

**A FURTHER INVESTIGATION OF GENERATING
FUNCTIONS RELATED TO PAIRS OF INVERSE
FUNCTIONS WITH APPLICATIONS TO GENERALIZED
DEGENERATE BERNOULLI POLYNOMIALS**

SEBASTIEN GABOURY AND RICHARD TREMBLAY

ABSTRACT. In this paper, we obtain new generating functions involving families of pairs of inverse functions by using a generalization of the Srivastava's theorem [H. M. Srivastava, Some generalizations of Carlitz's theorem, *Pacific J. Math.* 85 (1979), 471–477] obtained by Tremblay and Fugère [Generating functions related to pairs of inverse functions, *Transform methods and special functions*, Varna '96, Bulgarian Acad. Sci., Sofia (1998), 484–495]. Special cases are given. These can be seen as generalizations of the generalized Bernoulli polynomials and the generalized degenerate Bernoulli polynomials.

1. Introduction

In 1977, motivated by the work of Srivastava and Singhal [27] on the Jacobi polynomials, Carlitz obtained the following generating function for the Laguerre polynomials [2, p. 525, Eq.(5.5)]:

$$(1.1) \quad \sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x+ny)t^n = \frac{(1+\omega)^{\alpha+1} e^{-x\omega}}{1-\lambda\omega + \omega(1+\omega)y},$$

where α, λ are arbitrary complex numbers and ω is a function of t defined by

$$(1.2) \quad \omega = t(1+\omega)^{\lambda+1} e^{-y\omega}$$

with $\omega(0) = 0$.

In the same paper, Carlitz [2, p. 521, Theorem 1 and Eq.(2.10)] extended these results to the forms

$$\sum_{n=0}^{\infty} c_n^{(\alpha+\lambda n)} \frac{t^n}{n!} \quad \text{and} \quad \sum_{n=0}^{\infty} d_n^{(\alpha+\lambda n, \beta+\mu n)} \frac{t^n}{n!}$$

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respectively, where $c_n^{(\alpha+\lambda n)}$ and $d_n^{(\alpha+\lambda n, \beta+\mu n)}$ are general one- and two-parameter coefficients. Srivastava [23] proposed a generalization of Carlitz's theorem for the sequence of functions $\left\{f_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ and extended the result to the multivariable and multiparameter sequence of functions

$$\left\{g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_s)\right\}_{n=0}^{\infty}.$$

Explicitly, Srivastava obtained the following generating function [23, p. 472, Eq.(1.7)]:

$$(1.3) \quad A(z) \prod_{i=1}^r \{[B_i(z)]^{\alpha_i}\} \exp\left(\sum_{j=1}^s x_j C_j(z)\right) = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_s) \frac{z^n}{n!}$$

implies, with suitable conditions on variables and parameters, the following multivariable and multiparameter generating function [23, p. 472, Eq.(1.11)]:

$$(1.4) \quad \sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n, \dots, \alpha_r+\lambda_r n)}(x_1 + ny_1, \dots, x_s + ny_s) \frac{t^n}{n!} = \frac{A(\zeta) \prod_{i=1}^r \{[B_i(\zeta)]^{\alpha_i}\} \exp\left(\sum_{j=1}^s x_j C_j(\zeta)\right)}{\Lambda[\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; \zeta]},$$

where

$$(1.5) \quad \Lambda[\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; \zeta] = \left(1 - \zeta \left[\sum_{i=1}^r \lambda_i \left(\frac{B_i'(\zeta)}{B_i(\zeta)}\right) + \sum_{j=1}^s y_j C_j'(\zeta)\right]\right)$$

and

$$(1.6) \quad \zeta = t \prod_{i=1}^r \{[B_i(\zeta)]^{\lambda_i}\} \exp\left(\sum_{j=1}^s y_j C_j(\zeta)\right).$$

A large number of interesting papers treating many special cases of Srivastava's theorem exist in the literature [4, 24, 25, 30].

The proofs of these results are essentially based on the Lagrange's expansion theorem in the form [22, p. 146, Problem 207]:

$$(1.7) \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} D_z^n \{f(z) [\phi(z)]^n\}_{z=0} = \frac{f(\zeta)}{1 - t\phi'(\zeta)},$$

where the functions $f(z)$ and $\phi(z)$ are analytic about the origin, and ζ is given by

$$(1.8) \quad \zeta = t\phi(\zeta) \quad \text{and} \quad \phi(0) \neq 0.$$

Recently, Tremblay and Fugère [33] gave a further generalization of Srivastava's theorem by using a result due to Osler involving the fractional derivatives

[18, p. 290, Eq.(3.1)]:

$$(1.9) \quad D_{g(z)}^\alpha f(z) = D_{h(z)}^\alpha \left\{ \frac{f(z)g'(z)}{h'(z)} \left(\frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1} \right\} \Big|_{w=z}.$$

In particular, they considered the specific form:

$$(1.10) \quad D_{zG(z)}^n f(z) \Big|_{z=0} = D_{zH(z)}^n \left\{ f(z) \left(\frac{G(z) + zG'(z)}{H(z) + zH'(z)} \right) \left(\frac{H(z)}{G(z)} \right)^{n+1} \right\} \Big|_{z=0}.$$

They also gave many special cases. For further details on fractional derivatives, the interested reader should read [8, 9, 19, 20, 21, 34, 35, 36].

The aim of this paper is to further investigate the generalization of Srivastava’s theorem given by Tremblay and Fugère related to pairs of inverse functions. In Section 2, we recall the theorems recently obtained by Tremblay and Fugère. Section 3 is devoted to the obtention of new generating functions for the generalized degenerate Bernoulli polynomials introduced by Carlitz [3]. Finally, in Section 4, we introduce families of pairs of inverse functions by using a theorem of Donaghey [5], and we give some special cases of generating functions involving these pairs of inverse functions which can be seen as generalizations of the generalized Bernoulli polynomials and generalized degenerate Bernoulli polynomials.

2. Main theorems

In this section, we recall the generalization of Srivastava’s theorem as well as others theorems obtained by Tremblay and Fugère [33, Theorems 3.1 and 3.2].

Theorem 2.1. *Let the sequence of functions*

$$\left\{ h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_s; y_1, \dots, y_s; \lambda_1, \dots, \lambda_r | \sigma) \right\}_{n=0}^\infty$$

be defined by means of the generating function

$$(2.1) \quad \frac{A(z) \prod_{i=1}^r \{ [B_i(z)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(z) \right)}{\Lambda^\sigma [\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; z]} = \sum_{n=0}^\infty h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_s; y_1, \dots, y_s; \lambda_1, \dots, \lambda_r | \sigma) \frac{z^n}{n!}$$

with

$$(2.2) \quad \Lambda[\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; z] = \left(1 - z \left\{ \sum_{i=1}^r \lambda_i \left[\frac{B'_i(z)}{B_i(z)} \right] + \sum_{j=1}^s y_j C'_j(z) \right\} \right)$$

where the parameters $\sigma, \alpha_i, \lambda_i$ ($1 \leq i \leq r$) and x_j, y_j ($1 \leq j \leq s$) are arbitrary complex numbers independent of z (with suitable conditions on variables and

parameters). The functions $A(z)$, $B_i(z)$, $z^{-1}C_j(z)$ are analytic in the neighborhood of the origin, and assume that

$$(2.3) \quad A(0) = B_i(0) = C'_j(0) = 1; \quad i = 1, \dots, r; \quad j = 1, \dots, s.$$

Then we have

$$(2.4) \quad \sum_{n=0}^{\infty} h_n^{(\alpha_1+\theta_1n, \dots, \alpha_r+\theta_rn)}(x_1 + nw_1, \dots, x_s + nw_s; y_1, \dots, y_s; \lambda_1, \dots, \lambda_r | \sigma + \delta n) \frac{t^n}{n!} \\ = \frac{A(\zeta) \prod_{i=1}^r \{ [B_i(\zeta)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(\zeta) \right)}{\Lambda^\sigma [\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; \zeta] \Omega [\lambda_1, \dots, \lambda_r; \theta_1, \dots, \theta_r; y_1, \dots, y_s; w_1, \dots, w_s; \delta; \zeta]},$$

where

$$(2.5) \quad \Omega [\lambda_1, \dots, \lambda_r; \theta_1, \dots, \theta_r; y_1, \dots, y_s; w_1, \dots, w_s; \delta; \zeta] \\ = \Lambda [\theta_1, \dots, \theta_r; w_1, \dots, w_s; \zeta] - \delta \zeta \frac{d}{d\zeta} \ln(\Lambda [\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; \zeta])$$

and

$$(2.6) \quad \zeta = t \prod_{i=1}^r \{ [B_i(\zeta)]^{\theta_i} \} \exp \left(\sum_{j=1}^s w_j C_j(\zeta) \right) \Lambda^{-\delta} [\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; \zeta]$$

with all parameters δ , θ_i ($1 \leq i \leq r$) and w_j ($1 \leq j \leq s$) are independant of ζ .

Remark 2.2. If $\delta = 0$, $\theta_i = \lambda_i$ ($1 \leq i \leq r$) and $w_j = y_j$ ($1 \leq j \leq s$), the generating function (2.4) becomes

$$(2.7) \quad \sum_{n=0}^{\infty} h_n^{(\alpha_1+\lambda_1n, \dots, \alpha_r+\lambda_rn)}(x_1 + ny_1, \dots, x_s + ny_s; y_1, \dots, y_s; \lambda_1, \dots, \lambda_r | \sigma) \frac{t^n}{n!} \\ = \frac{A(\zeta) \prod_{i=1}^r \{ [B_i(\zeta)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(\zeta) \right)}{\Lambda^{\sigma+1} [\lambda_1, \dots, \lambda_r; y_1, \dots, y_s; \zeta]},$$

where $\zeta = t \prod_{i=1}^r \{ [B_i(\zeta)]^{\lambda_i} \} \exp \left(\sum_{j=1}^s y_j C_j(\zeta) \right)$.

In addition, if $\sigma = 0$, all variables y_j and parameters λ_i become irrelevant in (2.1) and (2.7). The sequence of functions

$$\left\{ h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_s; y_1, \dots, y_s; \lambda_1, \dots, \lambda_r | \sigma) \right\}_{n=0}^{\infty}$$

can be simply identified as $\left\{ g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_s) \right\}_{n=0}^{\infty}$. The generating function (2.7) would evidently reduce to Srivastava's theorem (1.4).

Tremblay and Fugère [33, p. 490, Eq.(3.10)] gave another interesting special form of Theorem 2.1. They obtained the following theorem.

Theorem 2.3. *If the sequence of functions $\left\{ k_n^{(\alpha_1, \dots, \alpha_r, \beta, \gamma)}(x_1, \dots, x_s | \sigma) \right\}_{n=0}^\infty$ is generated by a function of the form*

$$\begin{aligned}
 & A(z) \prod_{i=1}^r \{ [B_i(z)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(z) \right) \left(\frac{E(z)}{z} \right)^{\beta+\sigma} (E'(z))^{\gamma-\sigma} \\
 (2.8) \quad & = \sum_{n=0}^\infty k_n^{(\alpha_1, \dots, \alpha_r, \beta, \gamma)}(x_1, \dots, x_s | \sigma) \frac{z^n}{n!},
 \end{aligned}$$

where the parameters α_i ($1 \leq i \leq r$), β , γ and x_j ($1 \leq j \leq s$) are independent of z , the functions $A(z)$, $B_i(z)$, $z^{-1}C_j(z)$ and $E(z)$ are analytic at the origin and such that

$$A(0) = B_i(0) = C'_j(0) = 1; \quad i = 1, \dots, r; \quad j = 1, \dots, s$$

and

$$\lim_{z \rightarrow 0} \frac{E(z)}{z} = 1,$$

then the following generating function holds

$$\begin{aligned}
 (2.9) \quad & \sum_{n=0}^\infty k_n^{(\alpha_1 + \theta_1 n, \dots, \alpha_r + \theta_r n, \beta + \phi n, \gamma + \kappa n)}(x_1 + n\omega_1, \dots, x_s + n\omega_s | \sigma + \delta n) \frac{t^n}{n!} \\
 & = \frac{A(\zeta) \prod_{i=1}^r \{ [B_i(\zeta)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(\zeta) \right) \left(\frac{E(\zeta)}{\zeta} \right)^{\beta+\sigma} (E'(\zeta))^{\gamma-\sigma}}{\Lambda[\theta_1, \dots, \theta_r; \omega_1, \dots, \omega_s; \zeta] + (\phi + \delta) - \zeta(\phi + \delta) \frac{E'(\zeta)}{E(\zeta)} - \zeta(\kappa - \delta) \frac{E''(\zeta)}{E'(\zeta)}},
 \end{aligned}$$

where

$$(2.10) \quad \zeta = t \prod_{i=1}^r \{ [B_i(\zeta)]^{\theta_i} \} \exp \left(\sum_{j=1}^s w_j C_j(\zeta) \right) \left(\frac{E(\zeta)}{\zeta} \right)^{\phi+\delta} (E'(\zeta))^{\kappa-\delta}$$

and

$$(2.11) \quad \Lambda[\theta_1, \dots, \theta_r; \omega_1, \dots, \omega_s; \zeta] = \left(1 - \zeta \left\{ \sum_{i=1}^r \theta_i \left[\frac{B'_i(\zeta)}{B_i(\zeta)} \right] + \sum_{j=1}^s \omega_j C'_j(\zeta) \right\} \right).$$

Finally, another important theorem for the sequel, given also in [33, Theorem 3.2, p. 491], involves generating functions of pair of inverse functions. Specifically, we have the following result.

Theorem 2.4. *If the sequence of functions $\left\{ k_n^{(\alpha_1, \dots, \alpha_r, \beta, \gamma)}(x_1, \dots, x_s) \right\}_{n=0}^\infty$ is generated by a function of the form*

$$\begin{aligned}
 & A(z) \prod_{i=1}^r \{ [B_i(z)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(z) \right) \left(\frac{E(z)}{z} \right)^\beta (E'(z))^\gamma \\
 (2.12) \quad &= \sum_{n=0}^\infty k_n^{(\alpha_1, \dots, \alpha_r, \beta, \gamma)}(x_1, \dots, x_s) \frac{z^n}{n!},
 \end{aligned}$$

where the parameters α_i ($1 \leq i \leq r$), β , γ and x_j ($1 \leq j \leq s$) are independent of z , the functions $A(z)$, $B_i(z)$, $z^{-1}C_j(z)$ and $E(z)$ are analytic at the origin and such that

$$A(0) = B_i(0) = C'_j(0) = 1; \quad i = 1, \dots, r; \quad j = 1, \dots, s$$

and

$$\lim_{z \rightarrow 0} \frac{E(z)}{z} = 1,$$

then we have the following generating function

$$\begin{aligned}
 & A(E^{-1}(z)) \prod_{i=1}^r \{ [B_i(E^{-1}(z))]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(E^{-1}(z)) \right) \\
 & \times \left(\frac{E^{-1}(z)}{z} \right)^{-\beta} \left(\frac{d}{dz} E^{-1}(z) \right)^{1-\gamma} \\
 (2.13) \quad &= \sum_{n=0}^\infty k_n^{(\alpha_1, \dots, \alpha_r, \beta-n-1, \gamma)}(x_1, \dots, x_s) \frac{z^n}{n!}
 \end{aligned}$$

and we also have that

$$\begin{aligned}
 & \sum_{n=0}^\infty k_n^{(\alpha_1 + \lambda_1 n, \dots, \alpha_r + \lambda_r n, \beta - 1 - n + \phi n, \gamma + \kappa n)}(x_1 + ny_1, \dots, x_s + ny_s) \frac{t^n}{n!} \\
 &= \frac{A(\Psi) \left(\frac{d}{d\zeta} \Psi \right)^{1-\gamma} \left(\frac{\Psi}{\zeta} \right)^{-\beta} \prod_{i=1}^r \{ [B_i(\Psi)]^{\alpha_i} \} \exp \left(\sum_{j=1}^s x_j C_j(\Psi) \right)}{1 - \phi + \phi \zeta \frac{d}{d\zeta} \Psi + \kappa \zeta \frac{d^2}{d\zeta^2} \Psi - \zeta \frac{d}{d\zeta} \Psi \left[\sum_{i=1}^r \lambda_i \left[\frac{B'_i(\Psi)}{B_i(\Psi)} \right] + \sum_{j=1}^s y_j C'_j(\Psi) \right]},
 \end{aligned}$$

where

$$\zeta = t \left(\frac{\Psi}{\zeta} \right)^{-\phi} \left(\frac{d}{d\zeta} \Psi \right)^{-\kappa} \prod_{i=1}^r \{ [B_i(\Psi)]^{\lambda_i} \} \exp \left(\sum_{j=1}^s y_j C_j(\Psi) \right),$$

$\Psi = E^{-1}(\zeta)$ and $E^{-1}(z)$ denotes the inverse function of $E(z)$.

3. Applications to generalized degenerate Bernoulli polynomials

The generalized degenerate Bernoulli polynomials

$$\beta_n^{(\alpha)}(\lambda, \mu; x)$$

which contain, as special cases, the Nörlund polynomials [17] and the generalized Bernoulli polynomials [6, 10], have been introduced by Carlitz [3]. These polynomials are defined by the following generating function:

$$(3.1) \quad \left(\frac{z}{(1 + \lambda z)^\mu - 1} \right)^\alpha (1 + \lambda z)^{\mu x} = \sum_{n=0}^{\infty} \beta_n^{(\alpha)}(\lambda, \mu; x) \frac{z^n}{n!},$$

where $\lambda\mu = 1$ and $\alpha \in \mathbb{C}$.

By taking the limit as $\lambda \rightarrow 0$ in (3.1), we have

$$(3.2) \quad \begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \left\{ \beta_n^{(\alpha)} \left(\lambda, \frac{1}{\lambda}; x \right) \right\} \frac{z^n}{n!} &= \lim_{\lambda \rightarrow 0} \left\{ \left(\frac{z}{(1 + \lambda z)^{\frac{1}{\lambda}} - 1} \right)^\alpha (1 + \lambda z)^{\frac{x}{\lambda}} \right\} \\ &= \left(\frac{z}{e^z - 1} \right)^\alpha e^{xz} \\ &= \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!}, \end{aligned}$$

where $B_n^{(\alpha)}(x)$ denotes the well-known and largely investigated generalized Bernoulli polynomials [7, 11, 12, 13, 14, 15, 26, 28, 29, 31, 37].

The generalized degenerate Bernoulli polynomials satisfy the next properties [3]:

$$(3.3) \quad \beta_n^{(\alpha)}(\lambda, \mu, x) = (-1)^n \beta_n^{(\alpha)}(-\lambda, -\mu, \alpha - x),$$

$$(3.4) \quad \beta_n^{(a+b)}(\lambda, \mu, x + y) = \sum_{k=0}^n \beta_{n-k}^{(a)}(\lambda, \mu, x) \beta_k^{(b)}(\lambda, \mu, y),$$

$$(3.5) \quad \beta_n^{(a+b)}(\lambda, \mu, x + y) = \sum_{k=0}^n \binom{n}{k} \lambda^{-2b-m+k} \beta_{n-k}^{(-b)}(\lambda, 1/\mu, \mu x) \beta_k^{(a-m+k)}(\lambda, \mu, y).$$

and

$$(3.6) \quad \beta_n^{(\alpha)}(\lambda, \mu, x + 1) - \beta_n^{(\alpha)}(\lambda, \mu, x) = n \beta_{n-1}^{(\alpha-1)}(\lambda, \mu, x).$$

Another interesting property for the degenerate polynomials $\beta_n^{(1)}(\lambda, \mu; 0)$ that has been proved by Carlitz in [1] is the analog of the Staudt-Clausen theorem [16, Chap. 13]. Especially, Carlitz obtained the following result:

Let $\lambda = a/b$ with a, b relatively primes integers. Then for n even

$$(3.7) \quad \beta_n^{(1)}(\lambda, \mu; 0) = A_n - \sum_{\substack{p-1|n \\ p|a}} \frac{1}{p},$$

where A_n is a rational number whose denominator contains only primes occurring in b . For n odd, $\beta_1^{(1)}(\lambda, \mu; 0) = \frac{\lambda-1}{2}$ and

$$(3.8) \quad \beta_n^{(1)}(\lambda, \mu; 0) = A_n - \frac{1}{2} \quad (n > 1)$$

provided $2 \mid a, 4 \nmid a$; if $2 \nmid a$ or $4 \mid a$, then $\beta_n^{(1)}(\lambda, \mu; 0) = A_n$. In particular when λ is a integer then A_n is also an integer.

The following theorems are obtained by suitably applying Theorem 2.3 and Theorem 2.4.

Theorem 3.1. *The following generating function holds for the generalized degenerate Bernoulli polynomials $\beta_n^{(\alpha)}(\lambda, \mu, x)$:*

$$(3.9) \quad \frac{\left(\frac{\zeta}{(1+\lambda\zeta)^\mu - 1}\right)^\alpha (1 + \lambda\zeta)^{\mu x}}{1 - \tau + \frac{\tau\zeta(1+\lambda\zeta)^{\mu-1}}{(1+\lambda\zeta)^\mu - 1} - \frac{\zeta y}{1+\lambda\zeta}} = \sum_{n=0}^{\infty} \beta_n^{(\alpha+\tau n)}(\lambda, \mu, x + ny) \frac{t^n}{n!},$$

where

$$\zeta = t \left(\frac{\zeta}{(1 + \lambda\zeta)^\mu - 1} \right)^\tau (1 + \lambda z)^{\mu y}.$$

Proof. Setting $A(z) = B_i(z) = 1, x_j = \omega_j = 0 (1 \leq j \leq s), \sigma = \delta = 0, \theta_i = 0 (1 \leq i \leq r)$ and $E(z) = \frac{(1+\lambda z)^\mu - 1}{\mu\lambda}$, we thus have

$$E'(z) = (1 + \lambda z)^{\mu-1} \quad \text{and} \quad E''(z) = (\mu - 1)\lambda(1 + \lambda z)^{\mu-2}.$$

Finally, by making the appropriate substitutions in conjunction with Theorem 2.3, the result follows easily. \square

Theorem 3.2. *The following generating functions hold for the generalized degenerate Bernoulli polynomials $\beta_n^{(\alpha)}(\lambda, \mu, x)$:*

$$(3.10) \quad \left(\frac{(1+z)^{\frac{1}{\mu}} - 1}{\lambda z} \right)^\alpha (1+z)^{x-1+\frac{1}{\mu}} = \sum_{n=0}^{\infty} \beta_n^{(1+\alpha+n)}(\lambda, \mu, x) \frac{z^n}{n!}$$

and

$$(3.11) \quad \frac{\left(\frac{(1+\zeta)^{\frac{1}{\mu}} - 1}{\lambda\zeta}\right)^\alpha (1 + \zeta)^{x-1+\frac{1}{\mu}}}{1 + \varphi - \varphi\zeta^2\lambda \left(\frac{(1+\zeta)^{\frac{1}{\mu}} - 1}{(1+\zeta)^{\frac{1}{\mu}} - 1}\right) - \frac{y\zeta}{1+\zeta}} = \sum_{n=0}^{\infty} \beta_n^{(1+\alpha+n+\varphi n)}(\lambda, \mu, x + ny) \frac{t^n}{n!},$$

where

$$(3.12) \quad \zeta = t \left(\frac{(1 + \zeta)^{\frac{1}{\mu}} - 1}{\lambda \zeta} \right)^\varphi (1 + \zeta)^y.$$

Proof. Setting $A(z) = B_i(z) = 1$, $x_j = y_j = 0$ ($1 \leq j \leq s$), $\sigma = \delta = 0$, $\lambda_i = 0$ ($1 \leq i \leq r$) and $E(z) = \frac{(1+\lambda z)^\mu - 1}{\mu \lambda}$, we thus have

$$E^{-1}(z) = \frac{(1+z)^{\frac{1}{\mu}} - 1}{\lambda}, \frac{d}{dz} E^{-1}(z) = (1+z)^{\frac{1}{\mu}-1} \text{ and}$$

$$\frac{d^2}{dz^2} E^{-1}(z) = \frac{(1-\mu)(1+z)^{\frac{1}{\mu}-2}}{\mu}.$$

Finally, by making the appropriate substitutions in conjunction with Theorem 2.4, the result follows easily. \square

It is worthy to mention that taking the limit as $\lambda \rightarrow 0$ ($\lambda\mu = 1$) in (3.10), we have

$$(3.13) \quad \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \left\{ \beta_n^{(\alpha+1+n)} \left(\lambda, \frac{1}{\lambda}; x \right) \right\} \frac{z^n}{n!} = \lim_{\lambda \rightarrow 0} \left\{ \left(\frac{(1+z)^\lambda - 1}{\lambda z} \right)^\alpha (1+z)^{x-1+\lambda} \right\}$$

$$= \left(\frac{\ln(1+z)}{z} \right)^\alpha (1+z)^{x-1}$$

$$= \sum_{n=0}^{\infty} B_n^{(\alpha+1+n)}(x) \frac{z^n}{n!},$$

where $B_n^{(\alpha)}(x)$ denotes the well-known generalized Bernoulli polynomials. This last generating function is a well-known result for the Bernoulli polynomials.

4. Applications to special pairs of inverse functions

In this section, we first introduce some families of pairs of inverse functions with the help of a result given by Donaghey [5]. These families are listed into two tables. Next, we give some applications of Theorem 2.4 to some of these pairs of inverse functions which can be seen as extensions of the generalized Bernoulli polynomials and the generalized degenerate Bernoulli polynomials.

First of all, let us recall a result obtained by Donaghey [5].

Theorem 4.1. *Let $f_k(z) = zF_k(z)$ and $g_k(z) = zG_k(z)$ be functions generated by $F_k(z) = \frac{1}{1+kz} F_0 \left(\frac{z}{1+kz} \right)$ (k -th Euler transformation of a series [32]) and $G_k(z) = \frac{G_0(z)}{1-kzG_0(z)}$ (k -th star transformation of a series [32]) with $f_0(g_0(z)) = g_0(f_0(z)) = z$, then $f_k(z)$ and $g_k(z)$ are two families of inverse functions with $k = 0, \pm 1, \pm 2, \dots$*

TABLE 1. Pairs of inverse functions with 1 index

$F_0(z)$	$G_0(z)$	$f_k(z)$	$g_k(z)$
$\frac{e^z - 1}{z}$	$\frac{\ln(1+z)}{z}$	$e^{\frac{z}{1+kz}} - 1$	$\frac{\ln(1+z)}{1 - k \ln(1+z)}$
$\frac{\ln(1+z)}{z}$	$\frac{e^z - 1}{z}$	$\ln\left(1 + \frac{z}{1+kz}\right)$	$\frac{e^z - 1}{1 + k - ke^z}$
$\frac{(1+\lambda z)^\mu - 1}{z}$	$\frac{(1+z)^\frac{1}{\mu} - 1}{\lambda z}$	$\left(1 + \frac{\lambda z}{1+kz}\right)^\mu - 1$	$\frac{(1+z)^\frac{1}{\mu} - 1}{\lambda + k - k(1+z)^\frac{1}{\mu}}$
$\frac{(1+z)^\frac{1}{\mu} - 1}{\lambda z}$	$\frac{(1+\lambda z)^\mu - 1}{z}$	$\frac{\left(1 + \frac{z}{1+kz}\right)^\frac{1}{\mu} - 1}{\lambda}$	$\frac{(1+\lambda z)^\mu - 1}{1 + k - k(1+\lambda z)^\mu}$
$\frac{1}{z(e^z - 1)}$	$\frac{\ln(1+z) - \ln z}{z}$	$\frac{1}{e^{\frac{z}{1+kz}} - 1}$	$\frac{\ln(1+z) - \ln z}{1 - k(\ln(1+z) - \ln z)}$
$\frac{\ln(1+z) - \ln z}{z}$	$\frac{1}{z(e^z - 1)}$	$\ln(1+z+kz) - \ln z$	$\frac{1}{e^z - 1 - k}$

TABLE 2. Pairs of inverse functions with 2 indices

$F_{0,k}(z)$	$G_{0,k}(z)$	$f_{j,k}(z)$	$g_{j,k}(z)$
$\frac{\ln(1+z)}{z(1 - k \ln(1+z))}$	$\frac{e^{\frac{z}{1+kz}} - 1}{z}$	$\frac{\ln\left(1 + \frac{z}{1+jz}\right)}{1 - k \ln\left(1 + \frac{z}{1+jz}\right)}$	$\frac{e^{\frac{z}{1+kz}} - 1}{1 + j - je^{\frac{z}{1+kz}}}$
$\frac{e^z - 1}{z(1 + k - ke^z)}$	$\frac{\ln\left(1 + \frac{z}{1+kz}\right)}{z}$	$\frac{e^{\frac{z}{1+jz}} - 1}{1 + k - ke^{\frac{z}{1+jz}}}$	$\frac{\ln\left(1 + \frac{z}{1+kz}\right)}{1 - j \ln\left(1 + \frac{z}{1+kz}\right)}$
$\frac{(1+z)^\frac{1}{\mu} - 1}{z(\lambda + k - k(1+z)^\frac{1}{\mu})}$	$\frac{\left(1 + \frac{\lambda z}{1+kz}\right)^\mu - 1}{z}$	$\frac{\left(1 + \frac{z}{1+jz}\right)^\frac{1}{\mu} - 1}{\lambda + k - k\left(1 + \frac{z}{1+jz}\right)^\frac{1}{\mu}}$	$\frac{\left(1 + \frac{\lambda z}{1+kz}\right)^\mu - 1}{1 + j - j\left(1 + \frac{\lambda z}{1+kz}\right)^\mu}$

For example, let $f_0(z) = e^z - 1$ and $g_0(z) = \ln(1+z)$, we generate the family of pairs of inverse functions

$$f_k(z) = e^{\frac{z}{1+kz}} - 1 \quad \text{and} \quad g_k(z) = \frac{\ln(1+z)}{1 - k \ln(1+z)} \quad (k = 0, \pm 1, \pm 2, \dots)$$

which can also produce the pair of inverse functions with two indices

$$r_{k,j}(z) = \frac{\ln\left(1 + \frac{z}{1+jz}\right)}{1 - k \ln\left(1 + \frac{z}{1+jz}\right)} \quad \text{and}$$

$$s_{k,j}(z) = \frac{e^{\frac{z}{1+kz}} - 1}{1 + j - je^{\frac{z}{1+kz}}} \quad (k, j = 0, \pm 1, \pm 2, \dots).$$

The following tables contain a list of other possibilities of pairs of inverse functions (with $k, j = 0, \pm 1, \pm 2, \dots$).

It is worthy to mention that applying the same procedure to obtain pairs of inverse functions with 3 indices or more is irrelevant. By doing this, we recover the cases with one index.

Let us examine two examples of the application of Theorem 2.4 to a pair of inverse functions with 1 index and another with 2 indices.

Example 1. Setting $A(z) = B_i(z) = 1$, $x_j = y_j = 0$ ($1 \leq j \leq s$) and

$$E(z) = \left(1 + \frac{\lambda z}{1 + kz}\right)^\mu - 1$$

($\lambda\mu = 1$), we thus have

$$E'(z) = \frac{\left(1 + \frac{\lambda z}{1 + kz}\right)^{\mu-1}}{(1 + kz)^2}, \quad E^{-1}(z) = \frac{(1 + z)^{\frac{1}{\mu}} - 1}{\lambda + k - k(1 + z)^{\frac{1}{\mu}}},$$

$$\frac{d}{dz}E^{-1}(z) = \frac{\lambda(1 + z)^{\frac{1}{\mu}-1}}{\mu \left(\lambda + k - k(1 + z)^{\frac{1}{\mu}}\right)^2}$$

and

$$\frac{d^2}{dz^2}E^{-1}(z) = \frac{\lambda(1 + z)^{\frac{1}{\mu}-2} \left((\mu + 1)k(1 + z)^{\frac{1}{\mu}} + k(1 - \mu) + \lambda - 1\right)}{\mu^2 \left(\lambda + k - k(1 + z)^{\frac{1}{\mu}}\right)^3}.$$

Let us consider the sequence of functions $\left\{h_n^{(\beta, \gamma)}(0, \dots, 0)\right\}_{n=0}^\infty$ generated by

$$(4.1) \quad \left(\frac{\left(1 + \frac{\lambda z}{1 + kz}\right)^\mu - 1}{z}\right)^\beta \left(\frac{\left(1 + \frac{\lambda z}{1 + kz}\right)^{\mu-1}}{(1 + kz)^2}\right)^\gamma = \sum_{n=0}^\infty h_n^{(\beta, \gamma)}(0, \dots, 0) \frac{z^n}{n!}.$$

According to Theorem 2.4, we obtain the following generating function:

$$(4.2) \quad \left(\frac{(1 + z)^{\frac{1}{\mu}} - 1}{z \left(\lambda + k - k(1 + z)^{\frac{1}{\mu}}\right)}\right)^{-\beta} \left(\frac{\lambda(1 + z)^{\frac{1}{\mu}-1}}{\mu \left(\lambda + k - k(1 + z)^{\frac{1}{\mu}}\right)^2}\right)^{1-\gamma}$$

$$= \sum_{n=0}^\infty h_n^{(\beta-n-1, \gamma)}(0, \dots, 0) \frac{z^n}{n!}.$$

We also find that

$$(4.3) \quad \sum_{n=0}^\infty h_n^{(\beta-n-1+\phi n, \gamma+\kappa n)}(0, \dots, 0) \frac{t^n}{n!}$$

$$= \frac{\left(\frac{(1+\zeta)^{\frac{1}{\mu}} - 1}{\zeta \left(\lambda + k - k(1+\zeta)^{\frac{1}{\mu}}\right)}\right)^{-\beta} \left(\frac{\lambda(1+\zeta)^{\frac{1}{\mu}-1}}{\mu \left(\lambda + k - k(1+\zeta)^{\frac{1}{\mu}}\right)^2}\right)^{1-\gamma}}{1 - \phi + \frac{\phi\zeta\lambda(1+\zeta)^{\frac{1}{\mu}-1}}{\mu \left((1+\zeta)^{\frac{1}{\mu}} - 1\right) \left(\lambda + k - k(1+\zeta)^{\frac{1}{\mu}}\right)} + \frac{\kappa\zeta \left((1+\mu)k(1+\zeta)^{\frac{1}{\mu}} + k(1-\mu) + \lambda - 1\right)}{\mu(1+\zeta) \left(\lambda + k - k(1+\zeta)^{\frac{1}{\mu}}\right)}},$$

where

$$\zeta = t \left(\frac{(1 + \zeta)^{\frac{1}{\mu}} - 1}{\zeta (\lambda + k - k(1 + \zeta)^{\frac{1}{\mu}})} \right)^{-\phi} \left(\frac{\lambda(1 + \zeta)^{\frac{1}{\mu}-1}}{\mu (\lambda + k - k(1 + \zeta)^{\frac{1}{\mu}})^2} \right)^{-\kappa}.$$

Especially, if we set $k = 0$, $\beta = -\alpha$, $\phi = -\varphi$, $\gamma = \frac{\mu x}{\mu-1}$ and $\kappa = \frac{\mu y}{\mu-1}$ in (4.1), (4.2) and (4.3), we recover (3.10) to (3.12).

Example 2. Let $A(z) = B_i(z) = 1$, $x_j = y_j = 0$ ($1 \leq j \leq s$) and

$$E(z) = \frac{e^{\frac{z}{1+kz}} - 1}{1 + j - je^{\frac{z}{1+kz}}}.$$

Then we have

$$E'(z) = \frac{e^{\frac{z}{1+kz}}}{(1 + kz)^2 (1 + j - je^{\frac{z}{1+kz}})^2} \quad \text{and} \quad E^{-1}(z) = \frac{\ln \left(1 + \frac{z}{1+jz} \right)}{1 - k \ln \left(1 + \frac{z}{1+jz} \right)}.$$

Now consider the sequence of functions $\left\{ k_n^{(\beta, \gamma)}(0, \dots, 0) \right\}_{n=0}^{\infty}$ generated by

(4.4)

$$\left(\frac{e^{\frac{z}{1+kz}} - 1}{z(1 + j - je^{\frac{z}{1+kz}})} \right)^{\beta} \left(\frac{e^{\frac{z}{1+kz}}}{(1 + kz)(1 + j - je^{\frac{z}{1+kz}})^2} \right)^{\gamma} = \sum_{n=0}^{\infty} k_n^{(\beta, \gamma)}(0, \dots, 0) \frac{z^n}{n!}.$$

Using the fact that

$$\frac{d}{dz} E^{-1}(z) = \frac{1}{(1 + jz + z)(1 + jz) \left(1 - k \ln \left(1 + \frac{z}{1+jz} \right) \right)^2}$$

and

$$\frac{d^2}{dz^2} E^{-1}(z) = \frac{k [2j(1 + z + jz) + 1] \ln \left(1 + \frac{z}{1+jz} \right) - 2j(1 + z + jz) + 2k - 1}{(1 + jz + z)^2 (1 + jz)^2 \left(1 - k \ln \left(1 + \frac{z}{1+jz} \right) \right)^3},$$

we have the following generating function

(4.5)

$$\begin{aligned} & \left(\frac{\ln \left(1 + \frac{z}{1+jz} \right)}{z \left(1 - k \ln \left(1 + \frac{z}{1+jz} \right) \right)} \right)^{-\beta} \left(\frac{1}{(1 + jz + z)(1 + jz) \left(1 - k \ln \left(1 + \frac{z}{1+jz} \right) \right)^2} \right)^{1-\gamma} \\ &= \sum_{n=0}^{\infty} k_n^{(\beta-n-1, \gamma)}(0, \dots, 0) \frac{z^n}{n!} \end{aligned}$$

and we also have

$$\sum_{n=0}^{\infty} k_n^{(\beta-1-n+\phi n, \gamma+\kappa n)}(0, \dots, 0) \frac{t^n}{n!}$$

$$(4.6) \quad = \frac{\left(\frac{\ln\left(1+\frac{\zeta}{1+j\zeta}\right)}{\zeta\left(1-k\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\right)}\right)^{-\beta} \left(\frac{1}{(1+j\zeta+\zeta)(1+j\zeta)\left(1-k\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\right)^2}\right)^{1-\gamma}}{1-\phi+\frac{\phi\zeta}{(1+j\zeta)(1+j\zeta+\zeta)\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\left(1-k\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\right)}+\Omega},$$

where

$$\Omega = \frac{\kappa\zeta \left[k(2j+2j^2\zeta+2j\zeta+1)\ln\left(1+\frac{\zeta}{1+j\zeta}\right) - 2j(1+\zeta+j\zeta) + 2k - 1 \right]}{(1+j\zeta+\zeta)(1+j\zeta)\left(1-k\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\right)}$$

and

$$\zeta = t \left(\frac{\ln\left(1+\frac{\zeta}{1+j\zeta}\right)}{\zeta\left(1-k\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\right)} \right)^{-\phi} \left(\frac{1}{(1+j\zeta+\zeta)(1+j\zeta)\left(1-k\ln\left(1+\frac{\zeta}{1+j\zeta}\right)\right)^2} \right)^{-\kappa}.$$

Setting $k = j = 0$, $\beta = -\alpha$ and $\gamma = x$ in (4.4), (4.5) and (4.6), we find respectively

$$(4.7) \quad \left(\frac{z}{e^z-1}\right)^\alpha e^{xz} = \sum_{n=0}^\infty k_n^{(-\alpha,x)}(0,\dots,0) \frac{z^n}{n!} = \sum_{n=0}^\infty B_n^{(\alpha)}(x) \frac{z^n}{n!},$$

$$(4.8) \quad \left(\frac{\ln(1+z)}{z}\right)^\alpha (1+z)^{x-1} = \sum_{n=0}^\infty k_n^{(-\alpha-n-1,x)}(0,\dots,0) \frac{z^n}{n!} = \sum_{n=0}^\infty B_n^{(\alpha+n+1)}(x) \frac{z^n}{n!}$$

and replacing ϕ by $-\beta$ and κ by y

$$(4.9) \quad \frac{\left(\frac{\ln(1+\zeta)}{\zeta}\right)^\alpha (1+\zeta)^{x-1}}{1+\beta-\frac{\beta\zeta}{(1+\zeta)\ln(1+\zeta)}-\frac{y\zeta}{(1+\zeta)}} = \sum_{n=0}^\infty k_n^{(-\alpha-1-n-\beta n,x+ny)}(0,\dots,0) \frac{t^n}{n!} = \sum_{n=0}^\infty B_n^{(\alpha+1+n+\beta n)}(x+ny) \frac{t^n}{n!},$$

where

$$\zeta = t \left(\frac{\ln(1+\zeta)}{\zeta} \right)^\beta \left(\frac{1}{(1+\zeta)} \right)^{-y}.$$

These generating functions (Equations (4.7) to (4.9)) have been given in [33, p. 493].

In light of these two examples, it is obvious that the functions given in Tables 1 and 2 can be seen as generalizations of the generalized Bernoulli polynomials and generalized degenerate Bernoulli polynomials. For example, it could be

interesting to study the following extensions:

$$(4.10) \quad \left(\frac{z}{\left(1 + \frac{\lambda z}{1+kz}\right)^\mu - 1} \right)^\alpha \left(\frac{\left(1 + \frac{\lambda z}{1+kz}\right)^{\mu-1}}{(1+kz)^2} \right)^{\frac{\mu x}{\mu-1}} = \sum_{n=0}^{\infty} \beta_n^{(\alpha)}(k; \lambda, \mu, x) \frac{z^n}{n!},$$

with $\lambda\mu = 1$, and

$$(4.11) \quad \left(\frac{z(1+j - je^{\frac{z}{1+kz}})}{e^{\frac{z}{1+kz}} - 1} \right)^\alpha \left(\frac{e^{\frac{z}{1+kz}}}{(1+kz)^2 (1+j - je^{\frac{z}{1+kz}})^2} \right)^x = \sum_{n=0}^{\infty} B_n^{(\alpha)}(k, j; x) \frac{z^n}{n!}.$$

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SEBASTIEN GABOURY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF QUEBEC AT CHICOUTIMI
QUEBEC, G7H 2B1, CANADA
E-mail address: s1gabour@uqac.ca

RICHARD TREMBLAY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF QUEBEC AT CHICOUTIMI
QUEBEC, G7H 2B1, CANADA
E-mail address: rtrembla@uqac.ca