Kim and Kim Advances in Difference Equations 2012, **2012**:44
http://www.advancesindifferenceequations.com/content/2012/1/44

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

A note on Carlitz q-Bernoulli numbers and polynomials

Daeyeoul Kim[1] and Min-Soo Kim[2*]

* Correspondence:
mskim@kyungnam.ac.kr
[2]Division of Cultural Education, Kyungnam University, Changwon 631-701, South Korea
Full list of author information is available at the end of the article

Abstract

In this article, we first aim to give simple proofs of known formulae for the generalized Carlitz q-Bernoulli polynomials $\beta_{m,\chi}(x, q)$ in the p-adic case by means of a method provided by Kim and then to derive a complex, analytic, two-variable q-L-function that is a q-analog of the two-variable L-function. Using this function, we calculate the values of two-variable q-L-functions at nonpositive integers and study their properties when q tends to 1.

Mathematics Subject Classification (2000): 11B68; 11S80.

Keywords: Carlitz q-Bernoulli numbers, Carlitz q-Bernoulli polynomials, Dirichlet q-L-functions

1. Introduction

Let p be a fixed prime. We denote by $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of a q-extension, q can be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \le 1$.

Let d be a fixed positive integer. Let

$$\begin{aligned} X = X_d &= \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \tag{1.1}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.2}$$

Hence $\lim_{q\to 1} [x]_q = x$ for any $x \in \mathbb{C}$ in the complex case and any x with $|x|_p \le 1$ in the present p-adic case. This is the hallmark of a q-analog: The limit as $q \to 1$ recovers the classical object.

In 1937, Vandiver [1] and, in 1941, Carlitz [2] discussed generalized Bernoulli and Euler numbers. Since that time, many authors have studied these and other related subjects (see, e.g., [3-6]). The final breakthrough came in the 1948 article by Carlitz [7]. He defined inductively new q-Bernoulli numbers $\beta_m = \beta_m(q)$ by

$$\beta_0(q) = 1, \quad q(q\beta(q) + 1)^m - \beta_m(q) = \begin{cases} 1 \text{ if } m = 1 \\ 0 \text{ if } m > 1, \end{cases} \tag{1.3}$$

with the usual convention of β^i by β_i. The q-Bernoulli polynomials are defined by

$$\beta_m(x, q) = (q^x\beta(q) + [x]_q)^m = \sum_{i=0}^{m} \binom{m}{i} \beta_i(q) q^{ix} [x]_q^{m-i}. \tag{1.4}$$

In 1954, Carlitz [8] generalized a result of Frobenius [3] and showed many of the properties of the q-Bernoulli numbers $\beta_m(q)$. In 1964, Carlitz [9] extended the Bernoulli, Eulerian, and Euler numbers and corresponding polynomials as a formal Dirichlet series. In what follows, we shall call them the Carlitz q-Bernoulli numbers and polynomials.

Some properties of Carlitz q-Bernoulli numbers $\beta_m(q)$ were investigated by various authors. In [10], Koblitz constructed a q-analog of p-adic L-functions and suggested two questions. Question (1) was solved by Satoh [11]. He constructed a complex analytic q-L-series that is a q-analog of Dirichlet L-function and interpolates Carlitz q-Bernoulli numbers, which is an answer to Koblitz's question. By using a q-analog of the p-adic Haar distribution (see (1.6) below), Kim [12] answered part of Koblitz's question (2) and constructed q-analogs of the p-adic log gamma functions $G_{p,q}(x)$ on $\mathbb{C}_p \setminus \mathbb{Z}_p$.

In [11], Satoh constructed the generating function of the Carlitz q-Bernoulli numbers $F_q(t)$ in $\mathbb{C}$ which is given by

$$F_q(t) = \sum_{m=0}^{\infty} q^m e^{[m]_q t}(1 - q - q^m t) = \sum_{m=0}^{\infty} \beta_m(q) \frac{t^m}{m!}, \tag{1.5}$$

where q is a complex number with $0 < |q| < 1$. He could not explicitly determine $F_q(t)$ in $\mathbb{C}_p$, see [11, p.347].

In [12], Kim defined the q-analog of the p-adic Haar distribution $\mu_{\text{Haar}}(a + p^N \mathbb{Z}_p) = 1/p^N$ by

$$\mu_q(a + p^N\mathbb{Z}_p) = \frac{q^a}{[p^N]_q}. \tag{1.6}$$

Using this distribution, he proved that the Carlitz q-Bernoulli numbers $\beta_m(q)$ can be represented as the p-adic q-integral on $\mathbb{Z}_p$ by μ_q, that is,

$$\beta_m(q) = \int_{\mathbb{Z}_p} [a]_q^m d\mu_q(a), \tag{1.7}$$

and found the following explicit formula

$$\beta_m(q) = \frac{1}{(q-1)^m} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \frac{i+1}{[i+1]_q}, \tag{1.8}$$

where $m \geq 0$ and $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < p^{-\frac{1}{p-1}}$.

Recently, Kim and Rim [13] constructed the generating function of the Carlitz q-Bernoulli numbers $F_q(t)$ in $\mathbb{C}_p$:

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{j+1}{[j+1]_q} (-1)^j \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!}, \tag{1.9}$$

where $q \in \mathbb{C}_p$ with $0 < |1-q|_p < p^{-\frac{1}{p-1}}$.

This article is organized as follows.

In Section 2, we consider the generalized Carlitz q-Bernoulli polynomials in the p-adic case by means of a method provided by Kim. We obtain the generating functions of the generalized Carlitz q-Bernoulli polynomials. We shall provide some basic formulas for the generalized Carlitz q-Bernoulli polynomials which will be used to prove the main results of this article.

In Section 3, we construct the complex, analytic, two-variable q-L-function that is a q-analog of the two-variable L-function. Using this function, we calculate the values of two-variable q-L-functions at nonpositive integers and study their properties when q tends to 1.

2. Generalized Carlitz q-Bernoulli polynomials in the p-adic (and complex) case

For any uniformly differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic q-integral on $\mathbb{Z}_p$ is defined to be the limit $\frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a) q^a$ as $N \to \infty$. The uniform differentiability guarantees the limit exists. Kim [12,14-16] introduced this construction, denoted $I_q(f)$, where $|1 - q|_p < p^{-1/(p-1)}$.

The construction of $I_q(f)$ makes sense for many q in $\mathbb{C}_p$ with the weaker condition $|1 - q|_p < 1$. Indeed, when $|1 - q|_p < 1$ the function q^x is uniformly differentiable and the space of uniformly differentiable functions $\mathbb{Z}_p \to \mathbb{C}_p$ is closed under multiplication, so we can make sense of its p-adic q-integral $I_q(f)$ for $|1 - q|_p < 1$.

Lemma 2.1. *For $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < 1$ and $x \in \mathbb{Z}_p$, we have*

$$\lim_{N\to\infty} \frac{1}{1-q^{p^N}} \sum_{a=0}^{p^N-1} q^{ax} = \frac{x}{1-q^x}.$$

Proof. We assume that $q \in \mathbb{C}_p$ satisfies the condition $0 < |1 - q|_p < 1$. Then it is known that

$$q^x = \sum_{m=0}^{\infty} \binom{x}{m} (q-1)^m$$

for any $x \in \mathbb{Z}_p$ (see [[17], Lemma 3.1 (iii)]). Therefore, we obtain

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{1-q^{p^N}} \sum_{a=0}^{p^N-1} q^{ax} &= \frac{1}{1-q^x} \lim_{N\to\infty} \frac{\left(q^{p^N}\right)^x - 1}{q^{p^N}-1} \\
&= \frac{1}{1-q^x} \lim_{N\to\infty} \frac{\sum_{m=1}^{\infty} \binom{x}{m} \left(q^{p^N}-1\right)^m}{q^{p^N}-1} \\
&= \frac{1}{1-q^x} \lim_{N\to\infty} \sum_{m=0}^{\infty} \binom{x}{m+1} \left(q^{p^N}-1\right)^m \\
&= \frac{x}{1-q^x}.
\end{aligned}$$

This completes the proof.

Definition 2.2 ([12, §2, p. 323]). Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$ and let $x \in \mathbb{Z}_p$. For $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < 1$ and an integer $m \geq 0$, the generalized Carlitz q-Bernoulli polynomials $\beta_{m,\chi}(x, q)$ are defined by

$$\begin{aligned}\beta_{m,\chi}(x,q) &= \int_X \chi(a)[x+a]_q^m d\mu_q(a) \\ &= \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} \chi(a)[x+a]_q^m q^a.\end{aligned} \tag{2.1}$$

Remark 2.3. If $\chi = \chi^0$, the trivial character and $x = 0$, then (2.1) reduces to (1.7) since $d = 1$. In particular, Kim [12] defined a class of p-adic interpolation functions $G_{p,q}(x)$ of the Carlitz q-Bernoulli numbers $\beta_m(q)$ and gave several interesting applications of these functions.

By Lemma 2.1, we can prove the following explicit formula of $\beta_{m,\chi}(x, q)$ in $\mathbb{C}_p$.

Proposition 2.4. *For $q \in \mathbb{C}_p$ with $0 < |1 - q|_p < 1$ and an integer $m \geq 0$, we have*

$$\beta_{m,\chi}(x,q) = \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{[d(i+1)]_q}.$$

Proof. For $m \geq 0$, (2.1) implies

$$\begin{aligned}\beta_{m,\chi}(x,q) &= \lim_{N\to\infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{k=0}^{d-1} \sum_{a=0}^{p^N-1} \chi(k+da)[x+k+da]_q^m q^{k+da} \\ &= \lim_{N\to\infty} \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) \frac{q^k}{1-q^{dp^N}} \sum_{q=0}^{p^N-1} \left(1-q^{x+k+da}\right)^m q^{da} \\ &= \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \\ &\quad \times \lim_{N\to\infty} \frac{1}{1-(q^d)^{p^N}} \sum_{a=0}^{p^N-1} \left(q^d\right)^{a(i+1)} \\ &= \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{1-q^{d(i+1)}} \\ &\quad \text{(where we use Lemma 2.1).}\end{aligned}$$

This completes the proof.

Remark 2.5. We note here that similar expressions to those of Proposition 2.4 with $\chi = \chi^0$ are given by Kamano [[18], Proposition 2.6] and Kim [12, §2]. Also, Ryoo et al. [19, Theorem 4] gave the explicit formula of $\beta_{m,\chi}(0, q)$ in $\mathbb{C}$ for $m \geq 0$.

Lemma 2.6. *Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Then for $q \in \mathbb{C}$ with $|q| < 1$,*

$$\sum_{m=0}^{\infty} \chi(m) q^{mx} = \frac{1}{1-q^{dx}} \sum_{k=0}^{d-1} \chi(k) q^{kx}.$$

Proof. If we write $m = ad + k$, where $0 \leq k \leq d - 1$ and $a = 0,1, 2,\dots$, we have the desired result.

We now consider the case:

$$q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p, \quad 0 < |q| < 1, \quad 0 < |1-q|_p < 1. \tag{2.2}$$

For instance, if we set

$$q = \frac{1}{1-pz} \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$$

for each $z \neq 0 \in \mathbb{Z}$ and $p > 3$, we find $0 < |q| < 1$, $0 < |1 - q|_p < 1$.

Let $F_{q,\chi}(t, x)$ be the generating function of $\beta_{m,\chi}(x, q)$ defined in Definition 2.2. From Proposition 2.4, we have

$$\begin{aligned} F_{q,\chi}(t,x) &= \sum_{m=0}^{\infty} \beta_{m,\chi}(x,q)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{[d(i+1)]_q} \right) \frac{t^m}{m!} \\ &= P_{q,\chi}(t,x) + Q_{q,\chi}(t,x), \end{aligned} \tag{2.3}$$

where

$$P_{q,\chi}(t,x) = \sum_{m=0}^{\infty} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i}{[d(i+1)]_q} \frac{t^m}{m!}$$

and

$$Q_{q,\chi}(t,x) = \sum_{m=0}^{\infty} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{1}{[d(i+1)]_q} \frac{t^m}{m!}.$$

Then, noting that

$$e^{\frac{t}{1-q}} = \sum_{i=0}^{\infty} (-1)^i (q-1)^{-i} \frac{t^i}{i!},$$

we see that

$$\begin{aligned} P_{q,\chi}(t,x) &= \sum_{m=0}^{\infty} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^{m} \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i}{[d(i+1)]_q} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \frac{t^n}{n!} \sum_{j=0}^{\infty} \frac{1}{(q-1)^j} \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j}{[d(j+1)]_q} \frac{t^j}{j!} \\ &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1} \right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j}{[d(j+1)]_q} \frac{t^j}{j!}. \end{aligned} \tag{2.4}$$

Moreover, (2.4) now becomes

$$\begin{aligned} P_{q,\chi}(t,x) &= e^{\frac{t}{1-q}} \sum_{j=1}^{\infty} \left(\frac{1}{q-1}\right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{1}{[d(j+1)]_q} \frac{t^j}{(j-1)!} \\ &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^j q^{(j+1)x} \sum_{k=0}^{d-1} \chi(k) \frac{q^{k(j+2)}}{q^{d(j+2)}-1} \frac{t^{j+1}}{j!} \\ &= -te^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^j q^{(j+1)x} \sum_{n=0}^{\infty} \chi(n) q^{n(j+2)} \frac{t^j}{j!} \\ &\quad \text{(where we use Lemma 2.6)} \\ &= -te^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \chi(n) q^{x+2n} \sum_{j=0}^{\infty} \left(\frac{-q^{n+x}}{1-q}\right)^j \frac{t^j}{j!} \\ &= -te^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \chi(x) q^{x+2n} e^{\frac{(-q^{n+x})t}{1-q}} \\ &= -t \sum_{n=0}^{\infty} \chi(x) q^{x+2n} e^{[n+x]_q t} \end{aligned} \tag{2.5}$$

(cf. [13,16,20]). Similar arguments apply to the case $Q_{q,\chi}(t, x)$. We can rewrite

$$Q_{q,\chi}(t,x) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{1}{[d(j+1)]_q} \frac{t^j}{j!} \tag{2.6}$$

and

$$Q_{q,\chi}(t,x) = (1-q) \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]_q t}. \tag{2.7}$$

Then, by (2.4), (2.5), (2.6), and (2.7), we have the following theorem.

Theorem 2.7. *Let $q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$, $0 < |q| < 1$, $0 < |1-q|_p < 1$. Then the generalized Carlitz q-Bernoulli polynomials $\beta_{m,\chi}(x, q)$ for $m \leq 0$ is given by equating the coefficients of powers of t in the following generating function:*

$$\begin{aligned} F_{q,\chi}(t,x) &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{q-1}\right)^{j-1} \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j+1}{q^{d(j+1)}-1} \frac{t^j}{j!} \\ &= \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]_q t} (1-q-q^{n+x} t). \end{aligned} \tag{2.8}$$

Remark 2.8. If $\chi = \chi 0$, the trivial character, and $x = 0$, (2.8) reduces to (1.5).

3. *q*-analog of the two-variable *L*-function (in $\mathbb{C}$)

From Theorem 2.7, for $k \geq 0$, we obtain the following

$$\begin{aligned} \beta_{k,\chi}(x,q) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k F_{q,\chi}(t,x)\bigg|_{t=0} \\ &= (1-q) \sum_{m=0}^{\infty} \chi(m) q^m [m+x]_q^k - k \sum_{m=0}^{\infty} \chi(m) q^{x+2m} [m+x]_q^{k-1}. \end{aligned} \tag{3.1}$$

Hence we can define a q-analog of the L-function as follows:

Definition 3.1. Suppose that χ is a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Let q be a complex number with $0 < |q| < 1$, and let $L_q(s, x, \chi)$ be a function of two-variable $(s, x) \in \mathbb{C} \times \mathbb{R}$ defined by

$$L_q(s, x, \chi) = \frac{1-q}{s-1}\sum_{m=0}^{\infty}\frac{\chi(m)q^m}{[m+x]_q^{s-1}} + \sum_{m=0}^{\infty}\frac{\chi(m)q^{m+2x}}{[m+x]_q^s} \tag{3.2}$$

for $0 < x \leq 1$ (cf. [11,13,14,21-25]).

In particular, the two-variable function $L_q(s, x, \chi)$ is a generalization of the one-variable $L_q(s, \chi)$ of Satoh [11], yielding the one-variable function when the second variable vanishes.

Proposition 3.2. *For $k \in \mathbb{Z}$, $k \geq 1$, the limiting value* $\lim_{s \to k} L_q(1 - s, x, \chi) = L_q(1 - k, x, \chi)$ *exists and is given explicitly by*

$$L_q(1-k, x, \chi) = -\frac{1}{k}\beta_{k,\chi}(x, q).$$

Proof. The proof is clear by Proposition 2.4, Theorem 2.7 and (3.1).

The formula of Proposition 3.2 is slight extension of the result in [19] and [11, Theorem 2].

Theorem 3.3. *For any positive integer k, we have*

$$\begin{aligned}\lim_{q\to 1}\beta_{k,\chi}(x, q) &= \lim_{q\to 1}\frac{1}{(1-q)^m}\sum_{k=0}^{d-1}\chi(k)q^k\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{i(x+k)}\frac{i+1}{[d(i+1)]_q}\\ &= B_{k,\chi}(x),\end{aligned}$$

where the $B_{k,\chi}(x)$ are the kth generalized Bernoulli polynomials.

Proof. We follow the proof in [[26], Theorem 1] motivated by the study of a simple q-analog of the Riemann zeta function. Recall that the ordinary Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{i}{q^i-1}q^{ix} = \frac{1}{\log q}\frac{i\log q}{e^{i\log q}-1}e^{x(i\log q)} = \frac{1}{\log q}\sum_{k=0}^{\infty}B_k(x)i^k\frac{(\log q)^k}{k!}, \tag{3.3}$$

where it is noted that in this instance, the notation $B_k(x)$ is used to replace $B^k(x)$ symbolically. For each $m \geq 1$, let

$$(e^t-1)^m = \sum_{k=0}^{\infty}d_k^{(m)}\frac{t^k}{k!}. \tag{3.4}$$

Note that

$$(e^t-1)^m = \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}e^{it} = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}i^k\right)\frac{t^k}{k!}. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$d_k^{(m)} = \begin{cases}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}i^k, & m \leq k\\ 0, & 0 \leq k < m.\end{cases} \tag{3.6}$$

It is also clear from the definition that $d_0^{(0)} = 1, d_k^{(0)} = 0$ and $d_k^{(k)} = k!$ for $k \in \mathbb{N}$. From (2.3), (3.3), and (3.6), we obtain

$$
\begin{aligned}
\beta_{m,\chi}(x,q) &= \frac{q^{-x}}{(q-1)^m}\sum_{k=0}^{d-1}\chi(k)q^{k+x}\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}q^{i(k+x)}\frac{i+1}{[d(i+1)]_q} \\
&= \frac{q^{-x}}{(q-1)^{m-1}}\sum_{k=0}^{d-1}\chi(k)\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} \\
&\quad \times e^{d(i+1)\log q\frac{(k+x)}{d}}\frac{d(i+1)\log q}{e^{d(i+1)\log q-1}}\frac{1}{d\log q} \\
&= \frac{q^{-x}}{(q-1)^{m-1}}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}(i+1)^n\right) \\
&\quad \times d^{n-1}\sum_{k=0}^{d-1}\chi(k)B_n\left(\frac{k+x}{d}\right)\frac{(\log q)^{n-1}}{n!} \\
&= q^{-x}\frac{(\log q)^{m-1}}{(q-1)^{m-1}}d^{m-1}\sum_{k=0}^{d-1}\chi(k)B_m\left(\frac{k+x}{d}\right) \\
&\quad + q^{-x}\sum_{\sigma=1}^{\infty}\sum_{i=0}^{\sigma}\binom{m+\sigma}{i}d_{m+\sigma-i}^{(m)}\frac{1}{(m+\sigma)!}\frac{(\log q)^{m+\sigma-1}}{(q-1)^{m-1}} \\
&\quad \times d^{m+\sigma-1}\sum_{k=0}^{d-1}\chi(k)B_{m+\sigma}\left(\frac{k+x}{d}\right).
\end{aligned}
$$

Then, because

$$
\log q = \log(1+(q-1)) = (q-1) - \frac{(q-1)^2}{2} + \cdots = (q-1) + O((q-1)^2)
$$

as $q \to 1$, we find

$$
\lim_{q\to 1}\frac{(\log q)^{m+\sigma-1}}{(q-1)^{m-1}} = \begin{cases} 1, & \sigma = 0 \\ 0, & \sigma \geq 1, \end{cases}
$$

so

$$
\lim_{q\to 1}\beta_{m,\chi}(x,q) = d^{m-1}\sum_{k=0}^{d-1}\chi(k)B_m\left(\frac{k+x}{d}\right) = B_{m,\chi}(x),
$$

where the $B_{m,\chi}(x)$ are the mth generalized Bernoulli polynomials (e.g., [14,19]). This completes the proof.

Corollary 3.4. *For any positive integer k, we have*

$$
\lim_{q\to 1}L_q(1-k,x,\chi) = -\frac{1}{k}B_{k,x}(x).
$$

Remark 3.5. The formula of Theorem 3.3 is slight extension of the result in [[26], Theorem 1].

Remark 3.6. From Theorem 2.7, the generalized Bernoulli polynomials $B_{m,\chi}(x)$ are defined by means of the following generating function [[27], p. 8]

$$
\begin{aligned}
F_\chi(t,x) &:= \lim_{q\to 1} F_{q,\chi}(t,x) \\
&= -t\sum_{a=1}^{d}\sum_{l=0}^{\infty}\chi(a+dl)e^{(a+dl)t}e^{xt} \\
&= \sum_{a=1}^{d}\frac{\chi(a)te^{(a+x)t}}{e^{dt}-1} \\
&= \sum_{m=0}^{\infty}B_{m,\chi}(x)\frac{t^m}{m!}.
\end{aligned}
$$

Remark 3.7. If we substitute $\chi = \chi^0$, the trivial character, in Definition 3.1 and Corollary 3.4, we can also define a q-analog of the Hurwitz zeta function

$$\zeta(s,x) = \sum_{m=0}^{\infty}\frac{1}{(m+x)^s}$$

by

$$\zeta_q(s,x) = L_q(s,x,\chi^0) = \frac{1-q}{s-1}\sum_{m=0}^{\infty}\frac{q^{m+x}}{[m+x]_q^{s-1}} + \sum_{m=0}^{\infty}\frac{q^{2(m+x)}}{[m+x]_q^{s}}$$

and obtain the identity

$$\lim_{q\to 1}\zeta_q(s,x) = \zeta(s,x)$$

for all $s \neq 1$, as well as the formula

$$\lim_{q\to 1}\zeta_q(1-k,x) = -\frac{1}{k}B_k(x)$$

for integers $k \geq 1$ (cf. [11,13,19,22,24,25]).

Acknowledgements
This work was supported by the Kyungnam University Foundation Grant, 2012.

Author details
[1]National Institute for Mathematical Sciences, Doryong-dong, Yuseong-gu, Daejeon 305-340, South Korea [2]Division of Cultural Education, Kyungnam University, Changwon 631-701, South Korea

Authors' contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 21 December 2011 Accepted: 13 April 2012 Published: 13 April 2012

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doi:10.1186/1687-1847-2012-44
Cite this article as: Kim and Kim: **A note on Carlitz *q*-Bernoulli numbers and polynomials.** *Advances in Difference Equations* 2012 **2012**:44.