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Sums of Products of Bernoulli Numbers*

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Closed expressions are obtained for sums of products of Bernoulli numbers of the form $\sum \binom{2n}{2j_1, \ldots, 2j_N} B_{2j_1} \cdots B_{2j_N}$, where the summation is extended over all nonnegative integers $j_1, \ldots, j_N$ with $j_1 + j_2 + \cdots + j_N = n$. Corresponding results are derived for Bernoulli polynomials, and for Euler numbers and polynomials. As easy corollaries we obtain formulas for sums of products of the Riemann zeta function at even integers and of other related infinite series.

1. INTRODUCTION

The Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{1.1}$$

A well-known relation among the Bernoulli numbers is (for $n \geqslant 2$)

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1) B_{2n}. \tag{1.2}$$

This was found by many authors, including Euler; for references, see, e.g., [13]. Sitaramachandrarao and Davis [13] generalized (1.2) to sums of products of 3 and 4 Bernoulli numbers:

$$\sum \binom{2n}{2a, 2b, 2c} B_{2a} B_{2b} B_{2c}$$
$$= (n+1)(2n+1) B_{2n} + n\left(n - \frac{1}{2}\right) B_{2n-2}, \tag{1.3}$$

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$$\sum \binom{2n}{2a,\, 2b,\, 2c,\, 2d} B_{2a} B_{2b} B_{2c} B_{2d}$$
$$= -\binom{2n+3}{3} B_{2n} - \frac{4}{3} n^2 (2n-1)\, B_{2n-2}, \tag{1.4}$$

where $\binom{2n}{2a,\, 2b,\, 2c}$ and $\binom{2n}{2a,\, 2b,\, 2c,\, 2d}$ are multinomial coefficients. The sum in (1.3) ranges over all positive integers a, b, c with $a+b+c=n$ $(n \geqslant 3)$, and the sum in (1.4) ranges over all positive integers $a, ..., d$ with $a + \cdots + d = n$ $(n \geqslant 4)$. The identities (1.2)–(1.4) can be written in terms of the Riemann zeta function, via Euler's formula

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}. \tag{1.5}$$

Thus, (1.2) can be written as

$$\sum_{j=1}^{n-1} \zeta(2j)\, \zeta(2n-2j) = (n + \tfrac{1}{2})\, \zeta(2n), \qquad n \geqslant 2, \tag{1.6}$$

and similarly for (1.3) and (1.4).

Sitaramachandrarao and Davis remarked in [13] that it may be of interest to find formulas of the type (1.2)–(1.4), (1.6) for sums of products of $N \geqslant 5$ Bernoulli number, resp. zeta function factors.

This was achieved by Sankarayanan [12] for $N=5$ and by Zhang [16] for $N \leqslant 7$. Before [16] appeared, Ramachandra and Sankarayanan [11] proved that with

$$Y(z) := \sum_{n=1}^{\infty} (-1)^{n-1} B_{2n} \frac{(2z)^{2n}}{(2n)!}$$

we have

$$Y^N = \sum_{j=0}^{N-1} A_j(z)\, z^j \frac{d^j Y}{dz^j}, \tag{1.7}$$

where $A_j(z)$ are polynomials in z of degree at most $(N-1)\, N$, with rational coefficients. (In fact, a somewhat more general result was obtained). It is easy to see that (1.7) will lead to expressions of the type (1.2)–(1.4). It was remarked in [11] that the proof gives an algorithm for determining the $A_j(z)$; however, they are not explicitly given.

It is one purpose of this paper to obtain explicit expressions, of the kind given in [12], [13], and [16], and valid for all $N \geqslant 2$ and all $n \geqslant 1$ (not just for $n \geqslant N$). We will do this in Section 2. In Section 3 we prove corresponding results for the Bernoulli polynomials $B_n(x)$, and in Section 4 for the Euler polynomials $E_n(x)$. In both cases we obtain as easy consequences formulas for sums related to the Riemann zeta function. Section 5 contains some remarks on related sums that occur in the literature.

2. BERNOULLI NUMBERS

Before stating our first result, the generalization of (1.2)–(1.4), we introduce some notation. We define the sequence $b_k^{(N)}$ of rational numbers recursively by $b_0^{(1)} = 1$ and

$$b_k^{(N+1)} = -\frac{1}{N} b_k^{(N)} + \frac{1}{4} b_{k-1}^{(N-1)}, \tag{2.1}$$

with $b_k^{(N)} = 0$ for $k < 0$ and for $k > [(N-1)/2]$. (Here and in what follows $[x]$ denotes the greatest integer not exceceeding x). Furthermore, let

$$f(t) := \frac{1}{e^t - 1},$$

and $f^{(k)}(t)$ denote, as usual, the kth derivative of $f(t)$, with the convention that $f^{(0)}(t) = f(t)$.

Lemma 1. *For $N \geqslant 1$ we have*

$$\left(\frac{1}{2}\frac{e^t+1}{e^t-1}\right)^N - \left(\frac{1}{2}\right)^N = \sum_{k=0}^{[(N-1)/2]} b_k^{(N)} f^{(N-2k-1)}(t). \tag{2.2}$$

Proof. This is by induction on N. For $N = 1$ we have

$$\frac{1}{2}\frac{e^t+1}{e^t-1} - \frac{1}{2} = \frac{1}{e^t-1} = f(t),$$

which agrees with the right-hand side of (2.2). For $N = 2$, the left-hand side of (2.2) becomes

$$\frac{1}{4}\left(\frac{e^{2t}+2e^t+1}{e^{2t}-2e^t+1} - 1\right) = \frac{1}{4}\frac{4e^t}{(e^t-1)^2} = -f'(t),$$

which again agrees with the right-hand side of (2.2).

For the induction step we first note that we have the following identity (for $N \geqslant 1$) which is easy to verify:

$$\left(\frac{1}{2}\frac{e^t+1}{e^t-1}\right)^{N+1} - \left(\frac{1}{2}\right)^{N+1} = -\frac{1}{N}\frac{d}{dt}\left\{\left(\frac{1}{2}\frac{e^t+1}{e^t-1}\right)^{N} - \left(\frac{1}{2}\right)^{N}\right\} + \frac{1}{4}\left\{\left(\frac{1}{2}\frac{e^t+1}{e^t-1}\right)^{N} - \left(\frac{1}{2}\right)^{N}\right\}.$$

Assuming (2.2) is true for all exponents up to N, we thus obtain

$$\begin{aligned}
&\left(\frac{1}{2}\frac{e^t+1}{e^t-1}\right)^{N-1} - \left(\frac{1}{2}\right)^{N-1} \\
&\quad = -\frac{1}{N}\frac{d}{dt}\sum_{k=0}^{[(N-1)/2]} b_k^{(N)} f^{(N-2k-1)}(t) + \frac{1}{4}\sum_{k=0}^{[(N-2)/2]} b_k^{(N-1)} f^{(N-2k-2)}(t) \\
&\quad = -\sum_{k=0}^{[(N-1)/2]} \frac{1}{N} b_k^{(N)} f^{(N-2k)}(t) + \sum_{k=1}^{[N/2]} \frac{1}{4} b_{k-1}^{(N-1)} f^{(N-2k)}(t) \\
&\quad = \sum_{k=0}^{[N/2]} \left\{-\frac{1}{N} b_k^{(N)} + \frac{1}{4} b_{k-1}^{(N-1)}\right\} f^{(N-2k)}(t) \\
&\quad = \sum_{k=0}^{[((N+1)-1)/2]} b_k^{(N+1)} f^{(N+1-2k-1)}(t),
\end{aligned}$$

where the last step follows from (2.1). This completes the proof. ∎

Let $N \geqslant 1$ be given. Our aim is to evaluate the sum

$$S_N(n) := \sum \binom{2n}{2j_1, 2j_2, ..., 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N}, \tag{2.3}$$

where the sum is taken over all nonnegative integers $j_1, ..., j_N$ such that $j_1 + \cdots + j_N = n$, and where

$$\binom{2n}{2j_1, ..., 2j_N} = \frac{(2n)!}{(2j_1)! \cdots (2j_N)!}$$

is the multinomial coefficient.

THEOREM 1. *For $2n > N$ we have*

$$S_N(n) = \frac{(2n)!}{(2n-N)!} \sum_{k=0}^{[(N-1)/2]} b_k^{(N)} \frac{B_{2n-2k}}{2n-2k}. \tag{2.4}$$

Before proving the theorem, we use it to write down some special cases. First we tabulate, by way of the recursion (2.1), a few values of $b_k^{(N)}$:

k^N	1	2	3	4	5	6
0	1	-1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$\frac{-1}{120}$
1	0	0	$\frac{1}{4}$	$-\frac{1}{3}$	$\frac{5}{24}$	$-\frac{1}{12}$
2	0	0	0	0	$\frac{1}{16}$	$-\frac{23}{240}$

With this table and (2.4) we obtain for $N=1$ the trivial $S_1(n)=B_{2n}$, and further

$$S_2(n)=-(2n-1)\,B_{2n}; \tag{2.5}$$

$$S_3(n)=\tfrac{1}{2}(2n-1)(2n-2)\,B_{2n}+\tfrac{1}{2}n(2n-1)\,B_{2n-2}; \tag{2.6}$$

$$S_4(n)=-\tfrac{1}{6}(2n-1)(2n-2)(2n-3)\,B_{2n}-\tfrac{1}{3}2n(2n-1)(2n-3)\,B_{2n-2}; \tag{2.7}$$

these are equivalent to (1.2)–(1.4) if we take into account the slightly different ranges of summation. The first new identities are obtained for $N=5$ and $N=6$:

$$S_5(n)=\binom{2n-1}{4}B_{2n}+\frac{5}{12}n(2n-1)(2n-3)(2n-4)\,B_{2n-2}$$
$$+\frac{1}{8}n(2n-1)(2n-2)(2n-3)\,B_{2n-4}; \tag{2.8}$$

$$S_6(n)=-\binom{2n-1}{5}B_{2n}-\frac{1}{6}n(2n-1)(2n-3)(2n-4)(2n-5)\,B_{2n-2}$$
$$-\frac{23}{120}n(2n-1)(2n-2)(2n-3)(2n-5)\,B_{2n-4}. \tag{2.9}$$

Proof of Theorem 1. We rewrite (1.1) as

$$\frac{t}{e^t-1}=1-\frac{1}{2}t+\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}t^{2j}; \tag{2.10}$$

hence

$$f(t)=\frac{1}{t}-\frac{1}{2}+\sum_{j=1}^{\infty}\frac{B_{2j}}{2j}\frac{t^{2j-1}}{(2j-1)!}, \tag{2.11}$$

and therefore

$$f^{(N-2k-1)}(t) = (-1)^{N-1}\frac{(N-2k-1)!}{t^{N-2k}} + \sum_{j=[(N-2k+1)/2]}^{\infty} \frac{B_{2j}}{2j}\frac{t^{2j-1-(N-2k-1)}}{(2j-N+2k)!}, \tag{2.12}$$

except when $N-2k-1=0$ in which case we have just (2.11). Hence for $N>2k+1$ we have

$$t^N f^{(N-2k-1)}(t) = (-1)^{N-1}(N-2k-1)!\,t^{2k} + \sum_{j=[(N+1)/2]}^{\infty} \frac{B_{2j-2k}}{2j-2k}\frac{t^{2j}}{(2j-N)!}.$$

Now Lemma 1 gives

$$\left(\frac{t}{2}\frac{e^t+1}{e^t-1}\right)^N = \left(\frac{t}{2}\right)^N + \sum_{k=0}^{[(N-1)/2]} b_k^{(N)} \times \left\{(-1)^{N-1}(N-2k-1)!\,t^{2k} + \sum_{j=[(N+1)/2]}^{\infty} \frac{B_{2j-2k}}{2j-2k}\frac{t^{2j}}{(2j-N)!}\right\} - \frac{1}{2}a_N b_{(N-1)/2}^{(N)} t^N, \tag{2.13}$$

where $a_n=0$ when N is even, and $a_N=1$ when N is odd (from (2.11)). On the other hand, with (2.10) we get

$$\frac{t}{2}\frac{e^t+1}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_{2j}}{(2j)!}t^{2j},$$

and raising this to the Nth power gives, with (2.3),

$$\left(\frac{t}{2}\frac{e^t+1}{e^t-1}\right)^N = \sum_{n=0}^{\infty} S_N(n)\frac{t^{2n}}{(2n)!}. \tag{2.14}$$

Finally, we compare coefficients of t^{2n} in (2.14) and (2.13), and thus obtain (2.4) for $2n>N$, i.e., for $n\geqslant[(N+2)/2]$. ∎

The proof of Theorem 1 immediately gives expressions for $S_N(n)$ also in the case $n\leqslant[N/2]$. First we need a lemma about some special values of the coefficients $b_k^{(N)}$.

Lemma 2. *For* $N \geqslant 1$ *we have*

(a) $b_0^{(N)} = (-1)^{N-1}/(N-1)!$;

(b) $b_{(N-1)/2}^{(N)} = 2^{-N+1}$ *for N odd.*

Proof. (a) By definition we have $b_0^{(1)} = 1$ and $b_0^{(N+1)} = -(1/N)\, b_0^{(N)}$; the result now follows immediately by induction. (b) In this case (2.1) gives $b_0^{(1)} = 1$ and $b_{(N+1)/2}^{(N+2)} = \frac{1}{4} b_{(N-1)/2}^{(N)}$. The result follows again by induction. ▮

Theorem 2.

$$\text{(a)} \qquad S_{2n}(n) = \frac{(2n)!}{4^n} + \sum_{k=0}^{n-1} \frac{(2n)!}{2n-2k}\, b_k^{(2n)} B_{2n-2k};$$

$$\text{(b)} \qquad S_N(n) = (-1)^{N-1}\,(N-2n-1)!\,(2n)!\, b_n^{(N)}, \qquad 0 \leqslant n \leqslant \left[\frac{N-1}{2}\right],$$

and in particular

(c) $S_{2n+1}(n) = (2n)!\, 4^{-n}$.

Proof. Parts (a) and (b) follow directly from (2.13) and (2.14), by comparing coefficients of t^N. Note that for odd N, the two terms involving t^N in (2.13) cancel each other, by Lemma 2(b). Part (c) follows from (b), with Lemma 2(b). ▮

Although the recurrence relation (2.1) provides a convenient way of determining the coefficients $b_k^{(N)}$ occurring in Theorems 1 and 2, it might be of interest of express the $b_k^{(N)}$ in closed form in terms of known functions. This can be done by first defining polynomials which have the $b_k^{(N)}$ as coefficients. It is then easy to establish a three-term recurrence between these polynomials, which in turn gives rise to a generating function. Standard methods concerning generating functions are then used to obtain the explicit expression.

Here and in the remainder of the paper we use the well-known Stirling numbers of the first kind $s(n, k)$. For definition an properties, see, e.g., [2, p. 212ff.]; a table can be found in [2, p. 310].

Lemma 3. *For* $N \geqslant 1$ *and* $0 \leqslant k \leqslant [(N-1)/2]$ *we have*

$$b_k^{(N)} = (-1)^{N-1}\, N \sum_{i=0}^{2k} \binom{N-1}{i} \frac{s(N-i, N-2k)}{(N-i)!} \left(\frac{1}{2}\right)^i. \tag{2.15}$$

Proof. While the procedure described above was used to obtain this expression, it now suffices to verify that the right-hand side of (2.15) satisfies the recursion (2.1). First we note that (2.15) gives $b_0^{(1)} = s(1, 1) = 1$. Also from (2.15) we get, after changing the order of summation,

$$\frac{1}{4} b_{k-1}^{(N-1)} = (-1)^N \sum_{i=2}^{2k} \frac{2^{-i}}{(N-i)!} \binom{N-1}{i-2} \times s(N+1-i, N+1-2k). \tag{2.16}$$

Using the "triangular" recurrence relation for the $s(n, k)$ (see, e.g., [2, p. 214]), we obtain

$$\begin{aligned} -\frac{1}{N} b_k^{(N)} &= (-1)^N \sum_{i=0}^{2k} \frac{2^{-i}}{(N-i)!} \binom{N-1}{i} \{s(N+1-i, N+1-2k) \\ &\quad + (N-i)\, s(N-i, N+1-2k)\} \\ &= (-1)^N \left\{ \sum_{i=0}^{2k} \frac{2^{-i}}{(N-i)!} \binom{N-1}{i} s(N+1-i, N+1-2k) \right. \\ &\quad \left. + \sum_{i=1}^{2k+1} \frac{2 \cdot 2^{-i}}{(N-i)!} \binom{N-1}{i-1} s(N+1-i, N+1-2k) \right\}. \end{aligned} \tag{2.17}$$

Now we note that

$$\begin{aligned} \binom{N-1}{i} + 2\binom{N-1}{i-1} + \binom{N-1}{i-2} &= \binom{N}{i} + \binom{N}{i-1} \\ &= \binom{N+1}{i} = \frac{N+1}{N+1-i} \binom{N}{N-i}. \end{aligned}$$

Hence with (2.16), (2.17) and (2.15) we obtain (2.1), which was to be shown. (Note that the terms for $i = 0, 1$ in (2.16) and for $i = 0, 2k+1$ in the last sum in (2.17) all vanish). ∎

3. BERNOULLI POLYNOMIALS

The Bernoulli polynomials $B_n(x)$ can be defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{3.1}$$

These polynomials satisfy a large number of identities (see e.g., [6, Ch. 50] or [9]), including the sum

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x)\, B_{n-k}(y)$$
$$= n(x+y-1)\, B_{n-1}(x+y) - (n-1)\, B_n(x+y). \tag{3.2}$$

In this section we will generalize (3.2) to sums of products of $N \geqslant 2$ Bernoulli polynomials, and derive some consequences.

In analogy to (2.3) we denote

$$S_N(n; x_1, x_2, ..., x_N)$$
$$:= \sum \binom{n}{j_1, ..., j_N} B_{j_1}(x_1)\, B_{j_2}(x_2) \cdots B_{j_N}(x_N), \tag{3.3}$$

where the sum is taken over all nonnegative integers $j_1, ..., j_N$ such that $j_1 + \cdots + j_N = n$.

It is possible to proceed just as in Section 2. However, here we can make use of the well-developed theory of higher-order Bernoulli polynomials $B_n^{(N)}(y)$ defined by the generating function

$$\frac{t^N e^{yt}}{(e^t-1)^N} = \sum_{n=0}^{\infty} B_n^{(N)}(y) \frac{t^n}{n!}, \qquad |t| = 2\pi; \tag{3.4}$$

see, e.g., [10, p. 145 ff.].

LEMMA 4. *Let* $y := x_1 + x_2 + \cdots + x_n$. *Then for* $n \geqslant N$ *we have*

$$S_N(n; x_1, ..., x_N) = (-1)^{N-1} N \binom{n}{N} \sum_{j=0}^{N-1} (-1)^j \binom{N-1}{j}$$
$$\times B_j^{(N)}(y) \frac{B_{n-j}(y)}{n-j}. \tag{3.5}$$

Proof. In (3.1) we replace x by $x_1, x_2, ..., x_n$ and multiply these N expressions together. With (3.3) and (3.4) we immediately obtain

$$S_N(n; x_1, ..., x_N) = B_n^{(N)}(y)(x_1 + x_2 + \cdots + x_N).$$

Now the right-hand side of (3.5) comes directly from Equation (87) in [10, p. 148]. ∎

THEOREM 3. *Let $y = x_1 + \cdots + x_n$. Then for $n \geqslant N$ we have*

$$S_N(n; x_1, ..., x_N) = (-1)^{N-1} N \binom{n}{N} \sum_{j=0}^{N-1} (-1)^j \times \left\{ \sum_{k=0}^{j} \binom{N-j-1+k}{k} s(N, N-j+k)\, y^k \right\} \frac{B_{n-j}(y)}{n-j}. \tag{3.6}$$

Proof. From Equation (52.2.21) in [6, p. 350] we obtain, after appropriate substitutions,

$$\binom{N-1}{j} B_j^{(N)}(y) = \sum_{k=0}^{j} \binom{N-j-1+k}{k} s(N, N-j+k)\, y^k. \tag{3.7}$$

Now (3.6) follows directly from (3.5). ∎

EXAMPLES. (i) $N=2$: With $s(2,2)=1$, $s(2,1)=-1$, (3.6) immediately gives (3.2).

(ii) $N=3$: With $s(3,1)=2$, $s(3,2)=-3$, and $s(3,3)=1$ we obtain for $n \geqslant 3$

$$\begin{aligned} \sum \frac{n!}{i!\, j!\, k!} B_i(x)\, B_j(y)\, B_k(z) &= \frac{n(n-1)}{2} \left[\left(x+y+z-\frac{3}{2} \right)^2 - \frac{1}{4} \right] B_{n-2}(x+y+z) - n(n-2) \\ &\quad \times \left(x+y+z-\frac{3}{2} \right) B_{n-1}(x+y+z) + \frac{(n-1)(n-2)}{2} B_n(x+y+z), \end{aligned} \tag{3.8}$$

where the sum is taken over all nonnegative integers, i, j, k with $i+j+k=n$.

COROLLARY 1. *If $x_1 + \cdots + x_N = 0$ then we have for $n \geqslant N$*

$$S_N(n; x_1, ..., x_N) = N \binom{n}{N} \sum_{j=0}^{N-1} (-1)^{N-1-j} s(N, N-j) \frac{B_{n-j}}{n-j}. \tag{3.9}$$

This follows immediately from (3.6) with $y=0$; note that $B_j(0) = B_j$. In particular, with $x_1 = x_2 = \cdots = x_N = 0$ the right-hand side of (3.9) gives an expression for

$$\sum \binom{n}{j_1, ..., j_N} B_{j_1} \cdots B_{j_N}.$$

Compare with (2.4); the difference lies in the fact that the above sum includes $B_1 = -\frac{1}{2}$.

Remark. For the special case of Bernoulli numbers, i.e., when $x_1 = \cdots = x_N = y = 0$, the identity (3.9) (or (3.6)) was proved by Vandiver [15, Eq. (140)]. The equation (3.5) in this particular case occurs in [15] as identity (142).

Apart from formulas for $\zeta(2n)$, such as (1.6) and its generalizations, Sitaramachandrarao and Davis [13] also deal with formulas involving the alternating sums (in the notation of [1])

$$\eta(n) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n}. \tag{3.10}$$

For instance, they show that

$$\sum_{j=1}^{n-1} \eta(2j)\, \eta(2n-2j) = (n - \tfrac{1}{2})\, \zeta(2n) - \eta(2n), \tag{3.11}$$

and a formula for a sum of products of 3 factors. Using Theorem 3, we can easily generalize this to any number of factors.

THEOREM 4. *For $2n \geqslant N$ we have*

$$\begin{aligned}\sum \eta(2j_1) \cdots \eta(2j_N) = {} & \frac{(-1)^{N+n-1}\,(2\pi)^{2n}}{(N-1)!\,(2n-N)!\,2^N} \\ & \times \sum_{j=0}^{[(N-1)/2]} \left\{ \sum_{k=0}^{2j} \binom{N-2j-1+k}{k} s(N, N-2j+k) \left(\frac{N}{2}\right)^k \right\} \\ & \times \frac{B_{2n-2j}(N/2)}{2n-2j},\end{aligned} \tag{3.12}$$

with the convention $\eta(0) = \frac{1}{2}$, and the sum on the left-hand side taken over all nonnegative integers $j_1, ..., j_N$ with $j_1 + \cdots + j_N = n$.

Proof. From the definition (3.10) it is clear that for $n > 1$ we have

$$\eta(n) = (1 - 2^{1-n})\, \zeta(n), \tag{3.13}$$

and with Euler's formula (1.5) and the well-known identity $B_n(\frac{1}{2}) = -(1 - 2^{1-n})\, B_n$ (see, e.g., [1, p. 805]) we obtain

$$\eta(2n) = (-1)^n \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}\left(\frac{1}{2}\right). \tag{3.14}$$

Hence we use Theorem 3 (or Lemma 4) with $x_1 = \cdots = x_N = \frac{1}{2}$, so that $y = N/2$. With (3.4) we have

$$\sum_{j=0}^{\infty} B_j^{(N)}\left(\frac{N}{2}\right)\frac{t^j}{j!} = \frac{t^N e^{Nt/2}}{(e^t - 1)^N} = \frac{t^N}{(e^{t/2} - e^{-t/2})^N},$$

and from this it is clear that the generating function is an even function; hence we have $B_j^{(N)}(N/2) = 0$ for odd j. Now (3.12) follows directly from (3.5) and (3.6). ∎

Remark. The right-hand side of (3.12) can be written again in terms of $\eta(2n)$ or $\zeta(2n)$. Indeed, the well-known difference equation $B_n(x+1) - B_n(x) = nx^{n-1}$ (see e.g., [1, p. 804]) implies

$$B_{2n-2j}\left(\frac{N}{2}\right) = (2n-2j)\sum_{k=1}^{(N/2)-1} k^{2n-2j-1} + B_{2n-2j}$$

for N even, and

$$B_{2n-2j}\left(\frac{N}{2}\right) = (2n-2j)\sum_{k=0}^{[N/2]-1}\left(k + \frac{1}{2}\right)^{2n-2j-1} + B_{2n-2j}\left(\frac{1}{2}\right)$$

for N odd. Now use (1.5), resp. (3.14)

Example. $N = 2$. Note that $s(2, 2) = 1$ and $B_{2n}(1) = B_{2n}$. Then (3.12) and (1.5) give

$$\sum_{j=0}^{n} \eta(2j)\,\eta(2n-2j) = (n - \tfrac{1}{2})\,\zeta(2n).$$

This is equivalent to (3.11) (via $\eta(0) = \frac{1}{2}$).

4. EULER NUMBERS AND POLYNOMIALS

In complete analogy to the method of Section 3 one can obtain results concerning the Euler polynomials. They are defined by the generating function

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \qquad |t| < \pi \tag{4.1}$$

(see, e.g., [1, p. 804]). Again, many identities are known; see, e.g., [9] or [6, Ch. 51]. For instance, we have in analogy to (3.2),

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x)\, E_{n-k}(y) = 2(1-x-y)\, E_n(x+y) + 2E_{n+1}(x+y). \tag{4.2}$$

In analogy to (3.3) and (2.3) we denote

$$T_N(n; x_1, ..., x_N) := \sum \binom{n}{j_1, ..., j_N} E_{j_1}(x_1) \cdots E_{j_N}(x_N), \tag{4.3}$$

where the sum is again taken over all nonnegative integers $j_1, ..., j_N$ such that $j_1 + \cdots + j_N = n$. We use the nth order Euler polynomials defined by

$$\left(\frac{2}{e^t+1}\right)^N e^{yt} = \sum_{n=0}^{\infty} E_n^{(N)}(y)\, \frac{t^n}{n!}, \qquad |t| < \pi, \tag{4.4}$$

(see, e.g., [9, p. 143 ff.]). The following lemma and theorem are proved in complete analogy to Lemma 4 and Theorem 3. Here Equation (88) in [10, p. 148] is used.

Lemma 5. *Let $y = x_1 + \cdots + x_N$. Then for $n \geqslant N$ we have*

$$T_N(n; x_1, ..., x_N) = \frac{2^{N-1}}{(N-1)!} \sum_{j=0}^{N-1} (-1)^j \binom{N-1}{j} B_j^{(N)}(y)\, E_{n+N-1-j}(y). \tag{4.5}$$

Theorem 5. *Let $y = x_1 + \cdots + x_N$. Then for $n \geqslant N$ we have*

$$\begin{aligned} T_N(n; x_1, ..., x_N) = \frac{2^{N-1}}{(N-1)!} &\sum_{j=0}^{N-1} (-1)^j \\ &\times \left\{ \sum_{k=0}^{j} \binom{N-j-1+k}{k} s(N, N-j+k)\, y^k \right\} \\ &\times E_{n+N-1-j}(y). \end{aligned} \tag{4.6}$$

The Euler numbers are defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \qquad |t| < \pi/2. \tag{4.7}$$

It follows from a comparison with (4.1) that

$$E_n(\tfrac{1}{2}) = 2^{-n} E_n. \tag{4.8}$$

The left-hand side of (4.7) is an even function. Hence the odd-index Euler numbers are zero, and we have

$$E_{2n}(\tfrac{1}{2}) = 2^{-2n} E_{2n}, \qquad E_{2n+1}(\tfrac{1}{2}) = 0.$$

If we now use (4.5) and (4.6) with $x_1 = \cdots = x_n = \frac{1}{2}$ and use the fact that $B_j^{(N)}(N/2) = 0$ for odd j, we obtain the following result.

THEOREM 6. *For $2n \geqslant N$ we have*

$$\sum \binom{2n}{2j_1, ..., 2j_N} E_{2j_1} \cdots E_{2j_N}$$
$$= \frac{2^{2n+N-1}}{(N-1)!} \sum_{j=0}^{[(N-1)/2]} \left\{ \sum_{k=0}^{2j} \binom{N-2j-1+k}{k} s(N, N-2j+k) \left(\frac{N}{2}\right)^k \right\}$$
$$\times E_{2n+N-1-2j}\left(\frac{N}{2}\right). \tag{4.9}$$

Remark. The case $N = 3$ was recently considered by Zhang [17].

Another sum dealt with in [13] is (in the notation of [1, p. 807])

$$\beta(n) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}.$$

It is known that

$$\beta(2n+1) = (-1)^n \frac{(\pi/2)^{2n+1}}{2(2n)!} E_{2n} \tag{4.10}$$

(see, e.g., [1, p. 807]). With this and (4.9) we can easily obtain an expression for

$$\sum_{j_1 + \cdots + j_N = n} \beta(2j_1 + 1) \cdots \beta(2j_1 + 1). \tag{4.11}$$

The right-hand side of (4.9) can be written in terms of $\beta(j)$ or $\zeta(j)$, depending on the parity of N. Indeed, the difference equation $E_\nu(x+1) + E_\nu(x) = 2x^\nu$ gives upon repeated application

$$E_\nu\left(\frac{N}{2}\right) = 2 \sum_{k=1}^{[N/2]} (-1)^{k-1} \left(\frac{N}{2} - k\right)^\nu + (-1)^{[N/2]} E_\nu\left(\frac{N}{2} - \left[\frac{N}{2}\right]\right). \tag{4.12}$$

Now for odd N, (4.8) and (4.10) give the appropriate $\beta(j)$, while for even N we use the formula

$$E_\nu(0) = -2\,\frac{2^{\nu+1}-1}{\nu+1}\,B_{\nu+1}, \qquad \nu \geqslant 1 \tag{4.13}$$

(see, e.g., [1, p. 805]) and Euler's formula (1.5).

A further obvious consequence of (4.6) is obtained when $y=0$. Upon changing the order of summation we get the following

COROLLARY 2. *If $x_1+x_2+\cdots+x_N=0$ then we have for $n \geqslant N$,*

$$T_N(n; x_1, ..., x_N) = \frac{(-2)^{N-1}}{(N-1)!} \sum_{j=0}^{N-1} (-1)^j\, s(N, j+1)\, E_{n+j}(0). \tag{4.14}$$

This corollary allows us to deal with yet another class of series related to the Riemann zeta function, some results for which were proved in [13]. Using the notation of [1, p. 807], we define

$$\lambda(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = (1-2^{-s})\,\zeta(s), \qquad s>1. \tag{4.15}$$

With (1.5) and (4.13) we immediately obtain

$$\lambda(2n) = (-1)^n \frac{\pi^{2n}}{4(2n-1)!}\, E_{2n-1}(0). \tag{4.16}$$

In [13] it was shown that

$$\sum_{\substack{i+j+k=n\\ i,j,k \geqslant 1}} \lambda(2i)\,\lambda(2j)\,\lambda(2k) = \frac{(n-1)(2n-1)}{4}\,\lambda(2n) - \frac{\pi^2}{16}\,\lambda(2n-2). \tag{4.17}$$

We will now prove an extension to an arbitrary number of factors.

THEOREM 7. *For $m \geqslant N$ we have*

$$\begin{aligned}\sum \lambda(2k_1)\cdot \cdots \cdot \lambda(2k_N) = {} & \frac{2^{1-N}}{(2m-N)!} \sum_{j=0}^{[(N-1)/2]} (-1)^j\, \pi^{2j} (2m-2j-1)! \\ & \times \left\{ \sum_{k=0}^{2j} \binom{N}{k} \frac{s(N-k, N-2j)}{2^k (N-1-k)!} \right\} \lambda(2m-2j)\end{aligned} \tag{4.18}$$

where the sum on the left-hand side is taken over all positive integers $k_1, ..., k_N$, with $k_1+\cdots+k_N=m$.

Proof. We use (4.14) with $x_1 = \cdots = x_N = 0$. To simplify notation, we set $T_N(n) := T_N(n; 0, ..., 0)$ (see (4.3)), and denote by $\widetilde{T}_N(n)$ the sum $T_N(n)$ with all those summands removed for which at least one of $j_1, ..., j_N$ is zero. Since by (4.1) we have $E_0(0) = 1$, the inclusion-exclusion principle (see, e.g., [2, p. 176 ff]) gives

$$\widetilde{T}_N(n) = T_N(n) - \binom{N}{1} T_{N-1}(n) + \binom{N}{2} T_{N-2}(n) + \cdots$$

$$= \sum_{k=0}^{N-1} (-1)^k \binom{N}{k} T_{N-k}(n).$$

Using (4.14) and changing the order of summation, we obtain

$$\widetilde{T}_N(n) = \sum_{k=0}^{N-1} (-1)^k \binom{N}{k} \frac{(-2)^{N-k-1}}{(N-k-1)!} \sum_{j=0}^{N-k-1} (-1)^j s(N-k, j+1) E_{n+j}(0)$$

$$= \sum_{j=0}^{N-1} (-1)^j \sum_{k=0}^{N-1-j} (-1)^k \binom{N}{k} \frac{(-2)^{N-k-1}}{(N-k-1)!} s(N-k, j+1) E_{n+j}(0).$$

We reverse the order of summation in the first sum and obtain

$$\widetilde{T}_N(n) = \sum_{j=0}^{N-1} (-1)^{N-1+j} \sum_{k=0}^{j} (-1)^k \binom{N}{k} \frac{(-2)^{N-k-1}}{(N-k-1)!}$$

$$\times s(N-k, N-j) E_{n+N-1-j}(0). \tag{4.19}$$

First we consider the left-hand side of (4.19). By (4.13) and the fact that $B_{2n+1} = 0$ for $n \geqslant 1$ it is clear that $E_{2n}(0) = 0$ for $n \geqslant 1$. Let $j_i = 2k_i - 1$ for $i = 1, 2, ..., N$. Then $n = j_1 + \cdots + j_N = 2(k_1 + \cdots + k_N) - N$ or equivalently $2(k_1 + \cdots + k_N) = n + N$. Now we use (4.16) with n replaced by $k_1, ..., k_N$. Then

$$\widetilde{T}_N(n) = (-1)^{(n+N)/2} \frac{n!\, 2^{2N}}{\pi^{n+N}} \sum \lambda(2k_1) \cdot \cdots \cdot \lambda(2k_N),$$

where the sum is taken over all positive integers $k_1, ..., k_N$ with $k_1 + \cdots + k_N = (n+N)/2$. If we set $n + N = 2m$, then

$$\widetilde{T}_N(n) = (-1)^m \frac{(2m-N)!\, 4^N}{\pi^{2m}} \sum \lambda(2k_1) \cdot \cdots \cdot \lambda(2k_N), \tag{4.20}$$

where $k_1 + \cdots + k_N = m$, $k_i \geqslant 1$ $(i = 1, ..., N)$.

To deal with the right-hand side of (4.19), we recall that $E_{2n}(0)=0$ for $n \geqslant 1$; then

$$\tilde{T}_N(n)=(-1)^{N-1} \sum_{j=0}^{[(N-1)/2]} \sum_{k=0}^{2j} (-1)^k \binom{N}{k} \frac{(-2)^{N-k-1}}{(N-k-1)!} \times s(N-k, N-2j)\, E_{2m-2j-1}(0). \tag{4.21}$$

Finally we use (4.16) with n replaced by $m-j$; then (4.20) combined with (4.21) gives (4.18). ∎

EXAMPLES. (i) With $N=2$ we get immediately

$$\sum_{k=1}^{m-1} \lambda(2k)\, \lambda(2m-2k)=(m-\tfrac{1}{2})\, \lambda(2m)$$

(see also [13, p. 1180]). For $N=3$ we obtain (4.17) if we note that $s(3, 1)=2$, $s(2, 1)=-1$, $s(1, 1)=1$.

(ii) $N=4$: With $s(4, 2)=11$, $s(3, 2)=-3$, $s(2, 2)=1$ we get the first apparently new formula

$$\sum \lambda(2k_1) \cdots \lambda(2k_4)=\frac{1}{8}\binom{2m-1}{3} \lambda(2m)-\frac{\pi^2}{24}(2m-3)\, \lambda(2m-2),$$

where $k_1+\cdots+k_4=m$, $k_i \geqslant 1$.

5. ADDITIONAL REMARKS

1. Theorem 1 with Lemma 3 can also be proved with (3.9), by careful use of the inclusion-exclusion principle to take care of the case $B_1=-\frac{1}{2}$. This would be similar to the proof of Theorem 7, and would be somewhat shorter than the proof in Section 1. However, Theorem 2 cannot be obtained in this way. On the other hand, the method of Section 1 could probably be used to prove the results in Sections 2–4, including analogues to Theorem 2.

2. Sums similar to the ones treated in this paper have occured in connection with the study of the Riemann zeta function at odd positive integer arguments. In particular, the well-known formula (1.2) occurs in [14], and the sum

$$\sum_{j=0}^{n} (-1)^j \binom{2n}{2j} B_{2j} B_{2n-2j} \tag{5.1}$$

occurs in the main result of [4]. A closed form for this alternating sum does not appear to be known.

Finally, a sum very similar to (2.3) plays a fundamental role in [3]; for each integer $r > 2$, $\zeta(r)$ is shown to be an infinite series involving $A_{2m}(r-2)$, where

$$A_{2m}(r) := \sum \frac{\binom{2m}{2i_1, ..., 2i_r} B_{2i_1} \cdot \cdots \cdot B_{2i_r}}{\{2i_1+1\}\{2(i_1+i_2)+1\} \cdot \cdots \cdot \{2(i_1+i_2+\cdots+i_{r-1})+1\}},$$

with the summation as in (2.3). In spite of the similarlity to the sum (2.3), it does not appear to be possible to treat this sum with the methods of the present paper.

3. Both Bernoulli and Euler numbers are special cases of the generalized Bernoulli numbers B_χ^n belonging to a residue class character χ. It may be of interest to have formulas of the type (1.2), and possibly of type (2.4), also for generalized Bernoulli numbers. We note that the corresponding alternating sum (of type (5.1)) occurs in [5], again in connection with integer values of the Riemann zeta function.

4. It is clear (via (1.5)) that Theorem 1 gives a formula for the sum

$$\sum_{j_1+\cdots+j_N=n} \zeta(2j_1) \cdot \cdots \cdot \zeta(2j_N), \tag{5.2}$$

with nonnegative integers $j_1, ..., j_N$ and with the convention $\zeta(0) = -\frac{1}{2}$. The product $\zeta(2j_1) \cdot \cdots \cdot \zeta(2j_N)$ is obviously related to the multiple sum

$$A(j_1, ..., j_N) := \sum_{n_1 > n_2 > \cdots > n_N \geqslant 1} \frac{1}{n_1^{j_1} n_2^{j_2} \cdot \cdots \cdot n_k^{j_k}}$$

(see [8]) and the sum (5.2) corresponds to

$$\sum_{j_1+\cdots+j_N=n} A(j_1, ..., j_N), \tag{5.3}$$

with positive integers $j_1, ..., j_N$, and $j_1 > 1$. It was conjetured by C. Moen that the sum (5.3) is equal to $\zeta(n)$ for all n and $N < n$; see [8], where this was verified for $n \leqslant 6$.

5. The following interesting "inverse problem" was discussed in [7]: Given an integer $k \geqslant 1$ and a sequence $\{s_n\}$ with $s_n \neq 0$ for some n, is there a sequence $\{r_k\}$ such that

$$s_n = \sum_{i_1+\cdots+i_k=n} \frac{n!}{i_1!\, i_2! \cdots i_k!} r_{i_1} r_{i_2} \cdots r_{i_k}?$$

Haukkanen [7] proved that this is the case if and only if the least n for which $s_n \neq 0$ is a nonnegative multiple of k; similarly for

$$s_n = \sum_{i_1 + \cdots + i_k = n} r_{i_1} r_{i_2} \cdots r_{i_k}.$$

Solutions to these questions are called kth roots of $\{s_n\}$ under the binomial, respectively the usual convolution.

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