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Research Article

Some Identities on the Generalized q-Bernoulli, q-Euler, and q-Genocchi Polynomials

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Mahmudov (2012, 2013) introduced and investigated some q-extensions of the q-Bernoulli polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ of order α, and the q-Genocchi polynomials $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$ of order α. In this paper, we give some identities for $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$, $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$, and $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ and the recurrence relations between these polynomials. This is an analogous result to the q-extension of the Srivastava-Pintér addition theorem in Mahmudov (2013).

1. Introduction, Definitions, and Notations

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1, \qquad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \tag{1}$$

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, and $a \in \mathbb{C}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}, \tag{2}$$

where $(q:q)_n = (1-q)\cdots(1-q^n)$. The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2} x^{n-k} y^k. \tag{3}$$

The q-binomial formula is known as

$$\begin{aligned}(n;q)_a &= (1-a)_q^n \\ &= \prod_{j=0}^{n-1}\left(1-q^j a\right) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2}(-1)^k a^k.\end{aligned} \tag{4}$$

The q-exponential functions are given by

$$\begin{aligned}e_q(z) &= \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \\ &= \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \\ &\qquad |z| < \frac{1}{|1-q|},\end{aligned}$$

$$E_q(z) = \sum_{n=0}^{\infty} q^{(n(n-1))/2} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q) q^k z\right), \quad 0 < |q| < 1, \ z \in \mathbb{C}. \tag{5}$$

From these forms, we easily see that $e_q(z)E_q(-z) = 1$. Moreover, $D_q e_q(z) = e_q(z)$ and $D_q E_q(z) = E_q(qz)$, where D_q is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \ 0 \neq z \in \mathbb{C}. \tag{6}$$

The previous q-standard notation can be found in [1, 2]. Carlitz firstly extended the classical Bernoulli numbers and polynomials and Euler numbers and polynomials [3, 4]. There are numerous recent investigations on this subject by many other authors. Among them are Cenkci et al. [5, 6], Choi et al. [1], Cheon [7], Kim [8], Kurt [9], Kurt [10], Luo and Srivastava [11–13], Srivastava et al. [14, 15], Natalini and Bernardini [16], Tremblay et al. [17, 18], Gaboury and Kurt [19], Mahmudov [2, 20, 21], Araci et al. [22], and Kupershmidt [23].

Mahmudov defined and studied the properties of the following generalized q-Bernoulli polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x, y)$ of order α and q-Euler polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x, y)$ of order α as follows [2].

Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, and $0 < |q| < 1$. The q-Bernoulli numbers $\mathscr{B}_{n,q}^{(\alpha)}$ and polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x, y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha}, \quad |t| < 2\pi, \tag{7}$$

$$\sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < 2\pi. \tag{8}$$

The q-Euler numbers $\mathscr{E}_{n,q}^{(\alpha)}$ and polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x, y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha}, \quad |t| < \pi, \tag{9}$$

$$\sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < \pi. \tag{10}$$

The q-Genocchi numbers $\mathscr{G}_{n,q}^{(\alpha)}$ and polynomials $\mathscr{G}_{n,q}^{(\alpha)}(x, y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathscr{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha}, \quad |t| < \pi, \tag{11}$$

$$\sum_{n=0}^{\infty} \mathscr{G}_{n,q}^{(\alpha)}(x, y) \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < \pi. \tag{12}$$

The familiar q-Stirling numbers $S_{2,q}(n, k)$ of the second kind are defined by

$$\frac{\left(e_q(t) - 1\right)^k}{[k]_q!} = \sum_{n=0}^{\infty} S_{2,q}(n, k) \frac{t^n}{[n]_q!}. \tag{13}$$

It is obvious that

$$\begin{gathered}
\mathscr{B}_{n,q}^{(1)}(x, y) := \mathscr{B}_{n,q}(x, y), \qquad \mathscr{E}_{n,q}^{(1)}(x, y) := \mathscr{E}_{n,q}(x, y), \\
\mathscr{G}_{n,q}^{(1)}(x, y) := \mathscr{G}_{n,q}(x, y), \qquad \mathscr{B}_{n,q}(0, 0) := \mathscr{B}_{n,q}, \\
\mathscr{E}_{n,q}(0, 0) := \mathscr{E}_{n,q}, \qquad \mathscr{G}_{n,q}(0, 0) := \mathscr{G}_{n,q}, \\
\mathscr{B}_{n,q}^{(\alpha)} = \mathscr{B}_{n,q}^{(\alpha)}(0, 0), \\
\lim_{q \to 1^-} \mathscr{B}_{n,q}^{(\alpha)}(x, y) = \mathscr{B}_{n}^{(\alpha)}(x + y), \\
\lim_{q \to 1^-} \mathscr{B}_{n,q}^{(\alpha)} = \mathscr{B}_{n}^{(\alpha)}, \qquad \mathscr{E}_{n,q}^{(\alpha)} = \mathscr{E}_{n,q}^{(\alpha)}(0, 0), \\
\lim_{q \to 1^-} \mathscr{E}_{n,q}^{(\alpha)}(x, y) = \mathscr{E}_{n}^{(\alpha)}(x + y), \qquad \lim_{q \to 1^-} \mathscr{E}_{n,q}^{(\alpha)} = \mathscr{E}_{n}^{(\alpha)}, \\
\mathscr{G}_{n,q}^{(\alpha)} = \mathscr{G}_{n,q}^{(\alpha)}(0, 0), \qquad \lim_{q \to 1^-} \mathscr{G}_{n,q}^{(\alpha)}(x, y) = \mathscr{G}_{n}^{(\alpha)}(x + y), \\
\lim_{q \to 1^-} \mathscr{G}_{n,q}^{(\alpha)} = \mathscr{G}_{n}^{(\alpha)}.
\end{gathered} \tag{14}$$

From (8) and (10), it is easy to check that

$$\begin{aligned}
\mathscr{B}_{n,q}^{(\alpha)}(x, y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}_{n-k,q}(x, 0) \mathscr{B}_{k,q}^{(\alpha-1)}(0, y), \\
\mathscr{E}_{n,q}^{(\alpha)}(x, y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{n-k,q}^{(\alpha-1)}(x, 0) \mathscr{E}_{k,q}(0, y).
\end{aligned} \tag{15}$$

In this work, we give some identities for the q-Bernoulli polynomials. Also, we give some relations between the q-Bernoulli polynomials and q-Euler polynomials and the q-Genocchi polynomials and q-Bernoulli polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.

Theorem 1. *There are the following relations between the q-Bernoulli polynomials and q-Stirling numbers of the second kind:*

$$\mathscr{B}^{(\alpha)}_{n,q}(x,y) = \frac{[k]_q![n]_q!}{[n+k]_q!} \times \sum_{l=0}^{n+k} \begin{bmatrix} n+k \\ l \end{bmatrix}_q \mathscr{B}^{(\alpha+k)}_{l,q}(x,y) \times S_{2,q}(n+k-l,k), \quad (16)$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}^{(\alpha)}_{n-k,q}(x,y)[\alpha]_q! S_{2,q}(k,\alpha) = \sum_{l=0}^{n-\alpha} \begin{bmatrix} n-\alpha \\ l \end{bmatrix}_q \frac{[n]_q!}{[n-\alpha]_q!} x^{n-\alpha-l} y^l q^{\binom{l}{2}}, \quad (17)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 2. *The q-Stirling numbers of the second kind satisfy the following relations:*

$$\mathscr{E}^{(\alpha)}_{n,q}(x,y) = \sum_{j=0}^{\infty} \binom{-\alpha}{j} \frac{1}{2^j}[j]_q! \times \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix}_q S_{2,q}(n-p,j) \times \sum_{l=0}^{p} \begin{bmatrix} p \\ l \end{bmatrix}_q x^{p-l} y^l q^{\binom{l}{2}}, \quad (18)$$

$$\mathscr{B}^{(\alpha)}_{n,q} = [\alpha]_q! \sum_{j=0}^{\infty} \binom{-\alpha}{j} \times \sum_{k=0}^{j} \binom{j}{k} [k]_q! \frac{S_{2,q}(n+k,k)}{[n+k]_q!} [k]_q! (-1)^{j-k},$$

$$\mathscr{B}^{(-\alpha)}_{n,q}(x,y) = [\alpha]_q! \sum_{m=0}^{n+\alpha} \begin{bmatrix} n+\alpha \\ m \end{bmatrix}_q S_{2,q}(m,\alpha) \times (x+y)_q^{n+\alpha-m} \frac{[n]_q!}{[n+\alpha]_q!}, \quad (19)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 3. *The generalized q-Euler polynomials satisfy the following relation:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{k,q}(x,y) = 2(x+y)_q^n - \mathscr{E}_{n,q}(x,y), \quad (20)$$

where $q \in \mathbb{C}$, α, $n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 4. *The polynomials* $B_{n,q}(x,y)$ *and* $\mathscr{G}_{n,q}(x,y)$ *satisfy the following difference relationships:*

$$\mathscr{B}_{n,q}(x,y) = \sum_{\substack{l=0 \\ l\neq n}}^{n+1} \begin{bmatrix} n+1 \\ l \end{bmatrix}_q \frac{1}{[n+1]_q} \mathscr{G}_{l,q}(x,y) \mathscr{B}_{n+1-l,q}, \quad (21)$$

$$\mathscr{G}_{n,q}(x,y) = -2 \sum_{\substack{l=0 \\ l\neq n}}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{1}{[l+1]_q} \mathscr{G}_{l+1,q} \mathscr{B}_{n-l,q}(x,y), \quad (22)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

Theorem 5. *There is the following relation between the generalized q-Euler polynomials and generalized q-Bernoulli polynomials:*

$$\mathscr{E}^{(\alpha)}_{n,q}(x,y) = \left\{ \sum_{s=0}^{n+1} \begin{bmatrix} n+1 \\ s \end{bmatrix}_q \sum_{l=0}^{s} \begin{bmatrix} s \\ l \end{bmatrix}_q \mathscr{B}_{s-l,q}(mx,0) - \sum_{l=0}^{n+1} \begin{bmatrix} n+1 \\ l \end{bmatrix}_q \mathscr{B}_{n+1-l,q}(mx,0) \right\} \times \frac{m}{[n+1]_q!} \mathscr{E}^{(\alpha)}_{l,q}(0,y) m^{l-n-1}, \quad (23)$$

where $q \in \mathbb{C}$, $\alpha, n \in \mathbb{N}$, *and* $0 < |q| < 1$.

2. Proof of the Theorems

Lemma 6. *The generalized q-Bernoulli polynomials, q-Euler polynomials, and q-Genocchi polynomials satisfy the following relations:*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}^{(\alpha)}_{k,q}(x,y) \mathscr{B}^{(-\alpha)}_{n-k,q} = (x+y)_q^n,$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{B}^{(\alpha)}_{k,q}(0,y) \mathscr{B}^{(-\alpha)}_{n-k,q} = q^{(n(n-1))/2} y^n,$$

$$\mathscr{B}^{(\alpha)}_{n,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{B}^{(\alpha)}_{n-l,q}(0,y) \times \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \mathscr{E}^{(\alpha)}_{k,q}(x,0) \mathscr{E}^{(-\alpha)}_{l-k,q},$$

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{E}_{n-l,q}^{(\alpha)}(0,y) \times \sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_q \mathscr{B}_{k,q}^{(\alpha)}(x,0)\,\mathscr{B}_{l-k,q}^{(-\alpha)},$$

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{G}_{k,q}(x,y)+\mathscr{G}_{n,q}(x,y) = 2[n]_q(x+y)_q^{n-1},$$

$$\mathscr{G}_{n,q}^{(\alpha-\beta)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{G}_{k,q}^{(\alpha)}(x,0)\,\mathscr{G}_{n-k,q}^{(-\beta)}(0,y). \tag{24}$$

Proof. The proof of this lemma can be found from (7)–(12). □

Proof of Theorem 1. By (8) and (13) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{B}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} \\
&=\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx)\,E_q(ty) \times \frac{[k]_q!}{\left(e_q(t)-1\right)^k}\frac{\left(e_q(t)-1\right)^k}{[k]_q!} \\
&=[k]_q!\frac{t^{\alpha}}{\left(e_q(t)-1\right)^{\alpha+k}} e_q(tx)\,E_q(ty) \times \sum_{m=0}^{\infty}S_{2,q}(m,k)\frac{t^m}{[m]_q!} \\
&=[k]_q!t^{-k}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \times (x,y)\,S_{2,q}(n-l,k)\frac{t^n}{[n]_q!} \\
&=[k]_q!\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \times (x,y)\,S_{2,q}(n-l,k)\frac{t^{n-k}}{[n]_q!} \\
&=[k]_q!\sum_{n=-k}^{\infty}\sum_{l=0}^{n+k}\begin{bmatrix} n+k \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \times (x,y)\,S_{2,q}(n+k-l,k)\frac{t^{n-k}}{[n]_q!}.
\end{aligned} \tag{25}$$

Equating the coefficients of $(t^n/[n]_q!)$, we obtain (16).
Similarly, we have (17). □

Proof of Theorem 2. Combining (10) and (13), we obtain

$$\begin{aligned}
\left(\frac{2}{e_q(t)+1}\right)^{\alpha} &= \left(1+\frac{e_q(t)-1}{2}\right)^{(-\alpha)} \\
&= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)},
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} \\
&= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e_q(t)-1}{2}\right)^{(j)} e_q(tx)\,E_q(ty) \\
&= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\sum_{n=0}^{\infty}S_{2,q}(n,j)\frac{t^n}{[n]_q!} \times \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}y^k q^{\binom{k}{2}}\frac{t^n}{[n]_q!} \\
&= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j} \times \sum_{n=0}^{\infty}\sum_{p=0}^{n}\begin{bmatrix} n \\ p \end{bmatrix}_q [j]_q! S_{2,q}(n-p,j) \times \sum_{l=0}^{p}\begin{bmatrix} p \\ l \end{bmatrix}_q x^{p-l}y^l q^{\binom{l}{2}}\frac{t^n}{[n]_q!}.
\end{aligned} \tag{26}$$

Comparing the coefficients of $(t^n/[n]_q!)$, we find (18). Similarly, we have (19). □

Proof of Theorem 3. It is obvious that

$$\frac{-2}{\left(e_q(t)+1\right)e_q(t)}=\frac{2}{\left(e_q(t)+1\right)}-\frac{2}{e_q(t)}. \tag{27}$$

We write it as

$$\begin{aligned}
\frac{-2}{e_q(t)+1}\frac{e_q(tx)\,E_q(ty)}{e_q(t)} &= \frac{2}{e_q(t)+1}e_q(tx)\,E_q(ty) - \frac{2}{e_q(t)}e_q(tx)\,E_q(ty), \\
\frac{-2}{e_q(t)+1}e_q(tx)\,E_q(ty) &= \frac{2}{e_q(t)+1}e_q(tx)\,E_q(ty) - 2e_q(tx)\,E_q(ty)
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{n=0}^{\infty}\mathscr{E}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& =\sum_{n=0}^{\infty}\mathscr{E}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& \quad\times\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-2\sum_{n=0}^{\infty}(x+y)_q^n\frac{t^n}{[n]_q!}.
\end{aligned} \tag{28}
$$

Using the Cauchy product and comparing the coefficients of $(t^n/[n]_q!)$, we have

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{E}_{k,q}(x,y)=2(x+y)_q^n-\mathscr{E}_{k,q}(x,y). \tag{29}
$$

□

Finally, we consider the interesting relationships between the q-Bernoulli polynomials and q-Genocchi polynomials and the q-Euler polynomials and q-Bernoulli polynomials. These relations are q-analogues to the Srivastava-Pintér addition theorems.

Proof of Theorem 4. It follows immediately that

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& =\frac{1}{2}\frac{2te_q(tx)E_q(ty)}{e_q(t)+1} \\
& \quad+\frac{1}{t}\left(\frac{t}{e_q(t)-1}\right)\frac{2t}{e_q(t)+1}e_q(tx)E_q(ty) \\
& =\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}+\frac{1}{t} \\
& \quad\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& =\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& \quad+\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{1}{[n]_q}\mathscr{G}_{l,q}(x,y) \\
& \qquad\times\mathscr{B}_{n-l,q}\frac{t^{n-1}}{[n-1]_q!} \\
& =\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& \quad+\sum_{n=0}^{\infty}\left(-\frac{1}{2}\mathscr{G}_{n,q}(x,y)\right. \\
& \quad+\sum_{l=0}^{n+1}\begin{bmatrix} n+1 \\ l \end{bmatrix}_q\frac{1}{[n+1]_q} \\
& \quad\left.\times\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q}\right)\frac{t^n}{[n]_q!} \\
& =\sum_{n=0}^{\infty}\left(\sum_{\substack{l=0 \\ l\neq n}}^{n+1}\begin{bmatrix} n+1 \\ l \end{bmatrix}_q\frac{1}{[n+1]_q}\right. \\
& \quad\left.\times\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q}\right)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{30}
$$

Equating the coefficients of $(t^n/[n]_q!)$, we have (21).

In a similar fashion, (12) yields

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} \\
& =\frac{1}{t}\left(\frac{2t}{e_q(t)+1}\left(e_q(t)-1\right)\right)\left(\frac{te_q(tx)E_q(ty)}{e_q(t)-1}\right) \\
& =\frac{1}{t}\left(2t-2\frac{2t}{e_q(t)+1}\right)\left(\frac{t}{e_q(t)-1}e_q(tx)E_q(ty)\right) \\
& =\frac{1}{t}\left(2t-2\sum_{n=0}^{\infty}\mathscr{G}_{n,q}\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right) \\
& =\frac{1}{t}\left(-2\sum_{l=0}^{\infty}\frac{1}{[l+1]_q!}\mathscr{G}_{l+1,q}\frac{t^{l+1}}{[l]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right) \\
& =\sum_{n=1}^{\infty}\left(-2\sum_{\substack{l=0 \\ l\neq n}}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{\mathscr{G}_{l+1,q}}{[l+1]_q}\mathscr{B}_{n-l,q}(x,y)\right)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{31}
$$

Comparing the coefficients of $(t^n/[n]_q!)$, we have (22). □

Proof of Theorem 5. By (10), we write

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!} \\
& =\left(\frac{2}{e_q(t)+1}\right)^{\alpha} \\
& \quad\times E_q(ty)\frac{e_q(t/m)-1}{(t/m)}\frac{(t/m)}{e_q(t/m)\ -1}e_q((t/m)mx)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\right.\\
&\qquad\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\\
&\qquad\times\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}\\
&\qquad\left.\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\right\}\\
&= m\sum_{n=-1}^{\infty}\frac{1}{[n+1]_q}\\
&\qquad\times\left\{\sum_{s=0}^{n+1}\begin{bmatrix}n+1\\ s\end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix}s\\ l\end{bmatrix}_q\mathscr{B}_{s-l,q}(mx,0)\right.\\
&\qquad\qquad\left.-\sum_{l=0}^{n+1}\begin{bmatrix}n+1\\ l\end{bmatrix}_q\mathscr{B}_{n+1-l,q}(mx,0)\right\}\\
&\quad\times\frac{m}{[n+1]_q!}\mathscr{E}_{l,q}^{(\alpha)}(0,y)\,m^{l-n-1}\frac{t^n}{[n]_q!}.
\end{aligned}
\tag{32}
$$

By equating the coefficients of $(t^n/[n]_q!)$, we get the theorem. □

Remark 7. There are many different relationships which are analogues to the Srivastava-Pintér addition theorems at these polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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