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A NOTE ON BERNOULLI POLYNOMIALS

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1. Some General Remarks

Consider the function $x - [x] - \frac{1}{2}$ which is periodic with period 1. In the interval [0, 1] this function is simply $x - \frac{1}{2}$.

This function has the property that its integral in the interval [0, 1] is zero. Let us, then, with the same idea in mind define another function $\Phi_2(x)$, such that its derivative is $\Phi_1(x) = x - \frac{1}{2}$, and such that its integral in the interval [0, 1] is zero:

$$\int_0^1 \Phi_2(x)\,dx = 0.$$

Similarly, $\Phi_3'(x) = \Phi_2(x)$, and

$$\int_0^1 \Phi_3(x)\,dx = 0.$$

In general, we seek a sequence of functions $\Phi_n(x)$, $n = 1, 2, 3, \ldots$, such that

$$\Phi_1(x) = x - \tfrac{1}{2},\ \Phi_n'(x) = \Phi_{n-1}(x) \text{ for } n > 1,$$

and

$$\int_0^1 \Phi_n(x)\,dx = 0 \text{ for all } n \geq 1.$$

The constant multiples of these functions $n!\Phi_n(x) = B_n(x)$ are called Bernoulli polynomials after their discoverer [2]. They obey the relation

(1.1) $\quad B_n'(x) = nB_{n-1}(x),\ n \geq 1,\ B_0(x) = 1.$

The first few Bernoulli polynomials are

$$B_0(x) = 1,\ B_1(x) = x - 1/2,\ B_2(x) = x^2 - x + 1/6,$$
$$B_3(x) = x^3 - (3/2)x^2 + (1/2)x,\ B_4(x) = x^4 - 2x^3 + x^2 - 1/30, \text{ etc.}$$

It is clear from their construction that $B_n(x)$ is a polynomial of degree n. They are defined in the interval $0 \leq x \leq 1$. Their periodic continuation outside this interval are called Bernoulli functions.

The constant terms of the Bernoulli polynomials form a particularly interesting set of numbers. We set $B_n = B_n(0)$. It is obvious from the way the polynomials $B_n(x)$ are constructed that all the B_n are rational numbers. It can be shown that $B_{2n+1} = 0$ for $n \geq 1$, and is alternately positive and negative for even n. The B_n are called Bernoulli numbers, and the first few are

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,$$
$$B_8 = -1/30,\ B_{10} = 5/66,\ B_{12} = -691/2730,\ B_{14} = 7/6, \text{ etc.}$$

Bernoulli polynomials and numbers are intimately related to the sum of the powers of the natural numbers.

Bernoulli polynomials possess the following generating function [5, 3],

(1.2) $$te^{tx}(e^t - 1)^{-1} = \sum_{n=0}^{\infty} B_n(x)t^n/n!,$$

from which we find, on replacing x by $x + 1$ and then subtracting (1.2) from the resulting expression:

(1.3) $\sum_{n=0}^{\infty} [B_n(x + 1) - B_n(x)]t^n/n! = te^{tx}.$

Using the Maclaurin expansion on the right-hand side and comparing powers of t, we find

(1.4) $B_n(x + 1) - B_n(x) = nx^{n-1}, \quad n = 2, 3, \ldots .$

From (1.1) and (1.4) there follows

(1.5) $\int_x^{x+1} B_n(s)ds = x^n,$

from which we find [4]

(1.6) $$\sum_{k=0}^{r} k^n = \sum_{k=0}^{r}\int_k^{k+1} B_n(s)ds = \int_0^{r+1} B_n(s)ds = \frac{B_{n+1}(r + 1) - B_{n+1}}{n + 1}, \quad n = 2, 3, 4, \ldots .$$

In the next section we will make use of the following property of Bernoulli polynomials [8]:

(1.7) $$\int_0^1 B_n(s)B_m(s)ds = (-1)^{n-1}\frac{m!n!}{(m + n)!}B_{n+m},$$
$n = 1, 2, 3, \ldots; \; m = 1, 2, 3, \ldots .$

Formula (1.7) is only apparently unsymmetrical in m and n. The reader can convince him- or herself of the symmetry of it by trying the different combinations of even and odd values of m and n.

2. An Expansion for Products of Bernoulli Polynomials

We wish to expand a product of two Bernoulli polynomials in series of Bernoulli polynomials [7]. It will simplify matters if we use the functions $\Phi_n(x)$ defined at the beginning of Section 1. We want, then, an expression of the form

(2.1) $\Phi_n(x)\Phi_m(x) = \sum_{k=0}^{n+m} a_k\Phi_k(x),$

where the Φ_n's are, we recall, Bernoulli polynomials divided by $n!$.

We will make use of the properties

(2.2) $\int_0^1 \Phi_n(s)ds = 0$ for $n \geq 1,$

and (1.7), which now appears in the guise

(2.3) $\int_0^1 \Phi_n(s)\Phi_m(s)ds = (-1)^{n-1}b_{n+m}, \quad n, m = 1, 2, \ldots,$

where the b_n's are Bernoulli numbers divided by $n!$.

Also

(2.4) $D\Phi_n(x) = \Phi_n' = \Phi_{n-1}.$

Using Leibniz's theorem for the derivative of a product [1], we find from (2.1)

(2.5) $$D^s[\Phi_n(x)\Phi_m(x)] = \sum_{j=0}^{s}\binom{s}{j}D^j\Phi_n(x)D^{s-j}\Phi_m(x) = \sum_{k=0}^{n+m} a_kD^s\Phi_k(x).$$

That is,

$$(2.6)\qquad \sum_{k=s}^{n+m} a_k\Phi_{k-s}(x) = \sum_{k=0}^{n+m-s} a_{k+s}\Phi_k(x) = \sum \binom{s}{j}\Phi_{n-j}(x)\Phi_{m-s+j}(x),$$

with the restrictions that $n - j \geq 0$ and $m - s + j \geq 0$, i.e., $j \leq n$, $j \geq s - m$. Since the sum in (2.5) starts at $j = 0$ and ends at $j = s$, we must write (2.6) in the form

$$(2.7)\qquad \sum_{j=\max(0,\, s-m)}^{\min(s,\, n)} \binom{s}{j}\Phi_{n-j}(x)\Phi_{m-s+j}(x) = \sum_{k=0}^{n+m-s} a_{k+s}\Phi_k(x).$$

We now wish to integrate both sides of (2.7) from $x = 0$ to $x = 1$ and to apply properties (2.2) and (2.3). To do so, we must separate from the first sum in (2.7) the terms corresponding to $j = n$ and to $j = s - m$, since in both of these cases the corresponding index is zero and formula (2.3) does not apply.

This gives

$$(2.8)\qquad a_s = b_{n+m-s}(-1)^{n-1}\sum_{j=\max(0,\, s-m+1)}^{\min(s,\, n-1)} \binom{s}{j}(-1)^j, \quad s < m + n - 1.$$

If $s = m + n$, the first sum in (2.5) will contain only one term and we have

$$(2.9)\qquad a_{n+m} = \binom{n+m}{n}.$$

Similarly, if $s = m + n - 1$, then the sum will contain only two terms with nonzero index, both of which will integrate to zero and we have

$$(2.10)\qquad a_{n+m-1} = 0.$$

Expressing these results in terms of ordinary Bernoulli polynomials, we find, after dividing a_s by $s!$, the expressions

$$(2.11)\qquad B_n(x)B_m(x) = \sum_{k=0}^{n+m}\alpha_k B_k(x),$$

$$(2.12)\qquad \alpha_k = \frac{n!m!B_{n+m-k}}{(n+m-k)!}(-1)^{n-1}\sum_{j=\max(0,\, k-m+1)}^{\min(k,\, n-1)}\frac{(-1)^j}{(k-j)!j!}, \quad k < n + m - 1,\ m, n = 1, 2, \ldots,$$

$$(2.13)\qquad \alpha_{n+m-1} = 0,$$

$$(2.14)\qquad \alpha_{n+m} = 1.$$

Equations (2.11)-(2.14) are the desired results. The reader may wish to look at reference [6] to see alternate ways of expressing these coefficients.

Since Bernoulli numbers of odd index greater than one are zero, we see that if n and m are of the same parity, then expansion (2.11) will only involve Bernoulli polynomials of even index. If n and m are of opposite parity, then expansion (2.11) will only involve Bernoulli polynomials of odd index.

If we define

$$(2.15)\qquad S_n(r) = \sum_{k=1}^{r} k^n,$$

and make use of (1.6), we can express (2.11) in terms of the S_n's:

$$(n+1)(m+1)S_n(r)S_m(r) = \sum_{k=1}^{n+m+2} k\alpha_k S_{k-1}(r) - (n+1)B_{m+1}S_n(r) - (m+1)B_{n+1}S_m(r) - B_{m+1}B_{n+1} + \sum_{k=0}^{n+m+2}\alpha_k B_k.$$

Observe now that in the equation above $-B_{m+1}B_{n+1}$ cancels $\sum_{k=0}^{n+m+2}\alpha_k B_k$, since these expressions are the left- and right-hand sides of (2.11) with $x = 0$ and n and m replaced by $n + 1$ and $m + 1$, respectively.

The equation then takes the form

$$(2.16) \quad (n + 1)(m + 1)S_n(r)S_m(r) = \sum_{k=2}^{n+m+2} k\alpha_k S_{k-1}(r) - (n + 1)B_{m+1}S_n(r) - (m + 1)B_{n+1}S_m(r),$$

where the α_k's must now be written

$$(2.17) \quad \alpha_k = \frac{(n + 1)!(m + 1)!B_{n+m+2-k}}{(n + m + 2 - k)!}(-1)^n \sum_{j=\max(0,\,k-m)}^{\min(k,\,n)} \frac{(-1)^j}{(k - j)!j!},$$

$$k < n + m + 1,$$

$$(2.18) \quad \alpha_{n+m+1} = 0,$$

$$(2.19) \quad \alpha_{n+m+2} = 1,$$

and we have observed that $\alpha_1 = 0$.

Note now that the product of $S_n(r)$ and $S_m(r)$ will involve $S_k(r)$'s with odd index only if n and m are of the same parity, and $S_k(r)$'s with even index only if n and m are of opposite parity.

3. Some Examples

$$(3.1) \quad S_1(r)S_2(r) = \frac{5}{6}S_4(r) + \frac{1}{6}S_2(r),$$

$$(3.2) \quad S_1(r)S_3(r) = \frac{3}{4}S_5(r) + \frac{1}{4}S_3(r),$$

$$(3.3) \quad S_2(r)S_3(r) = \frac{7}{12}S_6(r) + \frac{5}{12}S_4(r),$$

$$(3.4) \quad S_2(r)S_4(r) = \frac{8}{15}S_7(r) + \frac{1}{2}S_5(r) - \frac{1}{30}S_3(r),$$

$$(3.5) \quad S_3(r)S_5(r) = \frac{5}{12}S_9(r) + \frac{2}{3}S_7(r) - \frac{1}{12}S_5(r),$$

$$(3.6) \quad S_3(r)S_7(r) = \frac{7}{24}S_{13}(r) + S_{11}(r) - \frac{3}{8}S_9(r) + \frac{1}{12}S_7(r),$$

$$(3.7) \quad S_1(r)S_3(r)S_5(r) = \frac{1}{4}S_{11} + \frac{35}{48}S_9(r) + \frac{1}{24}S_7(r) - \frac{1}{48}S_5(r).$$

Especially appealing are the formulas for powers of the $S_k(n)$'s. We obtain, for instance, the expressions

$$(3.8) \quad S_1(r)^2 = S_3(r),$$

$$(3.9) \quad S_2(r)^2 = \frac{2}{3}S_5(r) + \frac{1}{3}S_3(r),$$

$$(3.10) \quad S_3(r)^2 = \frac{1}{2}S_7(r) + \frac{1}{2}S_5(r),$$

$$(3.11) \quad S_4(r)^2 = \frac{2}{5}S_9(r) + \frac{2}{3}S_7(r) - \frac{1}{15}S_5(r),$$

$$(3.12) \quad S_5(r)^2 = \frac{1}{3}S_{11}(r) + \frac{5}{6}S_9(r) - \frac{1}{6}S_7(r),$$

(3.13) $S_1(r)^3 = \frac{3}{4}S_5(r) + \frac{1}{4}S_3(r),$

(3.14) $S_2(r)^3 = \frac{1}{3}S_8(r) + \frac{7}{12}S_6(r) + \frac{1}{12}S_4(r),$

(3.15) $S_3(r)^3 = \frac{3}{16}S_{11}(r) + \frac{5}{8}S_9(r) + \frac{3}{16}S_7(r),$

etc.

Formulas (3.8) through (3.11) have been known for a very long time. Formula (3.10) is attributed to Jacobi [9].

To the best of our knowledge, the only special case of (2.11) that is known is [10]

(3.16) $B_4(x) - B_4 = (B_2(x) - B_2)^2,$

and accounts for (3.8).

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