

# GENERALIZED BERNOULLI AND EULER POLYNOMIAL CONVOLUTION IDENTITIES

H. W. Gould

Department of Mathematics, West Virginia University, PO Box 6310,  
Morgantown, WV 26506-6310; Email: gould@math.wvu.edu

Our object is to prove some new convolution identities for the Bernoulli and Euler polynomials and numbers, and unify these with others that are not new but are not well known.

Nörlund [8, Chap. 6] defines generalized Bernoulli and Euler polynomials by the generating functions

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(z)}(x) = \frac{t^z e^{xt}}{(e^t - 1)^z}, \quad |t| < 2\pi, \quad (1)$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(z)}(x) = \frac{2^z e^{xt}}{(e^t + 1)^z}, \quad |t| < 2\pi, \quad (2)$$

These are actually special cases of more general functions he studies.

By the simple device of multiplying these generating functions together in different ways we may prove the following convolution identities

$$B_n^{(z+w)}(x+y) = \sum_{k=0}^n \binom{n}{k} B_k^{(z)}(x) B_{n-k}^{(w)}(y), \quad (3)$$

Nörlund [8, pp. 133, 139]

which follows by equating coefficients of  $t^n$  in the identity

$$\frac{t^{z+w} e^{(x+y)t}}{(e^t - 1)^{z+w}} = \frac{t^z e^{xt}}{(e^t - 1)^z} \frac{t^w e^{yt}}{(e^t - 1)^w} ,$$

$$E_n^{(z+w)}(x+y) = \sum_{k=0}^n \binom{n}{k} E_k^{(z)}(x) E_{n-k}^{(w)}(y) , \quad (4)$$

[Nörlund, p. 138] from the similar identity

$$\frac{2^{z+w} e^{(x+y)t}}{(e^t + 1)^{z+w}} = \frac{2^z e^{xt}}{(e^t + 1)^z} \frac{2^w e^{yt}}{(e^t + 1)^w}$$

and the new relations

$$2^n B_n^{(z)}(x+y) = \sum_{k=0}^n \binom{n}{k} B_k^{(z)}(x) E_{n-k}^{(z)}(y) , \quad (5)$$

from the identity

$$\frac{(2t)^z e^{(x+y)t}}{(e^{2t} - 1)^z} = \frac{t^z e^{xt}}{(e^t - 1)^z} \frac{2^z e^{yt}}{(e^t + 1)^z} ,$$

$$\sum_{k=0}^n \binom{n}{k} z^{n-k} B_k^{(2z)}(x-y) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k^{(z)}(x) B_{n-k}^{(z)}(y) , \quad (6)$$

from the identity

$$e^{tz} \frac{t^{2z} e^{(x-y)t}}{(e^t - 1)^{2z}} = \frac{t^z e^{xt}}{(e^t - 1)^z} \frac{(-t)^z e^{-yt}}{(e^{-t} - 1)^z} ,$$

and

$$\sum_{k=0}^n \binom{n}{k} E_k^{(2z)}(x-y) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_k^{(z)}(x) E_{n-k}^{(z)}(y) , \quad (7)$$

from the identity

$$e^{tz} \frac{2^{2z} e^{(x-y)t}}{(e^t + 1)^{2z}} = \frac{2^z e^{xt}}{(e^t + 1)^z} \frac{2^z e^{-yt}}{(e^{-t} + 1)^z} .$$

The familiar Bernoulli polynomials are given by  $B_n(x) = B_n^{(1)}(x)$ , and the ordinary Bernoulli numbers are given by  $B_n = B_n(0)$ . It is

well known that  $B_1 = -\frac{1}{2}$  and  $B_{2n+1} = 0$  for  $n = 1, 2, 3, \dots$ . Some

other values are  $B_0 = 1$ ,  $B_2 = \frac{1}{6}$ ,  $B_4 = -\frac{1}{30}$ ,  $B_6 = \frac{1}{42}$ ,  $B_8 = -\frac{1}{30}$ ,  $B_{10} = \frac{5}{66}$ ,  $B_{12} = -\frac{691}{2730}$ ,  $B_{14} = \frac{7}{6}$ .

N. B.: An explicit formula for the Bernoulli polynomials, discussed in [4] is

$$B_n(x) = \sum_{k=0}^n \frac{1}{k+1} \sum_{j=0}^k (-1)^j \binom{n}{k} (x+j)^n, \quad n \geq 0. \quad (8)$$

The familiar Euler numbers  $E_n$ , which occur in the Taylor series

$$\sec t = \sum_{n=0}^{\infty} \frac{t^n}{n!} i^n E_n = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} E_{2n},$$

are given by  $E_n = 2^n E_n^{(1)}(\frac{1}{2})$ . Here  $E_{2n+1} = 0$  for all  $n = 0, 1, 2, \dots$ , and a few other values are  $E_0 = 1$ ,  $E_2 = -1$ ,  $E_4 = 5$ ,  $E_6 = -61$ ,  $E_8 = 1385$ ,  $E_{10} = -50521$ ,  $E_{12} = 2702765$ ,  $E_{14} = -1993609891$ .

A corresponding formula for the Euler polynomials is

$$E_n(x) = \sum_{k=0}^n 2^{-k} \sum_{j=0}^k (-1)^j \binom{n}{k} (x+j)^n, \quad n \geq 0. \quad (9)$$

Nörlund [8, p. 145] also notes that by the simple device of

differentiating (1) and (2) with respect to  $t$ , we get the two difference equations (or three-term recurrence relations)

$$B_n^{(z+1)}(x) = \left(1 - \frac{n}{z}\right) B_n^{(z)}(x) + (x-z) \frac{n}{z} B_{n-1}^{(z)}(x) \quad (10)$$

and

$$E_n^{(z+1)}(x) = \frac{2}{z} E_n^{(z)}(x) - \frac{2}{z} (x-z) E_{n-1}^{(z)}(x) \quad (11)$$

Letting  $z = 1$  in (3), (4) and (5) gives us

$$B_n^{(2)}(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(y), \quad (12)$$

$$E_n^{(2)}(x+y) = \sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y), \quad (13)$$

and

$$2^n B_n^{(2)}(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) E_{n-k}(y) \quad (14)$$

but from (10) and (11) with  $z = 1$  we have the recurrences

$$B_n^{(2)}(x) = (1-n) B_n(x) + n(x-1) B_{n-1}(x) \quad (15)$$

and

$$E_n^{(2)}(x) = 2 E_n(x) - 2(x-1) E_{n-1}(x) \quad (16)$$

which we use to transform (12), (13) and (14) into convolution identities for the ordinary Bernoulli and Euler polynomials:

$$\sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(y) = (1-n)B_n(x+y) + n(x+y-1)B_{n-1}(x+y), \quad (17)$$

$$\sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) = 2E_n(x+y) - 2(x+y-1)E_{n-1}(x+y), \quad (18)$$

and

$$2^{-n} \sum_{k=0}^n \binom{n}{k} B_k(x) E_{n-k}(y) = (1-n)B_n(x+y) + n(x+y-1)B_{n-1}(x+y) \quad (19)$$

for the Bernoulli and Euler polynomials.

Formula (17) was the subject of a problem in the American Mathematical Monthly [9].

By setting  $x = y = 0$  in (17) we get the convolution recurrence

$$\sum_{k=0}^n \binom{n}{k} B_k B_{n-k} = (1-n)B_n - n B_{n-1} \quad (20)$$

for the ordinary Bernoulli numbers.

Somewhat more complicated formulas follow from (18) and (19). This is because if we set  $x = y = \frac{1}{2}$  in (18), we have to then find a way to express  $E_n(1)$  in the right hand side in terms of  $E_n$ .

And if we set  $x = 0$  and  $y = \frac{1}{2}$  in the left hand side of (19), we have

then to find a way to express  $B_n(\frac{1}{2})$  in the right hand side in terms of  $B_n$ . We have found some formulas for these cases. We return to these later. If we were working with numbers defined by  $E_n(0)$  then the results are very simple, just as in the case of (20), but the numbers  $E_n(0)$  are not as useful as the more well known "Euler" numbers  $E_n = 2^n E_n^{(1)}(\frac{1}{2}) = 2^n E_n^n(\frac{1}{2})$ .

The reader is cautioned not to confuse these standard "Euler" numbers with "Eulerian" numbers  $A_{k,n}$  which are given by

$$A_{k,n} = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k-j)^n, \quad 0 \leq k \leq n. \quad (21)$$

These have the three term recurrence relation

$$A_{k,n} = n A_{k-1,n} + (k-n+1) A_{k-1,n-1}. \quad (22)$$

With  $n$  as the row and  $k$  as the column numbers these form the array

1								
0	1							
0	1	1						
0	1	4	1					
0	1	11	11	1				
0	1	26	66	26	1			
0	1	57	302	302	57	1		
0	1	120	1191	2416	1191	120	1	
		...	...	...	...	...	...	...

The rows sum to give  $n!$ , i.e.

$$\sum_{k=0}^n A_{k,n} = n! .$$

The Eulerian numbers arise in many ways, for example,

$$x^n = \sum_{k=0}^n \binom{x+k-1}{n} A_{k,n} , \quad \text{for all complex } x, \quad (23)$$

and

$$(1-x)^{n+1} \sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^n A_{k,n} x^k , \quad \text{valid for } |x| < 1. \quad (24)$$

There is a large literature about these numbers. In Gould [3] these were studied as special cases of the more general Worpitzky-Nielsen numbers which may be defined by

$$B_{r,q}^n = \sum_{j=0}^r (-1)^j \binom{q}{j} (r-j)^n . \quad (25)$$

Then  $A_{k,n} = B_{k,n+1}^n$  ,

and, generalizing (23),

$$(-1)^{m+n} x^n = \sum_{k=0}^{m+1} \binom{x+k-1}{m} B_{k,m+1}^n , \quad (26)$$

valid for  $m \geq n \geq 0$  .

Carlitz [1] observed that Euler [2] introduced the array of numbers  $A_{k,n}$  as far back as 1755, which accounts for their being called Eulerian.

The numbers  $E_n(0)$  and Eulerian numbers are related, however, and there is the known interesting formula

$$E_n(0) = 2^{-n} \sum_{j=0}^n (-1)^j A_{j,n}, \quad (27)$$

which follows, of course, from (9) when  $x = 0$ .

Returning to the Bernoulli-Euler convolutions, we next note that when  $z = 1$  relation (6) yields

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k(x) B_{n-k}(y) &= \sum_{k=0}^n \binom{n}{k} B_k^{(2)}(x-y) \\ &= \sum_{k=0}^n \binom{n}{k} \{ (1-k)B_k(x-y) + k(x-y-1)B_{k-1}(x-y) \}, \end{aligned} \quad (28)$$

which may be considered the dual to (17). In the same way that (17) simplified when  $y = x$ , now (28) becomes

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k(x) B_{n-k}(x) &= \sum_{k=0}^n \{ (1-k)B_k - kB_{k-1} \} \\ &= \sum_{k=0}^n \binom{n}{k} B_k - \sum_{k=0}^n \binom{n}{k} kB_k - \sum_{k=0}^n \binom{n}{k} kB_{k-1} \\ &= \sum_{k=0}^n \binom{n}{k} B_k - \sum_{k=0}^n \binom{n}{k} kB_k - \sum_{k=0}^{n-1} \binom{n}{k+1} (k+1) B_{k-1} \\ &= \sum_{k=0}^n \binom{n}{k} B_k - \sum_{k=0}^n \left\{ \binom{n}{k} k + \binom{n}{k+1} (k+1) \right\} B_k \end{aligned}$$



$$\begin{aligned}
&= \sum_{k=0}^n \binom{n}{k} B_k - n \sum_{k=0}^n \binom{n}{k} B_k \\
&= (1-n) \sum_{k=0}^n \binom{n}{k} B_k = (-1)^{n-1} (n-1) B_n
\end{aligned}$$

because of the known linear recurrence

$$\sum_{k=0}^n \binom{n}{k} B_k = \begin{cases} (-1)^n B_n & \text{for all integers } n \geq 0 \\ B_n & \text{for } n \geq 0 \text{ \& } n \neq 1 \end{cases} \quad (29)$$

So we have the elegant formula

$$\sum_{k=0}^n (-1)^k \binom{n}{k} B_k(x) B_{n-k}(x) = (-1)^{n-1} (n-1) B_n, \quad (30)$$

which of course gives us the convolution recurrence

$$\sum_{k=0}^n (-1)^k \binom{n}{k} B_k B_{n-k} = (-1)^{n-1} (n-1) B_n \quad (31)$$

and this, in view of zero values of odd B's (except  $B_1$ ), reduces to

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1) B_{2n}, \quad n \geq 2. \quad (32)$$

Gould and Kerr [5, pp. 168–169] discuss some convolution recurrences for the Bernoulli numbers, and besides (28) the following two unusual forms are presented:

$$\begin{aligned}
&\sum_{k=1}^{n-1} \binom{2n}{2k} (2^{2k} - 1)(2^{2n-2k} - 1) B_{2k} B_{2n-2k} \\
&\quad = -(2n-1)(2n+2^{2n}) B_{2n}, \quad n \geq 2, \quad (33)
\end{aligned}$$

and

$$\sum_{k=1}^{n-1} \binom{2n}{2k} 2^{2k} B_{2k} B_{2n-2k} = -(2n + 2^{2n}) B_{2n}, \quad n \geq 2. \quad (34)$$

As is noted in [5], (34) follows from combining (32) and (33). And all recurrences for Bernoulli numbers that involve only even subscripts automatically give corresponding recurrences for Zeta functions of even argument since, of course

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n)!} B_{2n}, \quad n \geq 1. \quad (35)$$

The Bernoulli numbers satisfy another linear recurrence

$$B_n = - \sum_{k=0}^{n-1} \binom{n}{k} \frac{1}{n-k+1} B_k, \quad \text{for all } n \geq 1, \quad (36)$$

found by Gould and Alsardary [6] that appears to be new.

Relation (30) is interesting from a different point of view. The sum turned out to be independent of  $x$ , and this happens because the Bernoulli polynomials  $B_n(x)$  are binomial polynomials. A polynomial

$P_n(x)$  is called binomial if  $D_x P_n(x) = n P_{n-1}(x)$ . An old theorem

about such polynomials is that the function

$$S_n(x) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} P_k(x) P_{n-k}(x) \quad (37)$$

satisfies  $D_x S_n(x) = 0$  for all  $n = 0, 1, 2, \dots$  and so  $S_n(x)$  is

independent of the value of  $x$ .

Now the Euler polynomials  $E_n(x)$  are also binomial polynomials, so we know that the alternating convolution of these is independent of the value of  $x$ . But in fact from relation (7) with  $y = x$  we get

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_k(x) E_{n-k}(x) &= \sum_{k=0}^n \binom{n}{k} E_k(0) \\ &= \sum_{k=0}^n \binom{n}{k} 2^{-n} \sum_{j=0}^k (-1)^j A_{j,k}, \end{aligned} \quad (38)$$

by using relation (27) to express the  $E_n(0)$  in terms of Eulerian numbers. The formula does not appear to simplify much more than this.

Some further corollaries of our work may be indicated. Since

$$\sum_{0 \leq k \leq n} f(k) + \sum_{0 \leq k \leq n} (-1)^k f(k) = 2 \sum_{0 \leq k \leq n/2} f(2k)$$

with a similar formula for subtraction yielding a sum over odd indices, then formulas (3), (6), (4), and (7) yield convolution recurrences, where, in the convolution part, only even or only odd indices appear. For the ordinary Bernoulli polynomials we find then

$$\begin{aligned} 2 \sum_{0 \leq k \leq n/2} \binom{n}{2k} B_{2k}(x) B_{n-2k}(y) \\ = B_n^{(2)}(x+y) + \sum_{k=0}^n \binom{n}{k} B_k^{(2)}(x-y) \end{aligned} \quad (39)$$

and, of course, relation (15) may be used to rewrite the right hand side in terms of the ordinary Bernoulli polynomials, but it is not

especially elegant. Similarly, for the Euler polynomials,

$$\begin{aligned} 2 \sum_{0 \leq k \leq n/2} \binom{n}{2k} B_{2k}(x) B_{n-2k}(y) \\ = E_n^{(2)}(x+y) + \sum_{k=0}^n \binom{n}{k} E_k^{(2)}(x-y) . \end{aligned} \quad (40)$$

Relation (17) and similar formulas appear complicated because one has to use step-down formulas such as (15) and (16) to reduce polynomials of the form  $P_n^{(2)}(x)$  to ones of the form  $P_n^{(1)}(x)$  which may then be reduced to ordinary polynomials of some type. These step-down formulas necessarily exist in virtue of the fact that the basic convolutions (3) and (4) are addition theorems on the upper indices. In principle we may use iterations of the formulas to reduce any formula involving  $P_n^{(r)}(x)$  to one involving  $P_n^{(1)}(x)$ , but these iterated formulas are complicated and not expected to simplify far.

Formula (3) gives us

$$B_n^{(3)}(x+y) = \sum_{k=0}^n \binom{n}{k} B_k^{(2)}(x) B_{n-k}(y) , \quad (41)$$

and from this we can get

$$B_n^{(4)}(x+y) = \sum_{k=0}^n \binom{n}{k} B_{n-k}(y) \sum_{j=0}^k \binom{k}{j} B_k^{(2)}(x) B_{k-j}(y) , \quad (42)$$

and the process may be repeated continually, getting very

complicated iterated summation identities expressing  $B_n^{(r)}(x)$  in

terms of the ordinary Bernoulli polynomials.

Iteration of relations (10) and (11) affords another way to get  $B_n^{(r)}(x)$  in terms of the ordinary Bernoulli polynomials and  $E_n^{(r)}(x)$  in terms of ordinary Euler polynomials. Nörlund [8, p.148] does just this and obtains such formulas which we will not take space to write out here. Nörlund's application of such formulas is to devise useful numerical quadrature formulas for work with finite differences.

We wish finally to return to examine the possible cases of our formulas that might lead to special cases of any interest.

From Nörlund [9] we have the following facts:

$$B_n(1-x) = (-1)^n B_n(x) \text{ , so that } B_n(1) = (-1)^n B_n \text{ ;} \quad (43)$$

$$E_n(1-x) = (-1)^n E_n(x) \text{ , so that } E_n(1) = (-1)^n E_n \text{ ;} \quad (44)$$

Multiplication theorem:

$$B_n(rx) = r^{n-1} \sum_{k=0}^{r-1} B_n\left(x + \frac{k}{r}\right) \text{ , for } r \geq 1, n \geq 0. \quad (45)$$

which gives us

$$B_n\left(\frac{1}{2}\right) = -\left(1 - \frac{1}{2^{n-1}}\right) B_n \text{ .} \quad (46)$$

A similar multiplication theorem is the source of the relation

$$E_n\left(\frac{1}{2}\right) = E_n^{(1)}\left(\frac{1}{2}\right) = 2^{-n} E_n \text{ .} \quad (47)$$

The formulas (43), (44), (46) and (48) suggest that we

investigate the special formulas arising from relations (17), (18), (19), (30) and (38) more closely. The following cases are suggested as interesting:

Relation (17) with  $x = 0, y = 0$  yields (20).

Relation (17) with  $x = 0, y = 1$  yields (31).

Relation (17) with  $x = 0, y = 1/2$  yields

$$\sum_{k=0}^n \binom{n}{k} B_k B_{n-k}(1/2) = (1-n)B_n(1/2) - (n/2)B_{n-1}(1/2),$$

whence

$$\sum_{k=0}^n \binom{n}{k} (1-2^{1-k}) B_k B_{n-k} = (1-n)B_n - \frac{n}{2} (1-2^{2-n}) B_{n-1}, \quad (48)$$

Relation (17) with  $x = 1/2, y = 1/2$  yields

$$\sum_{k=0}^n \binom{n}{k} B_k(1/2) B_{n-k}(1/2) = (1-n)B_n(1),$$

whence

$$\sum_{k=0}^n \binom{n}{k} (2^{k-1} - 1)(2^{n-k-1} - 1) B_k B_{n-k} = (-1)^{n-1} 2^n (n-1) B_n. \quad (49)$$

Relation (18) with  $x = 0, y = 0$  yields

$$\sum_{k=0}^n \binom{n}{k} E_k(0) E_{n-k}(0) = 2 E_n(0) + 2 E_{n-1}(0). \quad (50)$$

Relation (18) with  $x = 0, y = 1/2$  yields

$$\sum_{k=0}^n \binom{n}{k} E_k(0) E_{n-k}(1/2) = 2 E_n(1/2) + 2 E_{n-1}(1/2) ,$$

whence

$$\sum_{k=0}^n \binom{n}{k} 2^k E_k(0) E_{n-k} = 2 E_n + 2 E_{n-1} . \quad (51)$$

Relation (18) with  $x = 1/2, y = 1/2$  yields

$$\sum_{k=0}^n \binom{n}{k} E_k(1/2) E_{n-k}(1/2) = 2 E_n(1) = 2 (-1)^n E_n(0) = 0, \text{ for odd } n.$$

whence

$$\sum_{k=0}^n \binom{n}{k} E_k E_{n-k} = \sum_{j=0}^n (-1)^{n-j} A_{j,n} = 0 , \text{ for all odd } n. \quad (52)$$

Relation (19) with  $x = 0, y = 1/2$  yields

$$2^{-n} \sum_{k=0}^n \binom{n}{k} B_k E_{n-k}(1/2) = (1-n)B_n(1/2) - \frac{n}{2} B_{n-1}(1/2) ,$$

whence

$$2^{-2n} \sum_{k=0}^n \binom{n}{k} B_k 2^k E_{n-k} = (n-1)(1-2^{1-n})B_n + \frac{n}{2}(1-2^{2-n})B_{n-1} .(53)$$

Relation (19) with  $x = 1/2, y = 0$  yields

$$2^{-n} \sum_{k=0}^n \binom{n}{k} B_k(1/2) E_{n-k}(0) = (1-n)B_n(1/2) - \frac{n}{2} B_{n-1}(1/2)$$

whence

$$\begin{aligned}
2^{-n} \sum_{k=0}^n \binom{n}{k} (1 - 2^{1-k}) B_k E_{n-k}^{(0)} \\
= (1-n)(1 - 2^{-n}) B_n - \frac{n}{2} (1 - 2^{1-n}) B_{n-1} \quad (54)
\end{aligned}$$

Relation (19) with  $x = 1/2$ ,  $y = 1/2$  yields

$$2^{-n} \sum_{k=0}^n \binom{n}{k} B_k(1/2) E_{n-k}(1/2) = (1-n) B_n(1) ,$$

whence

$$2^{-2n} \sum_{k=0}^n \binom{n}{k} (1 - 2^{1-k}) B_k E_{n-k} = (-1)^n (n-1) B_n . \quad (55)$$

Relation (30) with  $x = 0$  gave us (31).

Relation (30) with  $x = 1$  yields (31) again.

Relation (38) with  $x = 1/2$  yields

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_k(1/2) E_{n-k}(1/2) = \sum_{k=0}^n \binom{n}{k} E_k(0) ,$$

whence

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_k E_{n-k} = 2^n \sum_{k=0}^n \binom{n}{k} E_k(0) , \quad (56)$$

Relation (38) with  $x = 1$  yields

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_k(1) E_{n-k}(1) = \sum_{k=0}^n \binom{n}{k} E_k(0) ,$$

whence



$$\sum_{k=0}^n (-1)^k \binom{n}{k} E_k(0) E_{n-k}(0) = \sum_{k=0}^n \binom{n}{k} E_k(0) . \quad (57)$$

Remarks. The simple formula (8) for Bernoulli polynomials (which gives a corresponding simple formula for Bernoulli numbers when  $x = 0$ ) has never been widely known. For some remarks about this, see my review [12] of a book on series by Stanaitis. Also, G. T. Williams [13] did not know about this formula. Williams obtained formula (32) by manipulations with the generating function for Bernoulli numbers.

#### REFERENCES

1. L. Carlitz, Note on a paper of Shanks, *Amer. Math. Monthly*, 69(1952), 239–241.
2. L. Euler, *Institutiones calculi differentialis*, Petrograd, 1755.
3. H. W. Gould, The Stirling numbers and generalized difference expansions, Master's Thesis, University of Virginia, 1956.
4. H. W. Gould, Explicit formulas for Bernoulli numbers, *Amer. Math. Monthly*, 79(1972), 44–51.
5. H. W. Gould and W. Y. Kerr, Characterization of polynomial systems satisfying generalized convolution differential–difference equations, *Journal of Mathematical Research and Exposition (Dalian, China)*, 9(1989), 159–172.

6. H. W. Gould and Salar Y. Alsardary, Special function inverse series pairs, *Kyungpook Math. Journal*, to appear.
7. Niels Nielsen, *Traité élémentaire de nombres de Bernoulli*, Paris, 1923.
8. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, *Grundlehren der Mathematischen Wissenschaften*, Bd. 13, Springer Verlag, Berlin, 1924. Reprinted by Chelsea Publ. Co., N.Y., 1954.
9. Problem 10416, Posed by Kwang-Wu Chen, *Amer. Math. Monthly*, 101(1994), p. 912.
10. John Riordan, *An Introduction to Combinatorial Analysis*, Wiley, N.Y., 1958.
11. Ludwig Saalschütz, *Vorlesungen über die Bernoulli'schen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen*, Springer, Berlin, 1893. Available since 1964 in Xeroographed form from Univesity Microfilms, Ann Arbor, Michigan, Order No. OP-17136.
12. O. E. Stanaitis, *An Introduction to Sequences, Series, and Improper Integrals*, Holden-Day, San Francisco, 1967. Reviewed by H. W. Gould, *Amer. Math. Monthly*, 76(1969), 210-211.
13. G. T. Williams, A new method of evaluating  $\zeta(2n)$ , *Amer. Math. Monthly*, 60(1953), 19-25.

-----

AMS Classification Numbers: 11B68, 05A19,

Keywords: Bernoulli polynomials, Euler polynomials, Zeta function.

Revised: December 1994 and 19 April 2010