ON HYPERGEOMETRIC BERNOULLI NUMBERS AND POLYNOMIALS

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ABSTRACT. In this note, we shall provide several properties of hypergeometric Bernoulli numbers and polynomials, including sums of products identity, differential equations and recurrence formulas. Our results generalize many previous authors' results for this kind of special functions.

1. INTRODUCTION

The Bernoulli polynomials is defined by the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

while the Bernoulli numbers is defined by $B_n=B_n(0)$.

The Bernoulli polynomials and numbers have many applications in number theory and numerical analysis. For example, many results on the solutions of Diophantine equations are based on the properties of Bernoulli polynomials [28, 3, 4, 11], and using Bernoulli numbers we also have the following well-known Euler-Maclaurin Summation (EMS)

$$\sum_{k=0}^{n}f(k)=\int_0^n f(x)dx+\frac{1}{2}[f(n)+f(0)]+\sum_{k=2}^{\infty}\frac{B_k}{k!}[f^{(k-1)}(n)-f^{(k-1)}(0)] \tag{1.2}$$

which has been extensively used in numerical analysis for approximating sums and integrals.

The Bernoulli numbers and polynomials satisfy many interesting identities. The most remarkable one is Euler's sums of products identity

$$\sum_{i=0}^{n}\binom{n}{i}B_iB_{n-i}=-nB_{n-1}-(n-1)B_n \quad (n\geq 1). \tag{1.3}$$

This identity has been generalized by many authors from different directions (see [2, 6, 7, 10, 21, 22, 26, 27, 31, 30]). In particular, Dilcher [10] provided explicit expressions for sums of products for arbitrarily many Bernoulli numbers and polynomials.

Another approach to Bernoulli polynomials is to define them as an Appell sequence with zero mean:

$$B_0(x)=1, \tag{1.4}$$

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$$B_n'(x) = nB_{n-1}(x), \tag{1.5}$$

$$\int_0^1 B_n(x)dx = \begin{cases} 1 & n = 0 \\ 0 & n > 0. \end{cases} \tag{1.6}$$

Based on the properties of Appell polynomials, He and Ricci [12] proved that the Bernoulli polynomials $B_n(x)$ satisfy the following differential equations:

$$\frac{B_n}{n!}y^{(n)} + \frac{B_{n-1}}{(n-1)!}y^{(n-1)} + \cdots + \frac{B_2}{2!}y'' - \left(x - \frac{1}{2}\right)y' + ny = 0. \tag{1.7}$$

Howard [18] gave a generalization of Bernoulli polynomials by considering the following generating function:

$$\frac{t^2e^{xt}/2}{e^t - 1 - t} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!} \tag{1.8}$$

and more generally, for all positive integer N

$$\frac{t^N e^{xt}/N!}{e^t - T_{N-1}(t)} = \sum_{n=0}^{\infty} B_{N,n}(x)\frac{t^n}{n!}, \tag{1.9}$$

where $T_N(t) = \sum_{n=0}^{N} \frac{t^n}{n!}$ is the N-th Taylor polynomials of e^t.

The polynomials $B_{N,n}(x)$ are named hypergeometric Bernoulli polynomials, while the numbers $B_{N,n} = B_{N,n}(0)$ are named hypergeometric Bernoulli numbers. The reason is that the generating function $f(z) = \frac{e^z - T_{N-1}(z)}{z^N/N!}$ can be expressed as ${}_1F_1(1, N+1; z)$, where the confluent hypergeometric function ${}_1F_1(a, b; z)$ is defined by the following infinite series

$${}_1F_1(a, b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}.$$

Here $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) & (n \geq 1) \\ 1 & (n = 0) \end{cases}$$

(see [20, p. 2261]).

As their classical counterparts, Bernoulli numbers and polynomials, the hypergeometric Bernoulli numbers and polynomials also satisfy many interesting properties ([5, 9, 15, 18, 19, 20, 25]).

In the analytical aspect, many properties of hypergeometric zeta functions analogues to the classical Riemann zeta functions, including their analytic continuations, the intimate connection to Bernoulli numbers, a functional inequality satisfied by second-order hypergeometric zeta functions, have been obtained by Hassen and Nguyen in [16], and using special values of multiple analogues of hypergeometric zeta functions, Kamano [20] proved the following result for sums of products of hypergeometric Bernoulli numbers, which is a generalization of Euler's sums of products identity of Bernoulli numbers (1.3).

Theorem 1.1 (Kamano)**.** *Let N and r be positive integers. For any integer $n \ge r-1$, we have*

$$\sum_{\substack{i_1,\dots,i_r\ge 0\\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\cdots i_r!} B_{N,i_1}\cdots B_{N,i_r} = \frac{1}{N^{r-1}}\sum_{i=0}^{r-1} A_r^{(N)}(i;1+N(r-1)-n)(-1)^i\binom{n}{i} i! B_{N,n-i}, \tag{1.10}$$

where $A_r^{(N)}(i;s) \in \mathbb{Q}[s] (0 \le i \le r-1)$ are polynomials defined by the following recurrence relation:

$$\begin{aligned} &A_1^{(N)}(0;s) = 1\\ &A_r^{(N)}(i;s) = \frac{s-1}{r-1} A_{r-1}^{(N)}(i;s-N) + A_{r-1}^{(N)}(i-1;s-N+1). \end{aligned} \tag{1.11}$$

Here $r \ge 2$ and $A_r^{(N)}(i;s)$ are defined to be zero for $i \le -1$ and $i \ge r$.

As for Bernoulli polynomials, another approach to hypergeometric Bernoulli polynomials is to define them in terms of Appell sequence with zero mean (comparing with (1.4), (1.5) and (1.6) above):

$$B_{N,0}(x) = 1, \tag{1.12}$$

$$B'_{N,n}(x) = nB_{N,n-1}(x), \tag{1.13}$$

$$\int_0^1 (1-x)^{N-1} B_{N,n}(x)dx = \begin{cases} \frac{1}{N} & n=0\\ 0 & n>0 \end{cases} \tag{1.14}$$

(see [15, p. 768]).

In this paper, we shall further provide several properties for hypergeometric Bernoulli numbers and polynomials.

First, by using the generating function and purely combinational methods, we shall give a direct proof of the following result for sums of products of hypergeometric Bernoulli polynomials, which generalize Kamano's results (Theorem 1.1) for sums of products of hypergeometric Bernoulli numbers.

Theorem 1.2. *Let N and r be positive integers and let $x = x_1 + \dots + x_r$. For any integer $n \ge r-1$, we have*

$$\sum_{\substack{i_1,\dots,i_r\ge 0\\ i_1+\dots+i_r=n}} \frac{n!}{i_1!\cdots i_r!} B_{N,i_1}(x_1)\cdots B_{N,i_r}(x_r) = \frac{1}{N^{r-1}}\sum_{i=0}^{r-1} A_r^{(N)}(i,x;1+N(r-1)-n)(-1)^i\binom{n}{i} i! B_{N,n-i}(x), \tag{1.15}$$

where $A_r^{(N)}(i,x;s)\in\mathbb{Q}[x,s](0\leq i\leq r-1)$ are polynomials defined by the following recurrence relation:

(1.16)

$$\begin{aligned}&A_1^{(N)}(0,x;s)=1\\&A_r^{(N)}(i,x;s)=\frac{s-1}{r-1}A_{r-1}^{(N)}(i,x;s-N)-\frac{x-(r-1)}{r-1}A_{r-1}^{(N)}(i-1,x;s-N+1).\end{aligned}$$

Here $r\geq 2$ and $A_r^{(N)}(i,x;s)$ are defined to be zero for $i\leq -1$ and $i\geq r$.

Remark 1.3. Letting $x=0$ in Theorem 1.15, we obtain Theorem 1.1.

Remark 1.4. Nguyen and Cheong [25, Theorem 20] obtained another type of sums of products identity of hypergeometric Bernoulli polynomials by using a result of Nguyen [24] on two-dimensional recurrence sequences.

Remark 1.5. Letting $r=2,3$ in Theorem 1.2, from (1.16), we have the following identities,

(1.17)

$$\begin{aligned}&A_2^{(N)}(0,x;1+N-n)=N-n,\\&A_2^{(N)}(1,x;1+N-n)=-(x-1),\\&A_3^{(N)}(0,x;1+2N-n)=\frac{1}{2}(2N-n)(N-n),\\&A_3^{(N)}(1,x;1+2N-n)=-\frac{1}{2}(2N-n)(x-1)-\frac{1}{2}(x-2)(N-n+1),\\&A_3^{(N)}(2,x;1+2N-n)=\frac{1}{2}(x-2)(x-1),\end{aligned}$$

thus

$$\begin{aligned}&\sum_{\substack{i_1+i_2=n\\i_1,i_2\geq 0}}\frac{n!}{i_1!i_2!}B_{N,i_1}(x_1)B_{N,i_2}(x_2)\\&\quad=\frac{1}{N}(N-n)B_{N,n}(x)+\frac{n}{N}(x-1)B_{N,n-1}(x)\\&\quad\quad\text{(where } x=x_1+x_2 \text{ and } n\geq 1),\end{aligned}\tag{1.18}$$

and

$$\begin{aligned}&\sum_{\substack{i_1+i_2+i_3=n\\i_1,i_2,i_3\geq 0}}\frac{n!}{i_1!i_2!i_3!}B_{N,i_1}(x_1)B_{N,i_2}(x_2)B_{N,i_3}(x_3)\\&\quad=\frac{1}{2N^2}\Big[(N-n)(2N-n)B_{N,n}(x)\\&\quad+n((2N-n)(x-1)+(x-2)(N-n+1))B_{N,n-1}(x)\\&\quad+n(n-1)(x-1)(x-2)B_{N,n-2}(x)\Big]\\&\quad\text{(where } x=x_1+x_2+x_3 \text{ and } n\geq 2).\end{aligned}\tag{1.19}$$

These are Example 18 of [25]. Nguyen and Cheong in [25] also obtained (1.18) and (1.19) in a different way.

Remark 1.6. If $N=1$, $x=0$ and $r=2$, (1.18) becomes the classical sums of products identity for Bernoulli numbers (see (1.3)).

For positive integers N and r, the higher order hypergeometric Bernoulli polynomials are defined by the generating function

$$\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x)\frac{t^n}{n!}, \tag{1.20}$$

where $T_N(t)=\sum_{n=0}^{N}\frac{t^n}{n!}$ is the N-th Tayler polynomials of e^t, and the higher order hypergeometric Bernoulli numbers are defined by $B_{N,n}^{(r)}=B_{N,n}^{(r)}(0)$ (see [20, 25]). In particular, $B_{N,n}^{(1)}(x)=B_{N,n}(x)$, the hypergeometric Bernoulli polynomials, and $B_{N,n}^{(1)}(0)=B_{N,n}$, the hypergeometric Bernoulli numbers.

In this paper, based on the properties of Appell polynomials, we also derive the following result on the differential equation satisfied by the higher order hypergeometric Bernoulli polynomials. This generalize He and Ricci's results [12] for classical Bernoulli polynomials.

Theorem 1.7. *The higher order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ satisfy the differential equation*

$$\frac{B_{N,n}}{n!}y^{(n)}+\frac{B_{N,n-1}}{(n-1)!}y^{(n-1)}+\cdots+\frac{B_{N,2}}{2!}y''-\left(\frac{x}{rN}-\frac{1}{N(N+1)}\right)y'+\frac{n}{rN}y=0.$$

Letting $N=r=1$ in Theorem 1.7, we have the following result.

Corollary 1.8 (See He and Ricci [12, Theorem 2.3])**.** *The classical Bernoulli polynomials $B_n(x)$ satisfy the differential equation*

$$\frac{B_n}{n!}y^{(n)}+\frac{B_{n-1}}{(n-1)!}y^{(n-1)}+\cdots+\frac{B_2}{2!}y''-\left(x-\frac{1}{2}\right)y'+ny=0.$$

Furthermore, as a byproduct of our proofs, we also obtain the following result on the linear recurrence relation for higher order hypergeometric Bernoulli polynomials. This generalize Lu's results [23] for classical Bernoulli polynomials.

Theorem 1.9. *For any integral $n\geq 1$, the following linear recurrence relation for higher order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ holds true:*

$$B_{N,n+1}^{(r)}(x)=\left(x-\frac{r}{N+1}\right)B_{N,n}^{(r)}(x)-rN\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{N,n-k+1}}{n-k+1}B_{N,k}^{(r)}(x).$$

In the special case of Theorem 1.9 when $N=1$, we have the following result.

Corollary 1.10 (See Lu [23, Theorem 2.1])**.** *For any integral $n\geq 1$, the following linear recurrence relation for higher order Bernoulli polynomials $B_n^{(r)}(x)$ holds true:*

$$B_{n+1}^{(r)}(x)=\left(x-\frac{r}{2}\right)B_n^{(r)}(x)-r\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_{n-k+1}}{n-k+1}B_k^{(r)}(x).$$

Remark 1.11. From the generating function of the classical Bernoulli numbers and polynomials, we have

$$B_n = (-1)^n B_n(1), \quad n \in \mathbb{N} \cup \{0\},$$

thus Corollary 1.10 is equivalent to Theorem 2.1 in [23].

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

For the simplification of notations, we denote by

$$F_{r,N}(t,x) = \left(\frac{t^N/N!}{e^t - T_{N-1}(t)}\right)^r e^{xt}, \tag{2.1}$$

$$F_{r,N}(t) = F_{r,N}(t,0) = \left(\frac{t^N/N!}{e^t - T_{N-1}(t)}\right)^r, \tag{2.2}$$

and

$$F_N(t) = F_{1,N}(t) = \frac{t^N/N!}{e^t - T_{N-1}(t)}. \tag{2.3}$$

To derive our main theorem, we need the following lemmas.

Lemma 2.1.

$$\frac{\mathrm{d}}{\mathrm{d}t}F_N(t) = \frac{N}{t}F_N(t) - F_N(t) - \frac{N}{t}F_{2,N}(t).$$

Proof. Differentiation of both sides of (2.3) with respect to t, yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_N(t) &= \frac{Nt^{N-1}/N!(e^t - T_{N-1}(t)) - t^N/N!(e^t - T_{N-2}(t))}{(e^t - T_{N-1}(t))^2} \\ &= \frac{N}{t}\frac{t^N/N!}{(e^t - T_{N-1}(t))} - \frac{t^N/N!(e^t - T_{N-1}(t) + t^{N-1}/(N-1)!)}{(e^t - T_{N-1}(t))^2} \\ &= \frac{N}{t}F_N(t) - F_N(t) - \frac{N}{t}F_{2,N}(t),\end{aligned}$$

the last equality follows from (2.2). □

Lemma 2.2.

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{r,N}(t,x) = \frac{rN}{t}F_{r,N}(t,x) + (x-r)F_{r,N}(t,x) - \frac{rN}{t}F_{r+1,N}(t,x). \tag{2.4}$$

Proof. Differentiation of both sides of (2.1) with respect to t, yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_{r,N}(t,x) &= rF_{r-1,N}(t,x)\frac{\mathrm{d}}{\mathrm{d}t}F_N(t) + xF_{r,N}(t,x) \\ &= rF_{r-1,N}(t,x)\left(\frac{N}{t}F_N(t) - F_N(t) - \frac{N}{t}F_{2,N}(t)\right) + xF_{r,N}(t,x) \\ &\quad \text{(Here we use Lemma 2.1)} \\ &= \frac{rN}{t}F_{r,N}(t,x) + (x-r)F_{r,N}(t,x) - \frac{rN}{t}F_{r+1,N}(t,x).\end{aligned}$$

□

By (1.20) and (2.1), we have

$$F_{r,N}(t,x)=\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x)\frac{t^n}{n!}. \tag{2.5}$$

Substituting the above equality in (2.4), then comparing the coefficients of the resulting equality, we have the following Lemma.

Lemma 2.3. *For any $n \geq 1$ we have*

$$B_{N,n}^{(r+1)}(x)=\frac{1}{N}\left(N-\frac{n}{r}\right)B_{N,n}^{(r)}(x)+\frac{1}{N}\frac{n}{r}(x-r)B_{N,n-1}^{(r)}(x).$$

Let

$$\binom{n}{i_1,\dots,i_r}=\frac{n!}{i_1!\cdots i_r!}$$

be the multinomial coefficient. We have the following equality.

Lemma 2.4.

$$B_{N,n}^{(r)}(x)=\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\geq 0}}\binom{n}{i_1,\dots,i_r}B_{N,i_1}(x_1)\cdots B_{N,i_r}(x_r), \tag{2.6}$$

where $x=x_1+\cdots+x_r$.

Proof. Letting $x=x_1+\cdots+x_r$ in (1.20), we get the following symbolic formula for higher order hypergeometric Bernoulli polynomials, we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x_1+\cdots+x_r)\frac{t^n}{n!}&=\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^r e^{(x_1+\cdots+x_r)t}\\&=e^{(B_N(x_1)+\cdots+B_N(x_r))t}\\&=\sum_{n=0}^{\infty}(B_N(x_1)+\cdots+B_N(x_r))^n\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Using multinomial formula, we expand the third identity in powers of $B_N(x)$, and each term $(B_N(x))^i$ is denoted by $B_{N,i}(x)$, then (2.7) becomes

$$\begin{aligned}\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x_1+\cdots+x_r)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}(B_N(x_1)+\cdots+B_N(x_r))^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\geq 0}}\binom{n}{i_1,\dots,i_r}B_{N,i_1}(x_1)\cdots B_{N,i_r}(x_r)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Finally by comparing the coefficients in the above equality, we get our result. □

Proof of the Theorem 1.2. We prove our result by induction.

First, this theorem is clearly true for $r=1$. Now assume that it is true for $r-1$. Then by Lemma 2.3, we have

$$\begin{aligned}
B_{N,n}^{(r)}(x) &= \frac{1}{N}\left(N-\frac{n}{r-1}\right)B_{N,n}^{(r-1)}(x)+\frac{1}{N}\frac{n}{r-1}(x-(r-1))B_{N,n-1}^{(r-1)}(x) \\
&= \frac{1}{N^{r-1}}\left(N-\frac{n}{r-1}\right)\sum_{i=0}^{r-2}A_{r-1}^{(N)}(i,x;1+N(r-2)-n) \\
&\quad\times(-1)^i\binom{n}{i}i!B_{N,n-i}(x) \\
&\quad+\frac{1}{N^{r-1}}\frac{n}{r-1}(x-(r-1))\sum_{i=0}^{r-2}A_{r-1}^{(N)}(i,x;1+N(r-2)-(n-1)) \\
&\quad\times(-1)^i\binom{n-1}{i}i!B_{N,n-1-i}(x) \\
&\quad\text{(Here we use induction)} \\
&= \frac{1}{N^{r-1}}\Bigg[\left(N-\frac{n}{r-1}\right)\sum_{i=0}^{r-2}A_{r-1}^{(N)}(i,x;1+N(r-2)-n) \\
&\quad\times(-1)^i\binom{n}{i}i!B_{N,n-i}(x) \\
&\quad+\frac{n}{r-1}(x-(r-1))\sum_{i=1}^{r-1}A_{r-1}^{(N)}(i-1,x;1+N(r-2)-(n-1)) \\
&\quad\times(-1)^{i-1}\binom{n-1}{i-1}(i-1)!B_{N,n-i}(x)\Bigg] \\
&= \frac{1}{N^{r-1}}\sum_{i=0}^{r-1}\Bigg[\left(N-\frac{n}{r-1}\right)A_{r-1}^{(N)}(i,x;1+N(r-2)-n) \\
&\quad\times(-1)^i\binom{n}{i}i! \\
&\quad+\frac{n}{r-1}(x-(r-1))A_{r-1}^{(N)}(i-1,x;1+N(r-2)-(n-1)) \\
&\quad\times(-1)^{i-1}\binom{n-1}{i-1}(i-1)!\Bigg]B_{N,n-i}(x) \\
&\quad\text{(Here we use } A_r^{(N)}(i,x;s) \text{ are zero for } i\le -1 \text{ and } i\ge r\text{)} \\
&= \frac{1}{N^{r-1}}\sum_{i=0}^{r-1}\Bigg[\frac{N(r-1)-n}{r-1}A_{r-1}^{(N)}(i,x;1+N(r-2)-n) \\
&\quad-\frac{1}{r-1}(x-(r-1))A_{r-1}^{(N)}(i-1,x;1+N(r-2)-(n-1))\Bigg] \\
&\quad\times(-1)^i\binom{n}{i}i!B_{N,n-i}(x) \\
&= \frac{1}{N^{r-1}}\sum_{i=0}^{r-1}A_r^{(N)}(i,x;1+N(r-1)-n)(-1)^i\binom{n}{i}i!B_{N,n-i}(x)
\end{aligned}$$

(Here we use (1.16) with $s=1+N(r-1)-n$).

Finally, by (2.6), we get our result. □

3. Differential equations

3.1. Appell polynomials. In order to obtain the differential equation satisfied by the higher order hypergeometric Bernoulli polynomials, in this subsection, we recall the definition and some basic facts on Appell polynomials which have been stated in [12].

The Appell polynomials [1] can be defined by the following generating function:

$$A(t)e^{xt}=\sum_{n=0}^{\infty}\frac{R_n(x)}{n!}t^n, \tag{3.1}$$

where

$$A(t)=\sum_{n=0}^{\infty}\frac{R_n}{n!}t^n, \quad A(0)\neq 0 \tag{3.2}$$

is an analytic function at $t=0$.

A polynomial $p_n(x)$ ($n\in\mathbb{N}, x\in\mathbb{C}$) is said to be a quasi-monomial [23] whenever two operators $\hat{M},\hat{P}$, called the multiplicative and derivative (or lowing) operators respectively, can be defined in such a way that

$$\hat{M}(p_n(x))=p_{n+1}(x), \tag{3.3}$$

and

$$\hat{P}(p_n(x))=np_{n-1}(x). \tag{3.4}$$

The operators $\hat{M}$ and $\hat{P}$ must satisfy the commutation relation

$$[\hat{P},\hat{M}]=\hat{P}\hat{M}-\hat{M}\hat{P}=\hat{1} \tag{3.5}$$

and thus display a Weyl group structure.

He and Ricci [12] showed that the multiplicative and derivative operators of $R_n(x)$ are

$$\hat{M}=(x+\alpha_0)+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}D_x^{n-k}, \tag{3.6}$$

$$\hat{P}=D_x, \tag{3.7}$$

where

$$\frac{A'(t)}{A(t)}=\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!}. \tag{3.8}$$

3.2. **Proofs of Theorems 1.7 and 1.9.** In this subsection, we shall prove Theorems 1.7 and 1.9, respectively.

Lemma 3.1.

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p}B_{N,n}^{(r)}(x)=\frac{n!}{(n-p)!}B_{N,n-p}^{(r)}(x),\quad n\geq p.$$

Proof. Obviously

$$\begin{aligned}\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^r\frac{\mathrm{d}^p}{\mathrm{d}x^p}e^{xt}&=\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^r(e^{xt}t^p)\\&=\left(\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^re^{xt}\right)t^p.\end{aligned}\tag{3.9}$$

Substituting (1.20) into the right hand side of (3.9), we have

$$\begin{aligned}\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^r\frac{\mathrm{d}^p}{\mathrm{d}x^p}e^{xt}&=\left(\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x)\frac{t^n}{n!}\right)t^p\\&=\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x)\frac{t^{n+p}}{n!}\\&=\sum_{n=p}^{\infty}B_{N,n-p}^{(r)}(x)\frac{t^n}{(n-p)!}\\&=\sum_{n=p}^{\infty}\frac{B_{N,n-p}^{(r)}(x)}{(n-p)!}t^n.\end{aligned}\tag{3.10}$$

It is obviously that

$$\frac{\mathrm{d}^p}{\mathrm{d}x^p}\sum_{n=0}^{\infty}B_{N,n}^{(r)}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{\mathrm{d}^p}{\mathrm{d}x^p}\frac{B_{N,n}^{(r)}(x)}{n!}t^n.\tag{3.11}$$

Thus differentiation of both sides of (1.20) with respect to x, from (3.10) and (3.11), we have

$$\sum_{n=p}^{\infty}\frac{B_{N,n-p}^{(r)}(x)}{(n-p)!}t^n=\sum_{n=0}^{\infty}\frac{\mathrm{d}^p}{\mathrm{d}x^p}\frac{B_{N,n}^{(r)}(x)}{n!}t^n.\tag{3.12}$$

Comparing the coefficients of the both sides of the above equality, we get our result. □

Proof of the Theorem 1.9. From (1.20), we know that the hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ are Appell polynomials with

$$A(t)=\left(\frac{t^N/N!}{e^t-T_{N-1}(t)}\right)^r.$$

And by using Lemma 2.2 with $x = 0$, we obtain

$$
\begin{aligned}
\frac{A'(t)}{A(t)} &= \frac{rN}{t} - r - \frac{rN}{t}\left(\frac{t^N/N!}{e^t - T_{N-1}(t)}\right)\\
&= \frac{rN}{t}\left(1 - \frac{t}{N} - \sum_{n=0}^{\infty} B_{N,n}\frac{t^N}{n!}\right)\\
&= \frac{rN}{t}\left(-\frac{1}{N(N+1)}t - \sum_{n=2}^{\infty} B_{N,n}\frac{t^N}{n!}\right) \qquad (3.13)\\
&\quad \text{(Here we use } B_{N,0} = 1 \text{ and } B_{N,1} = -1/(N+1))\\
&= r\left(-\frac{1}{N+1} - N\sum_{n=1}^{\infty}\frac{B_{N,n+1}}{n+1}\frac{t^N}{n!}\right).
\end{aligned}
$$

Therefore, by (3.6)–(3.8), we have the following multiplicative and derivative operators of the hypergeometric Bernoulli polynomials:

$$\hat{M} = \left(x - \frac{r}{N+1}\right) - rN\sum_{k=0}^{n-1}\frac{B_{N,n-k+1}}{(n-k+1)!}D_x^{n-k}, \tag{3.14}$$

$$\hat{P} = D_x. \tag{3.15}$$

By (3.14), we obtain

$$\hat{M}B_{N,n}^{(r)}(x) = \left[\left(x - \frac{r}{N+1}\right) - rN\sum_{k=0}^{n-1}\frac{B_{N,n-k+1}}{(n-k+1)!}D_x^{n-k}\right]B_{N,n}^{(r)}(x). \tag{3.16}$$

From (3.3) (here $p_n(x) = B_{N,n}^{(r)}(x)$), we have $\hat{M}B_{N,n}^{(r)}(x) = B_{N,n+1}^{(r)}(x)$, thus by Lemma 3.1, (3.16) implies

$$B_{N,n+1}^{(r)}(x) = \left(x - \frac{r}{N+1}\right)B_{N,n}^{(r)}(x) - rN\sum_{k=0}^{n-1}\frac{B_{N,n-k+1}}{(n-k+1)!}\frac{n!}{(n-(n-k))!}B_{N,k}^{(r)}(x).$$

Since

$$\frac{1}{(n-k+1)!}\frac{n!}{(n-(n-k))!} = \frac{1}{n-k+1}\frac{n!}{k!(n-k)!} = \frac{1}{n-k+1}\binom{n}{k},$$

we get our result. □

Proof of the Theorem 1.7. Replacing n by $n+1$ in Theorem 1.9, then multiplying $\frac{1}{rN}$ on both sides of the resulting equality, we obtain the following identity

$$
\begin{aligned}
&\frac{1}{rN}B_{N,n}^{(r)}(x) - \left(\frac{x}{rN} - \frac{1}{N(N+1)}\right)B_{N,n-1}^{(r)}(x)\\
&\qquad + \sum_{k=0}^{n-2}\binom{n-1}{k}\frac{B_{N,n-k}}{n-k}B_{N,k}^{(r)}(x) = 0.
\end{aligned} \tag{3.17}
$$

In Lemma 3.1, denote by $y = B_{N,n}^{(r)}(x)$, we have

$$B_{N,n-p}^{(r)}(x) = \frac{(n-p)!}{n!} y^{(p)}, \quad p = 1, \ldots, n. \tag{3.18}$$

In (3.18), letting $k = n - p$, then

$$B_{N,k}^{(r)}(x) = \frac{k!}{n!} y^{(n-k)}, \quad k = 0, \ldots, n. \tag{3.19}$$

Thus

$$\begin{aligned} &\frac{1}{rN} y - \left(\frac{x}{rN} - \frac{1}{N(N+1)} \right) \frac{1}{n} y' \\ &\quad + \sum_{k=0}^{n-2} \binom{n-1}{k} \frac{B_{N,n-k}}{n-k} \frac{k!}{n!} y^{(n-k)} = 0. \end{aligned} \tag{3.20}$$

This is equivalent to

$$\frac{1}{rN} y - \frac{1}{n} \left(\frac{x}{rN} - \frac{1}{N(N+1)} \right) y' + \frac{1}{n} \sum_{k=0}^{n-2} B_{N,n-k} \frac{1}{(n-k)!} y^{(n-k)} = 0, \tag{3.21}$$

and we have the desired result.

□

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