Generalized Degenerated Bernoulli Numbers and Polynomials

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Abstract: The Generalized degenerate Bernoulli numbers $B_m^\alpha(\lambda)$ can be defined by means of the exponential generating function $(t)^\alpha / \left[(1+\lambda t)^{\frac{1}{\lambda}} - 1 \right]^\alpha$. As further applications we derive several identities,recurrences,andcongruences involving theGeneralized Bernoulli numbers, Generalizeddegenerate Bernoulli numbers and polynomials.

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1. Introduction

Carlitz[1] defined the generalized degenerate Bernoulli numbers $B_m^\alpha(\lambda)$ by means of the generating function

(1.1)
$$\frac{(t)^{\alpha}}{\left[\left(1+\lambda t\right)^{\frac{1}{\lambda}}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda)\frac{t^{m}}{m!}$$

we have, $B_{m}^{\alpha}(0)=B_{m}^{\alpha}$, the ordinary generalized Bernoulli number In[2] Carlitz proved many properties of $B_{m}^{\alpha}(\lambda)$. He also pointed out that $B_{m}^{\alpha}(\lambda)$ is a polynomials in λ with degree $\leq m$, we have

$$B_{0}^{\alpha}(\lambda)=1$$
$$B_{1}^{\alpha}(\lambda)=\frac{-\alpha}{2}+\frac{\alpha\lambda}{2}\quad\text{and so on.}$$

Carlitz[2] also defined the generalized degenerate Bernoulli polynomials $B_{m}^{\alpha}(\lambda,x)$ for $\lambda\neq0$ by means of the generating function.

(1.2)
$$\frac{(t)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}}{\left[\left(1+\lambda t\right)^{\frac{1}{\lambda}}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,x)\frac{t^{m}}{m!}$$

where $\lambda\mu=1$. These are polynomials in λ and x with rational coefficients. We often write $B_{m}^{\alpha}(\lambda)$ for $B_{m}^{\alpha}(\lambda,0)$, and refer to the polynomial $B_{m}^{\alpha}(\lambda)$ as ageneralized degenerate Bernoulli number. The first few are

$$B_{0}^{\alpha}(\lambda,x)=1$$
$$B_{1}^{\alpha}(\lambda,x)=x-\frac{\alpha}{2}+\frac{\alpha}{2}\lambda\quad\text{and so on}$$

Clearly, we have
$$B_{m}^{1}(\lambda,x)=B_{m}(\lambda,x)$$

2. A Recurrence Relation of $B_{m}^{\alpha}(\lambda,x)$

In this section, we derive the following recurrence relation for $B_{m}^{\alpha}(\lambda,x)$

(2.1)
$$B_m^{\alpha}(\lambda, x) = \sum_{k=0}^{m} \binom{m}{k} B_k^{\alpha}(\lambda) \left(\frac{x}{\lambda}\right)_{m-k}$$

Proof: We know that the generating function of generalized degenerate Bernoulli polynomial

$$\frac{(t)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - 1\right]^{\alpha}} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, x) \frac{t^m}{m!}$$

By (1.1) we get

$$\sum_{m=0}^{\infty} B_m^{\alpha}(\lambda) \frac{t^m}{m!} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, x) \frac{t^m}{m!}$$

By the Binomial expansion

$$[(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + ...]$$

$$\sum_{m=0}^{\infty} B_m^{\alpha}(\lambda) \frac{t^m}{m!} \sum_{m=0}^{\infty} \left(\frac{x}{\lambda}\right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, x) \frac{t^m}{m!}$$

By the Cauchy product rule

$$B_m^{\alpha}(\lambda, x) = \sum_{k=0}^{m} \binom{m}{k} B_k^{\alpha}(\lambda) \left(\frac{x}{\lambda}\right)_{m-k}$$

where $\left(\dfrac{x}{\lambda}\right)_m = [x(x-\lambda)(x-2\lambda).....(x-(m-1)\lambda]$

Particular case: It is interesting to note that (2.1) reduces to the well-known recurrence relation of degenerate Bernoulli polynomial for $\alpha = 1$.

(2.2)
$$B_m(\lambda, x) = \sum_{k=0}^{m} \binom{m}{k} B_k(\lambda) \left(\frac{x}{\lambda}\right)_{m-k}$$

3. Properties of generalized Degenerate Bernoulli polynomial

In this section, some of well-known properties ofgeneralized Degenerate Bernoulli polynomials are derived from the generating function (1.2)

Property-1

(3.1) $$B_m^\alpha(\lambda, x+y) = \sum_{k=0}^{m} \binom{m}{k} B_k^\alpha(\lambda, x) \left(\frac{y}{\lambda}\right)_{m-k}$$

Proof: Now put $x \to x+y$ in (1.2)

$$\frac{(t)^\alpha (1+\lambda t)^{\mu(x+y)}}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - 1\right]^\alpha} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x+y)\frac{t^m}{m!}$$

$$\frac{(t)^\alpha (1+\lambda t)^{\mu x}(1+\lambda t)^{\mu y}}{\left[(1+\lambda t)^{\frac{1}{\lambda}} - 1\right]^\alpha} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x+y)\frac{t^m}{m!}$$

By the equation (1.2)

$$\sum_{m=0}^{\infty} B_m^\alpha(\lambda, x)\frac{t^m}{m!}(1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x+y)\frac{t^m}{m!}$$

By the help of Binomial expansion

$$(1+\lambda t)^{\mu y} = \sum_{m=0}^{\infty} \left(\frac{y}{\lambda}\right)_m \frac{t^m}{m!}$$

Therefore

$$\sum_{m=0}^{\infty} B_m^\alpha(\lambda, x)\frac{t^m}{m!}\sum_{m=0}^{\infty} \left(\frac{y}{\lambda}\right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x+y)\frac{t^m}{m!}$$

By the Cauchy product rule

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

where
$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k . b_{n-k}$$

$$B_m^\alpha(\lambda, x+y) = \sum_{k=0}^{m} \binom{m}{k} B_k^\alpha(\lambda, x)\left(\frac{y}{\lambda}\right)_{m-k}$$

Particular case: When $\alpha = 1$, we get the ordinary Degenerate Bernoulli polynomial

(3.2)
$$B_m(\lambda, x+y) = \sum_{k=0}^{m} \binom{m}{k} B_k(\lambda, x)\left(\frac{y}{\lambda}\right)_{m-k}$$

Here $y=1$ then,

(3.3)
$$B_m^\alpha(\lambda, x+1) = \sum_{k=0}^{m} \binom{m}{k} B_k^\alpha(\lambda, x)\left(\frac{1}{\lambda}\right)_{m-k}$$

Property-2

(3.4)
$$\frac{d}{dx} B_m^\alpha(\lambda, x) = \lambda^{-1} B_m^\alpha(\lambda, x)$$

Proof:-By the generating function of generalized degenerate Bernoulli polynomials

$$\frac{(t)^\alpha (1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^\mu - 1\right]^\alpha} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x)\frac{t^m}{m!}$$

Differentiate above equation with respect to x

$$\frac{\mu(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}} = \sum_{m=0}^{\infty}\frac{d}{dx}B_m^{\alpha}(\lambda,x)\frac{t^m}{m!}$$

$$\mu.\sum_{m=0}^{\infty}B_m^{\alpha}(\lambda,x)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\frac{d}{dx}B_m^{\alpha}(\lambda,x)\frac{t^m}{m!}$$

Equating the coefficients

$$\frac{d}{dx}B_m^{\alpha}(\lambda,x) = \mu B_m^{\alpha}(\lambda,x)\text{ Where }\mu\lambda=1$$

$$\frac{d}{dx}B_m^{\alpha}(\lambda,x) = \lambda^{-1}B_m^{\alpha}(\lambda,x)$$

Property-3

(3.5)
$$B_m^{\alpha}(\lambda,\alpha-x) = \left(-1\right)^m B_m^{\alpha}(\lambda,x)$$

Proof: By equation (1.2)

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}} = \sum_{m=0}^{\infty}B_m^{\alpha}(\lambda,x)\frac{t^m}{m!}$$

Replace $x \rightarrow \alpha-x$ in the above equation

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu(\alpha-x)}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}} = \sum_{m=0}^{\infty}B_m^{\alpha}(\lambda,\alpha-x)\frac{t^m}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu\alpha}(1+\lambda t)^{-\mu x}}{\left[(1+\lambda t)^{\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{\mu\alpha}(1+\lambda t)^{-\mu x}}{(1+\lambda t)^{\mu\alpha}\left[1-(1+\lambda t)^{-\mu}\right]^{\alpha}}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

$$\frac{(t)^{\alpha}(1+\lambda t)^{-\mu x}}{\left[1-(1+\lambda t)^{-\mu}\right]^{\alpha}}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

$$-\frac{(t)^{\alpha}(1+\lambda t)^{-\mu x}}{\left[(1+\lambda t)^{-\mu}-1\right]^{\alpha}}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

$$\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,x)\frac{(-t)^{m}}{m!}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

$$\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,x)\frac{(-t)^{m}}{m!}=\sum_{m=0}^{\infty}B_{m}^{\alpha}(\lambda,\alpha-x)\frac{t^{m}}{m!}$$

Equating the coefficients

$$B_{m}^{\alpha}(\lambda,\alpha-x)=\left(-1\right)^{m}B_{m}^{\alpha}(\lambda,x)$$

Property-4

$$(3.6)\qquad B_{m}^{\alpha}(\lambda,x+1)-B_{m}^{\alpha}(\lambda,x)=mB_{m-1}^{\alpha-1}(\lambda,x)$$

Proof: By equation (1.2)

(3.7)
$$\frac{(t)^\alpha (1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^\mu -1\right]^\alpha} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x)\frac{t^m}{m!}$$

Replace $x \to x+1$ in the above equation.

(3.8)
$$\frac{(t)^\alpha (1+\lambda t)^{\mu(x+1)}}{\left[(1+\lambda t)^\mu -1\right]^\alpha} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x+1)\frac{t^m}{m!}$$

Subtracting (3.8)-(3.7)

$$\frac{(t)^\alpha (1+\lambda t)^{\mu(x+1)}}{\left[(1+\lambda t)^\mu -1\right]^\alpha} - \frac{(t)^\alpha (1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^\mu -1\right]^\alpha} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x+1)\frac{t^m}{m!} - \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x)\frac{t^m}{m!}$$

$$\frac{(t)^\alpha (1+\lambda t)^{\mu x}\left[(1+\lambda t)^\mu -1\right]}{\left[(1+\lambda t)^\mu -1\right]^\alpha} = \sum_{m=0}^{\infty}\left[B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x)\right]\frac{t^m}{m!}$$

$$\frac{(t)^{\alpha-1+1}(1+\lambda t)^{\mu x}\left[(1+\lambda t)^\mu -1\right]}{\left[(1+\lambda t)^\mu -1\right]^{\alpha-1+1}} = \sum_{m=0}^{\infty}\left[B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x)\right]\frac{t^m}{m!}$$

$$\frac{t.(t)^{\alpha-1}(1+\lambda t)^{\mu x}}{\left[(1+\lambda t)^\mu -1\right]^{\alpha-1}} = \sum_{m=0}^{\infty}\left[B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x)\right]\frac{t^m}{m!}$$

$$t\sum_{m=0}^{\infty} B_m^{\alpha-1}(\lambda, x)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left[B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x)\right]\frac{t^m}{m!}$$

$$\sum_{m=0}^{\infty} B_m^{\alpha-1}(\lambda, x) \frac{t^{m+1}}{m!} = \sum_{m=0}^{\infty} \left[B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x) \right] \frac{t^m}{m!}$$

Equating the coefficient of $\dfrac{t^m}{m!}$

$$B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x) = m B_{m-1}^{\alpha-1}(\lambda, x)$$

Property-4

(3.9)
$$B_m^{\alpha-1}(\lambda, x) = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k^\alpha(\lambda, x) \left(\frac{1}{\lambda} \right)_{m+1-k}$$

Proof:- By equation (3.6)

$$B_m^\alpha(\lambda, x+1) = \sum_{k=0}^m \binom{m}{k} B_k^\alpha(\lambda, x) \left(\frac{1}{\lambda} \right)_{m-k}$$

$$B_m^\alpha(\lambda, x+1) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^\alpha(\lambda, x) \left(\frac{1}{\lambda} \right)_{m-k} + \binom{m}{m} B_m^\alpha(\lambda, x) \left(\frac{1}{\lambda} \right)_{m-m}$$

$$B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^\alpha(\lambda, x) \left(\frac{1}{\lambda} \right)_{m-k}$$

But we know that by (3.5)

$$B_m^\alpha(\lambda, x+1) - B_m^\alpha(\lambda, x) = m B_{m-1}^{\alpha-1}(\lambda, x)$$

(3.10)
$$m B_{m-1}^{\alpha-1}(\lambda, x) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^\alpha(\lambda, x) \left(\frac{1}{\lambda} \right)_{m-k}$$

Now put $m \to m+1$ in the above equation

$$(3.11) \qquad B_m^{\alpha-1}(\lambda, x) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k^{\alpha}(\lambda, x) \left(\frac{1}{\lambda} \right)_{m+1-k}$$

Now put $x=0$ in equation (3.10), then

$$m B_{m-1}^{\alpha-1}(\lambda, 0) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^{\alpha}(\lambda, 0) \left(\frac{1}{\lambda} \right)_{m-k}$$

By definition

$$B_m^{\alpha}(\lambda, 0) = B_m^{\alpha}(\lambda)$$

Therefore

$$(3.12) \qquad m B_{m-1}^{\alpha-1}(\lambda) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^{\alpha}(\lambda) \left(\frac{1}{\lambda} \right)_{m-k}.$$

References

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