \leq 1,$$

where we use the assumption $q \in \overline{D}_1$. By [12, Chapter II, §2], if $k \equiv k'$ $\pmod{(p-1)p^N}$, then we have

$$|x^k - x^{k'}|_p \leq \frac{1}{p^N} \quad \text{for } x \in X^*.$$

Using the corollary at the end of [12, Chapter II, §5], we easily see that

$$\begin{aligned}\frac{\ell_{p,q}(-k,\chi\omega^k)}{1-q} &= \int_{X^*} \langle x\rangle^k \chi\omega^k(x)\,\frac{d\mu_q(x)}{1-q}\\ &= \int_{X^*} \chi(x)x^k\,\frac{d\mu_q(x)}{1-q}\\ &\equiv \int_{X^*} \chi(x)x^{k'}\,\frac{d\mu_q(x)}{1-q} \pmod{p^N}\\ &= \frac{\ell_{p,q}(-k',\chi\omega^{k'})}{1-q}.\end{aligned}$$

By Proposition 4.2 and (4.8), the result now follows easily. □

Remark. By the definition (2.2) and (4.2), we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^*(q)\frac{t^n}{n!} &= \frac{q-1}{\log q}\left(\frac{\log q+t}{qe^t-1}\right)\\ &= \frac{q-1}{qe^t-1} + \frac{t}{\log q}\frac{q-1}{qe^t-1}\\ &= \sum_{n=0}^{\infty} H_n^*(q)\frac{t^n}{n!} + \frac{t}{\log q}\sum_{n=0}^{\infty} H_n^*(q)\frac{t^n}{n!}.\end{aligned}$$

Equating the coefficient of t^n, we obtain the following relation between the q-analogue Bernoulli numbers $B^*(q)$ and the number $H^*(q)$

$$B_n^*(q) = H_n^*(q) + \frac{n}{\log q}H_{n-1}^*(q) \quad (n \geq 1).$$

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