q-BERNOULLI AND EULERIAN NUMBERS

BY
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1. **Introduction.** In a previous paper [2] the writer defined a set of rational functions η_m of the indeterminate q by means of

$$(1.1) \qquad (q\eta + 1)^m = \eta^m \ (m > 1), \quad \eta_0 = 1, \quad \eta_1 = 0,$$

and a set of polynomials

$$\eta_m(x) = \eta_m(x, q)$$

in q^x by

$$(1.2) \qquad \eta_m(x) = ([x] + q^x\eta)^m, \qquad \eta_m(0) = \eta_m,$$

where $[x] = (q^x - 1)/(q-1)$; also

$$(1.3) \qquad q^x\beta_m(x) = \eta_m(x) + (q-1)\eta_{m+1}(x), \qquad \beta_m(0) = \beta_m.$$

For $q=1$, β_m reduces to the Bernoulli number B_m, $\beta_m(x)$ reduces to the Bernoulli polynomial $B_m(x)$; η_m however does not remain finite for $m>1$.

In the present paper we first define polynomials $A_{ms} = A_{ms}(q)$ by means of

$$(1.4) \qquad [x]^m = \sum_{s=1}^{m} A_{ms} \begin{bmatrix} x+s-1 \\ m \end{bmatrix} \qquad (m \geqq 1),$$

where

$$\begin{bmatrix} x \\ m \end{bmatrix} = \frac{(q^x - 1)(q^{x-1} - 1)\cdots(q^{x-m+1} - 1)}{(q-1)(q^2-1)\cdots(q^m-1)}.$$

Alternatively if we define the rational function $H_m = H_m(x, q)$ by means of $H_0 = 1$, $H_1 = 1/(x-q)$,

$$(1.5) \qquad (qH + 1)^m = xH^m \qquad (m > 1),$$

then we have

$$(1.6) \qquad H_m(x, q) = A_m(x, q)/\prod_{s=1}^{m}(x - q^s),$$

where

$$(1.7) \qquad A_m(x, q) = \sum_{s=1}^{m} A_{ms}x^{s-1} \qquad (m \geqq 1),$$

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and the coefficients are the same as those occurring in (1.4). For $q=1$, A_{ms} and $H_m(x)$ reduce to well known functions; some of the properties of these quantities are stated in §2 below. As Frobenius [3] showed, many of the properties of the Bernoulli and related numbers can be derived from properties of H_m. We shall show that much the same is true in the case of the q analogues.

In [2] a theorem somewhat analogous to the Staudt-Clausen theorem was obtained for β_m (with q an indeterminate). We now show that if p is an odd prime and we put $q=a$, where the rational number a is integral (mod p), then if $a\equiv 1$ (mod p),

$$p\beta_m \equiv -1 \pmod p \tag{1.8}$$

provided $p-1\mid m$; otherwise β_m is integral (mod p). If $a\not\equiv 1$ (mod p) the situation is more complicated. In particular, if a is a primitive root (mod p^2), then β_m is integral (mod p) for $p-1\nmid m$, while for $p-1\mid m$ we have

$$p\beta_m \equiv -\frac{1}{k} \pmod p, \qquad (k=(a^{p-1}-1)/p).$$

In general the denominator of β_m may be divisible by arbitrarily high powers of p (see Theorem 4 below).

Finally we derive some congruences of Kummer's type for H_m, etc. For example if $q=a$ is integral (mod p) while x is an indeterminate, then

$$H^m(H^w-1)^r \equiv 0 \pmod{p^m,\ p^{re}} \qquad (p^{e-1}(p-1)\mid w),$$

where after expansion of the left member H^k is replaced by H_k. We also obtain simple congruences for the numbers A_{ms} defined in (1.4). The corresponding results for η_m and β_m are more complicated.

2. **Eulerian numbers.** To facilitate comparison we quote the following formulas from the papers of Frobenius [3] and Worpitzky [5].

$$x^m = \sum_{s=1}^{m} A_{ms}\binom{x+s-1}{m} \qquad (m\geqq 1), \tag{2.1}$$

$$A_{m+1,s} = (m+2-s)A_{m,s-1} + sA_{ms}, \tag{2.2}$$

$$A_{ms} = \sum_{r=0}^{s}(-1)^r\binom{m+1}{r}(s-r)^m, \tag{2.3}$$

$$B_m = \frac{1}{m+1}\sum_{r=1}^{m}(-1)^{m-r-1}\binom{m}{r-1}^{-1}A_{mr}. \tag{2.4}$$

In the next place if we put

$$A_m = A_m(x) = \sum_{s=1}^{m} A_{ms}x^{s-1}, \tag{2.5}$$

and let

$$H_m = H_m(x) = (x-1)^{-m}R_m(x),$$

then H_m satisfies

$$(2.6) \qquad (H+1)^m = xH^m \quad (m \geqq 1), \qquad H_0 = 1.$$

The connection between H_m and the Bernoulli numbers is furnished by

$$(2.7) \qquad \sum_{r=0}^{k-1} \zeta^{-r} B_m\left(\frac{r}{k}\right) = \frac{k^{1-m}\zeta}{1-\zeta} mH_{m-1}(\zeta),$$

where $\zeta^k = 1$, $\zeta \neq 1$. An immediate consequence of (2.7) is

$$(2.8) \qquad k^m B_m\left(\frac{r}{k}\right) - B_m = -m \sum_{\zeta \neq 1} \frac{1}{\zeta - 1} H_{m-1}(\zeta),$$

where ζ runs through the kth roots of unity distinct from 1.

We also mention

$$(2.9) \qquad H_m = \sum_{r=0}^{m} (x-1)^{-r} \Delta^r 0^m.$$

3. **Some preliminaries.** We shall use the notation of [2]; see in particular §2 of that paper. In addition the following remarks will be useful. Let $f(u)$ be a polynomial in q^u of degree $\leqq m$. Then the difference equation

$$(3.1) \qquad g(u+1) - cg(u) = f(u) \qquad (c \neq q^r)$$

has a unique polynomial solution $g(u)$, as can easily be proved by comparison of coefficients. To put the solution in more useful form we rewrite (3.1) as $(E-c)g(u) = f(u)$ and recall that

$$\Delta = E - 1,\ \Delta^2 = (E-1)(E-q),\ \Delta^3 = (E-1)(E-q)(E-q^2), \cdots.$$

In the identity

$$\frac{1}{t-z} = \frac{1}{t-z_1} + \frac{z-z_1}{t-z_1}\frac{1}{t-z_2} + \frac{(z-z_1)(z-z_2)}{(t-z_1)(t-z_2)}\frac{1}{t-z_3} + \cdots$$
$$+ \frac{(z-z_1)\cdots(z-z_n)}{(t-z_1)\cdots(t-z_n)}\frac{1}{t-z}$$

take $t = c$, $z = E$, $z_s = q^{s-1}$, so that we get

$$\frac{1}{c-E} = \frac{1}{c-1} + \frac{\Delta}{(c-1)(c-q)} + \frac{\Delta^2}{(c-1)(c-q)(c-q^2)} + \cdots$$
$$+ \frac{\Delta^n}{(c-1)\cdots(c-q^{n-1})}\frac{1}{c-E}.$$

Hence if we take $n>m$, we obtain the following formula for $g(u)$:

$$g(u) = \sum_{s=0}^{m} \frac{\Delta^s f(u)}{(c-1)(c-q)\cdots(c-q^s)}. \tag{3.2}$$

4. **The number** A_{ms}. We suppose A_{ms} defined by means of (1.4). Using the identity

$$(q^{m+1}-1)(q^x-1) = (q^{m+1-s}-1)(q^{x+s}-1) + q^{m+1-s}(q^s-1)(q^{x+s-m-1}-1)$$

and multiplying both members of (1.4) by $[x]$, we get

$$\begin{aligned}[x]^{m+1} &= \sum_s A_{ms}\left\{[m+1-s]\begin{bmatrix} x+s \\ m+1\end{bmatrix} + q^{m+1-s}[s]\begin{bmatrix} x+s-1 \\ m+1\end{bmatrix}\right\} \\ &= \sum_s \begin{bmatrix} x+s-1 \\ m+1\end{bmatrix}\{[m+2-s]A_{m,s-1} + q^{m+1-s}[s]A_{ms}\},\end{aligned}$$

which implies the recursion

$$A_{m+1,s} = [m+2-s]A_{m,s-1} + q^{m+1-s}[s]A_{ms}. \tag{4.1}$$

For $q=1$ it is evident that (4.1) reduces to (2.2). As an immediate consequence of (4.1) we infer that A_{ms} is a polynomial in q with positive integral coefficients.

It is easy to show that A_{ms} is divisible by $q^{(m-s)(m-s+1)/2}$. Indeed if we put

$$A_{ms} = q^{(m-s)(m-s+1)/2}A^*_{ms}, \tag{4.2}$$

then (4.1) becomes

$$A^*_{m+1,s} = [m+2-s]A^*_{m,s-1} + [s]A^*_{ms}, \tag{4.3}$$

which proves the stated property. Moreover it follows easily from (4.3) that

$$\deg A^*_{ms} = (s-1)(m-s). \tag{4.4}$$

Indeed assuming the truth of (4.4), we get

$$\begin{aligned}\deg([m+2-s]A^*_{m,s-1}) &= (m+1-s) + (s-2)(m+1-s) \\ &= (s-1)(m+1-s), \\ \deg([s]A^*_{ms}) &= (s-1) + (s-1)(m-s) = (s-1)(m+1-s),\end{aligned}$$

so that

$$\deg A^*_{m+1,s} = (s-1)(m+1-s),$$

which proves (4.4).

The symmetry properties

$$A^*_{m,m-s+1} = A^*_{ms} \tag{4.5}$$

and

$$A^*_{ms}(q) = q^{(s-1)(m-s)}A^*_{ms}(q^{-1}) \tag{4.6}$$

will be proved below.

Comparing coefficients of q^{mx} on both sides of (1.4) we get

$$\sum_{s=1}^{m} A_{ms} = [m]! = [m][m-1]\cdots[1]. \tag{4.7}$$

More generally if we expand both sides in powers of q^x and equate coefficients we get

$$\begin{bmatrix} m \\ s \end{bmatrix} \sum_{r=1}^{m} A_{mr}q^{rs} = \binom{m}{s} q^{ms-s(s-1)/2}[m]! \qquad (0 \leqq s \leqq m). \tag{4.8}$$

The following table of A^*_{ms}, $1 \leqq s \leqq m \leqq 5$, is easily computed by means of (4.3).

	1	2	3	4	5
1	1				
2	1	1			
3	1	$2(q+1)$	1		
4	1	$3q^2+5q+3$	$3q^2+5q+3$	1	
5	1	$4q^3+9q^2+9q+4$	$6q^4+16q^3+22q^2+16q+6$	$4q^3+9q^2+9q+4$	1

5. **A formula for** A^*_{mr}. It is easy to show that if $f(x)$ is a polynomial in q^x of degree $\leqq m$,

$$f(x) = \sum_{s=0}^{m} C_{ms} \begin{bmatrix} x+s-1 \\ m \end{bmatrix} \qquad (m \geqq 1), \tag{5.1}$$

then

$$C_{m0} = (-1)^m q^{m(m+1)/2} f(0), \tag{5.2}$$

$$C_{m,m-r} = \sum_{s=0}^{r} (-1)^s \begin{bmatrix} m+1 \\ s \end{bmatrix} f(r+1-s) q^{s(s-1)/2}. \tag{5.3}$$

Since

$$\sum_{s=0}^{m+1} (-1)^s q^{s(s-1)/2} \begin{bmatrix} m+1 \\ s \end{bmatrix} f(x+m+1-s) = 0,$$

we have in particular

$$\sum_{s=0}^{m+1} (-1)^s q^{s(s-1)/2} \begin{bmatrix} m+1 \\ s \end{bmatrix} f(r+1-s) = 0,$$

and (5.3) yields

$$C_{mr} = \sum_{s=0}^{r} (-1)^{m-s} q^{(m-s)(m+1-s)/2} \begin{bmatrix} m+1 \\ s \end{bmatrix} f(s-r), \tag{5.4}$$

which includes (5.2) also. Thus the coefficients in (5.1) are determined.

If we take $f(x) = [x]^m$, then $C_{mr} = A_{mr}$ and we get after a little manipulation

$$A^*_{mr} = q^{r(r-1)/2} \sum_{s=0}^{r} (-1)^s q^{s(s-1)/2} \begin{bmatrix} m+1 \\ s \end{bmatrix} [r-s]^m; \tag{5.5}$$

for $q=1$, (5.5) reduces to (2.3).

Replacing q by q^{-1}, (5.5) becomes

$$q^{(r-1)(m-r)} A^*_{mr}(q^{-1}) = q^{(r-1)m+r(r-1)/2} \sum_{s=0}^{r} (-1)^s q^{s(s-1)/2+e} \begin{bmatrix} m+1 \\ s \end{bmatrix} [r-s]^m,$$

where

$$e = -\frac{s(s-1)-(m+1)(m+2)}{2} + \frac{(m+1-s)(m+2-s)}{2} + \frac{s(s+1)}{2} - (r-s+1)m = -(r-1)m.$$

Hence

$$A^*_{mr}(q) = q^{(r-1)(m-r)} A^*_{mr}(q^{-1}),$$

which is identical with (4.6).

In the next place we observe that exactly as in the proof of (6.2) of [2] we have

$$\sum_{i=0}^{m} \binom{m}{i} [x]^{i+1} q^{(m-i)x} \frac{\eta_{m-i}}{i+1} + (q^{(m+1)x} - 1) \frac{\eta_{m+1}}{m+1} = \sum_{s=1}^{m} A_{ms} q^{m-s+1} \begin{bmatrix} x+s-1 \\ m+1 \end{bmatrix}.$$

Divide both sides of this identity by $[x]$ and then put $x=0$. We find that

$$\beta_m = \frac{1}{[m+1]} \sum_{s=1}^{m} (-1)^{m-s-1} q^{-(m-s)(m-s+1)/2} \begin{bmatrix} m \\ s-1 \end{bmatrix}^{-1} A_{ms}.$$

Using (4.2) and (4.5) this becomes

$$\beta_m = \frac{1}{[m+1]} \sum_{s=1}^{m} (-1)^{m-s-1} \begin{bmatrix} m \\ s-1 \end{bmatrix}^{-1} A^*_{ms}$$
$$= \frac{1}{[m+1]} \sum_{s=1}^{m} (-1)^{s} \begin{bmatrix} m \\ s \end{bmatrix}^{-1} A^*_{ms},$$

the first of which may be compared with (2.4).

6. **The polynomial $A_m(x)$.** The polynomial $A_m(x) = A_m(x, q)$ is defined in (1.7) for $m \geqq 1$; we put $A_0(x) = 1$. Put

$$\phi_m(x) = \prod_{s=0}^{m} (x - q^s) \tag{6.1}$$

and apply the Lagrange interpolation formula at the points $x = q^s$, $s = 0, 1, \cdots, m$. Since

$$\phi'(q^s) = \prod_{i=0}^{s-1} (q^s - q^i) \prod_{j=s+1}^{m} (q^s - q^j)$$
$$= (-1)^{m-s} q^{ms-s(s-1)/2} (q-1)^m [s]! [m-s]!,$$

we get using (4.8)

$$A_m(x) = \frac{\phi_m(x)}{(q-1)^m} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{1}{x - q^s}. \tag{6.2}$$

As a first application of (6.2) consider

$$A_m(x^{-1}, q^{-1}) = \frac{x^{-m-1} q^{-m(m+1)/2} \phi_m(x)}{q^{-m}(q-1)^m} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{x q^s}{x - q^s},$$

which gives

$$x^{m-1} q^{m(m-1)/2} A_m(x^{-1}, q^{-1}) = A_m(x, q) \qquad (m \geqq 1). \tag{6.3}$$

Substituting from (1.7) in (6.3) we get

$$q^{m(m-1)/2} \sum_{s=1}^{m} A_{ms}(q^{-1}) x^{m-s} = \sum_{s=1}^{m} A_{ms}(q) x^{s-1},$$

which implies

$$q^{m(m-1)/2} A_{ms}(q^{-1}) = A_{m,m-s+1}(q). \tag{6.4}$$

Hence by (4.2) and (4.6), (6.4) becomes

$$q^{m(m-1)/2-(m-s)(m-s+1)/2-(s-1)(m-s)} A^*_{ms}(q^{-1}) = q^{s(s-1)/2} A^*_{m,m-s+1}(q),$$

which is the same as (4.5).

7. **The functions $H_m(x)$ and $H_m(u, x)$.** Using (6.2) and (1.6) we get

$$(7.1)\qquad (q-1)^m H_m(x) = (x-1)\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\frac{1}{x-q^s}\cdot$$

We remark that (7.1) implies

$$(7.2)\qquad H_m(x) = (x-1)\sum_{r=0}^{\infty} x^{-r-1}[r]^m$$

for $|x|>|q^s|$, $0\leqq s\leqq m$. It is also evident that

$$\begin{aligned}(1+qH)^m &= (x-1)\sum_{r=0}^{m}\binom{m}{r}q^r(q-1)^{-r}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{1}{x-q^s}\\ &= (x-1)\sum_{s=0}^{m}\binom{m}{s}\frac{1}{x-q^s}\sum_{r=s}^{m}(-1)^{r-s}\binom{m-s}{r-s}q^r(q-1)^{-r}\\ &= (x-1)\sum_{s=0}^{m}\binom{m}{s}\frac{1}{x-q^s}\,\frac{q^s}{(q-1)^s}\left(\frac{-1}{q-1}\right)^{m-s}\\ &= (q-1)^{-m}(x-1)\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\frac{q^s}{x-q^s},\end{aligned}$$

which implies

$$(7.3)\qquad (1+qH)^m = xH^m \qquad (m>1).$$

We have therefore proved (1.5). Alternatively taking (7.3) as definition of H_m one can work back to the earlier formulas obtained for A_{ms} above.

For some purposes it is convenient to define $H_m(u;\,x)=H_m(u;\,x,\,q)$, a polynomial in q^u. We put

$$(7.4)\qquad (q-1)^m H_m(u;\,x) = (x-1)\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\frac{q^{su}}{x-q^s},$$

so that $H_m(0;\,x)=H_m(x)$. It follows at once from (7.4) that

$$(7.5)\qquad H_m(1-u;\,x^{-1},\,q^{-1}) = (-q)^m H_m(u;\,x,\,q)$$

and that

$$(7.6)\qquad xH_m(u;\,x) - H_m(u+1;\,x) = (x-1)[u]^m.$$

We have also

$$\sum_{r=0}^{m}\binom{m}{r}q^r H_r(u;\,x) = H_m(u+1;\,x),$$

which becomes, using (7.6),

$$(7.7)\qquad (1+qH(u;\,x))^m = xH_m(u;\,x) - (x-1)[u]^m.$$

For $u=0$, (7.7) reduces to (7.3).

Clearly (7.6) implies

$$\sum_{i=0}^{k-1} x^{k-i}[u+i]^m = x^k H_m(u;x) - H_m(u+k;x), \tag{7.8}$$

which includes (7.2) as a special case.

Since $H_m(u;x)$ is a polynomial in q^u of degree m, the remarks in §3 apply to the difference equation (7.6). In particular, application of (3.2) leads to

$$H_m(u;x) = \sum_{s=0}^{m} \frac{\Delta^s[u]^m}{\psi_s(x)} \qquad \left(\psi_s(x) = \prod_{r=1}^{s}(x-q^r)\right), \tag{7.9}$$

provided $x \neq q^r$, $r=0, 1, \cdots, m$. To simplify the right member of (7.9), we used (2.6) and (3.1) of [2]; thus

$$\Delta^s[u]^m = \sum_{r=s}^{m} q^{r(r-1)/2} a_{m,r}[r]_s[u]_{r-s} q^{s(u-r+s)}$$

and (7.9) becomes after a little manipulation

$$H_m(u;x) = \sum_{r=0}^{m} q^{r(r-1)/2} a_{m,r} \sum_{s=0}^{r} \frac{q^{s(u-r+s)}}{\psi_s(x)} [r]_s[u]_{r-s}. \tag{7.10}$$

If we let $G_r(u)$ denote the inner sum it is clear from (3.2) that

$$xG_r(u) - G_r(u+1) = [u]_r.$$

In (7.10) put $u=0$, then

$$H_m(x) = \sum_{r=0}^{m} q^{r(r-1)/2} \frac{a_{m,r}[r]!}{\psi_r(x)}, \tag{7.11}$$

which for $q=1$ reduces to (2.9).

Using (7.1) and (7.4) it is easy to verify the formula

$$H_m(u;x) = \sum_{r=0}^{m} \binom{m}{r} q^{ru} H_r [u]^{m-r} = (q^u H + [u])^m. \tag{7.12}$$

Next using (7.12), (7.11), and the explicit formula [2, (6.2)] for $a_{m,s}$ we get

$$H_m(u;x) = \sum_{r=0}^{m} \frac{1}{\psi_r(x)} \sum_{s=0}^{r} (-1)^s q^{s(s-1)/2} \begin{bmatrix} r \\ s \end{bmatrix} [u+r-s]^m, \tag{7.13}$$

which is useful later.

8. Connection with $\eta_m(u)$. Using the formula [2, (4.7)]

$$(q-1)^m \eta_m(u) = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{s}{[s]} q^{su},$$

we find that

$$(q^k - 1)^{m-1}\sum_{r=0}^{k-1} \zeta^{-r}\eta_m\left(u + \frac{r}{k}, q^k\right) = \sum_{s=0}^{m} (-1)^{m-s}\binom{m}{s}\frac{sq^{ksu}}{\zeta^{-1}q^s - 1}$$

$$= m\sum_{s=0}^{m-1} (-1)^{m-1-s}\binom{m-1}{s}\frac{\zeta q^{ksu+ku-1}}{q^s - \zeta q^{-1}},$$

where

$$\zeta^k = 1, \qquad \zeta \neq 1.$$

Comparing with (7.4) we have therefore

$$[k]^{m-1}\sum_{r=0}^{k-1} \zeta^{-r}\eta_m\left(u + \frac{r}{k}, q^k\right) = \frac{m\zeta q^{ku-1}}{1 - \zeta} H_{m-1}(ku; \zeta q^{-1}), \tag{8.1}$$

and in particular for $u=0$

$$[k]^{m-1}\sum_{r=0}^{k-1} \zeta^{-r}\eta_m\left(\frac{r}{k}, q^k\right) = \frac{m\zeta q^{-1}}{1 - \zeta} H_{m-1}(\zeta q^{-1}), \tag{8.2}$$

which may be compared with (2.7).

Next using the multiplication formula (see [2, (4.12)]; note that a term is missing in that formula)

$$[k]^{m-1}\sum_{r=0}^{k-1} \eta_m\left(u + \frac{r}{k}, q^k\right) = \eta_m(ku, q) + (-1)^m \frac{k - [k]}{(q - 1)^m} \tag{8.3}$$

together with (8.1) we get

$$\begin{aligned} k[k]^{m-1}\eta_m\left(u + \frac{r}{k}, q^k\right) - \eta_m(ku, q) - (-1)^m \frac{k - [k]}{(q - 1)^m} \\ = \frac{m}{q}\sum_{\zeta\neq 1} \frac{\zeta^{r+1}}{1 - \zeta} H_{m-1}(ku; \zeta q^{-1}), \end{aligned} \tag{8.4}$$

and in particular for $u=0$,

$$\begin{aligned} k[k]^{m-1}\eta_m\left(\frac{r}{k}, q^k\right) - \eta_m - (-1)^m \frac{k - [k]}{(q - 1)^m} \\ = \frac{m}{q}\sum_{\zeta\neq 1} \frac{\zeta^{r+1}}{1 - \zeta} H_{m-1}(\zeta q^{-1}). \end{aligned} \tag{8.5}$$

By means of (1.3) it is easy to write down formulas like (8.1), $\cdots$, (8.5) involving β_m.

9. **Multiplication formulas.** For the polynomial $H_m(u; x)$ we have, using (7.4),

$$(q^k-1)^m\sum_{r=0}^{k-1}\zeta^{-r}q^{rt}H_m\left(u+\frac{r}{k};\zeta q^{-kt},q^k\right)$$
$$=(\zeta q^{-kt}-1)\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\frac{q^{ksu}}{\zeta q^{-kt}-q^{ks}}\sum_{r=0}^{k-1}\zeta^{-r}q^{r(s+t)}$$
$$=(\zeta-q^{kt})\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}\frac{q^{ksu}}{\zeta-q^{k(s+t)}}\ \frac{1-\zeta^{-k}q^{k(s+t)}}{1-\zeta^{-1}q^{s+t}}.$$

Consequently if $\zeta^{k-1}=1$, $\zeta\neq 1$, we get

$$(9.1)\qquad [k]^m\sum_{r=0}^{k-1}\zeta^{-r}q^{rt}H_m\left(u+\frac{r}{k};\zeta q^{-kt},q^k\right)=\frac{\zeta-q^{kt}}{\zeta-q^t}H_m(ku;\zeta q^{-t},q),$$

analogous to (8.3).

In the special case $x=-q^{-1}$, the polynomial $\epsilon_m(u)$ of [2, §8] satisfies

$$(9.2)\qquad \epsilon_m(u)=H_m(u;-q^{-1},q);$$

in this case (9.1) becomes ($\zeta=-1$, $t=1$)

$$(9.3)\qquad [k]^m\sum_{r=0}^{k-1}(-q)^r\epsilon_m\left(u+\frac{r}{k},q^k\right)=\frac{q^k+1}{q+1}\epsilon_m(ku,q)$$

for k odd; note that (8.6) of [2] requires a slight correction.

10. **Staudt-Clausen theorems for β_m.** In [2] a theorem analogous to the Staudt-Clausen theorem was proved for β_m with q indeterminate. Now on the other hand we replace q by a rational number a which is assumed to be integral modulo a fixed prime p. We shall use the representation [2, (6.2)]

$$(10.1)\qquad \beta_m=\sum_{s=0}^{m}(-1)^s a_{m,s}[s]!/[s+1],$$

where

$$(10.2)\qquad a_{m,s}=\frac{q^{-s(s-1)/2}}{s!}\sum_{r=0}^{s}(-1)^r q^{r(r-1)/2}\begin{bmatrix} s\\ r\end{bmatrix}[s-r]^m;$$

the quantity $a_{m,s}$ is a polynomial in q and has occurred in (7.10) and (7.11) above.

Suppose first that $a\equiv 1 \pmod p$. Then from (10.1) or [2, §7] it is clear that the sth term in the right member of (10.1) is of the form $u_s=N_s(a)/F_{s+1}(a)$, where $F_{s+1}(x)$ is the cyclotomic polynomial and $N_s(x)$ is a polynomial with integral coefficients. If we recall that $F_k(1)=p$ when $k=p^e$, $e\geqq 1$, but $F_k(1)=1$ otherwise, it is clear that u_s is integral (mod p) except possibly when $s+1=p^e$; the same holds also for $F_k(a)$. Now let $s+1=p^e$. Then by a simple computation it is seen that $[s]!$ is divisible by exactly p^f, where

$$f = (p^e - 1)/(p - 1) - e,$$

while the denominator is divisible by exactly p^e. Since $(p^e-1)/(p-1) \geqq 2e$ for $e \geqq 2$, $p \geqq 3$, it follows that u_s is integral in this case. If $e=1$, $p \geqq 3$, we have first $p(a-1)/(a^p-1) \equiv 1 \pmod p$. As for the numerator of u_{p-1}, it follows readily from (10.2) and

$$\begin{bmatrix} p-1 \\ r \end{bmatrix} \equiv \binom{p-1}{r} \pmod p$$

that

$$\begin{aligned}[][p-1]!a_{m,p-1} &\equiv \sum_{r=0}^{p-1} (-1)^r \binom{p-1}{r} r^m \\ &\equiv \sum_{r=0}^{p-1} r^m \equiv \begin{cases} -1 \pmod p & (p-1 \mid m), \\ 0 \pmod p & (p-1 \nmid m). \end{cases}\end{aligned}$$

We have therefore proved

THEOREM 1. *Let* $p \geqq 3$, $q=a\equiv 1 \pmod p$. *Then*

$$p\beta_m \equiv \begin{cases} -1 \pmod p & (p-1 \mid m), \\ 0 \pmod p & (p-1 \nmid m). \end{cases} \tag{10.3}$$

For $p=2$, the preceding argument shows that all terms in (10.1) are integral (mod 2) except perhaps u_1 and u_3. Now

$$u_1 = \frac{a_{m,1}}{[2]} = \frac{1}{a+1},$$

while

$$u_3 = \frac{[3]!a_{m,3}}{[4]} = \frac{a^{-3}}{(a+1)(a^2+1)} \sum_{r=0}^{3} (-1)^r a^{r(r-1)/2} \begin{bmatrix} 3 \\ r \end{bmatrix} [3-r]^m.$$

Let $2^e \mid (a+1)$, $2^{e+1} \nmid (a+1)$; then $(a^2+a+1)^2 \equiv 1 \pmod{2^{e+1}}$ and

$$\begin{aligned}&\sum_{r=0}^{3} (-1)^r a^{r(r-1)/2} \begin{bmatrix} 3 \\ r \end{bmatrix} [3-r]^m \\ &\qquad \equiv (a^2+a+1)^m - (a^2+a+1)(a+1)^m + a(a^2+a+1) \\ &\qquad \equiv \begin{cases} 0 \pmod{2^{e+1}} & (m \text{ even}), \\ a+1 \pmod{2^{e+1}} & (m \text{ odd}). \end{cases}\end{aligned}$$

Consequently u_3 is integral (mod 2) for m even while for m odd $2u_3 \equiv 1 \pmod 2$. This yields the following supplement to Theorem 1.

THEOREM 2. *Let* $p=2$, $q=a\equiv 1 \pmod 2$; *also let* $2^e \mid (a+1)$, $2^{e+1} \nmid (a+1)$.

Then if $e=1$ *we have* $2\beta_m \equiv 1$ (mod 2) *for* m *even* $\geqq 2$, $2\beta_1 \equiv 1$ (mod 2), *while* β_m *is integral* (mod 2) *for* m *odd* $\geqq 3$. *If* $e>1$ *then*

$$2^e\beta_m \equiv 1 \pmod 2 \tag{10.4}$$

for all $m \geqq 1$.

In particular it is evident from (10.4) that the denominator of β_m may be divisible by arbitrarily high powers of 2.

In the next place we suppose $q=a\not\equiv 1 \pmod p$, $p>2$. It is now convenient to use [2, (5.3)]

$$(q-1)^m\beta_m = \sum_{s=0}^{m} (-1)^{m-s}\binom{m}{s}\frac{s+1}{[s+1]}\,. \tag{10.5}$$

We shall assume first that a is a primitive root (mod p^2). Clearly in the right member of (10.5) we need consider only those terms in which $p-1 \mid s+1$. Put $a^{p-1}=1+kp$, $p \nmid k$. Then

$$a^{(p-1)r} = 1 + rkp + \binom{r}{2}k^2p^2 + \cdots,$$

$$\frac{a^{(p-1)r}-1}{rp} = k + \frac{1}{2}(r-1)k^2p + \cdots \equiv k \pmod p.$$

Thus (10.5) implies

$$(a-1)^m p\beta_m \equiv (-1)^m \frac{1}{k}\sum_{r>0}\binom{m}{r(p-1)-1} \pmod p. \tag{10.6}$$

But it is known [4, p. 255] that

$$\sum_{0<r(p-1)\leqq m}\binom{m}{r(p-1)-1} \equiv \begin{cases} -1 \pmod p & (p-1 \mid m), \\ 0 \pmod p & (p-1 \nmid m). \end{cases}$$

Hence (10.6) implies that p is integral when $m\not\equiv 0 \pmod{p-1}$. This proves

THEOREM 3. *Let* $p \geqq 3$, $q=a$, *a primitive root* (mod p^2); *then* β_m *is integral* (mod p) *for* $p-1 \nmid m$, *while*

$$p\beta_m \equiv -\frac{1}{k} \pmod p \qquad (p-1 \mid m), \tag{10.7}$$

where $k=(a^{p-1}-1)/p$.

It is now clear how to handle the general situation. We may state

THEOREM 4. *Let* $p \geqq 3$, $q=a$, *where* a *belongs to the exponent* e (mod p), $e>1$. *Put*

$$a^e = 1 + p^l k \qquad (p \nmid k). \tag{10.8}$$

Then

$$(a-1)^m p^l \beta_m \equiv \frac{e}{k} \sum_{r>0} (-1)^{m-re} \binom{m}{re-1} \pmod{p}. \tag{10.9}$$

In particular if $e = p-1$, *then*

$$(a-1)^m p^l \beta \equiv \begin{cases} 0 \pmod{p} & (p-1 \nmid m), \\ -\dfrac{1}{k} \pmod{p} & (p-1 \mid m). \end{cases} \tag{10.10}$$

To prove (10.9) it is only necessary to observe that (10.8) implies

$$p^l \frac{re}{a^{re}-1} = p^l \frac{re}{(1+p^l k)^r - 1} \equiv \frac{e}{k} \pmod{p}.$$

It is clear from (10.10) that the denominator of β_m may be divisible by arbitrarily high powers of p. We also remark that theorems like Theorems 3 and 4 can be framed for η_m.

When $p^e \mid a$, it is evident from (10.5) that

$$\beta_m \equiv \sum_{s=0}^{m} (-1)^s \binom{m}{s} (s+1) \equiv 0 \pmod{p^e} \qquad (m > 1). \tag{10.11}$$

11. **Congruences.** The formula (7.11) together with (10.2) makes it possible to derive certain congruences satisfied by $H_m(x, q)$. We observe, to begin with, that if $q = a$ is integral (mod p) then $u_m = [s]! a_{m,s}$ satisfies, for s fixed and $(p-1)p^{e-1} \mid w$,

$$u^m(u^w - 1)^r \equiv 0 \pmod{p^m, p^{re}}, \tag{11.1}$$

where after expansion of the left member, u^n is replaced by u_n. To prove (11.1) we need only remark that

$$u^m(u^w-1)^r = a^{-s(s-1)/2} \sum_{r=0}^{s} (-1)^r a^{r(r-1)/2} \begin{bmatrix} s \\ r \end{bmatrix} [s-r]^m([s-r]^w - 1)^r.$$

If we look on x in (7.11) as an indeterminate and apply (11.1), we can assert that

$$H^m(H^w - 1)^r \equiv 0 \pmod{p^m, p^{re}}. \tag{11.2}$$

We interpret this congruence in the following manner. The left member of (11.2) is a rational function of x such that the coefficient of each term in the numerator $\equiv 0 \pmod{p^m, p^r}$. We may call (11.2) Kummer's congruence for H_m. Using (7.13) we can prove like results for $H_m(u; x)$, where u is now an integer.

In view of (1.6) the result (11.2) can be restated in terms of $A_m(x)$:

$$(11.3)\qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}A_{m+sw}(x)\prod_{i=m+sw+1}^{m+rw}(x-a^i)\equiv 0 \pmod{p^m,\ p^{re}}.$$

We may state

THEOREM 5. *Let $q=a$ be integral* (mod p), *x an indeterminate, and $r\geqq 1$; then* (11.3) *holds.*

In the next place (11.3) implies congruences for the $A_{m,s}$ of (1.7). (For the case $q=1$, compare [1].) Since

$$\prod_{i=m+sw+1}^{m+rw}(x-a^i)=\prod_{i=0}^{(r-s)w}(-1)^{(r-s)w-i}a^{i(i+1)/2+i(m+sw)}\begin{bmatrix}(r-s)w\\ i\end{bmatrix}x^i,$$

examination of the coefficient of x^{k-1} in (11.3) implies

$$(11.4)\qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\sum_i(-1)^{(r-s)w-i}A_{m+sw,k-i}a^{i(i+1)/2+i(m+sw)}\begin{bmatrix}(r-s)w\\ i\end{bmatrix}\equiv 0 \pmod{p^m,\ p^{re}}.$$

In order to obtain simpler results we consider some special values of the parameters. In the first place we take $r=1$, so that (11.4) becomes

$$(11.5)\qquad A_{m+w,k}-\sum_i(-1)^{w-i}A_{m,k-i}a^{i(i+1)/2+im}\begin{bmatrix}w\\ i\end{bmatrix}\equiv 0 \pmod{p^m,\ p^e}.$$

Now suppose first that $a\equiv 1 \pmod p$. It is necessary to examine

$$(11.6)\qquad \begin{bmatrix}w\\ i\end{bmatrix}=\frac{(a^w-1)\cdots(a^{w-i+1}-1)}{(a-1)\cdots(a^i-1)}.$$

We assume from now on that $p>2$. If we put $a=1+p^lh$, $p\nmid h$, then as in the proof of Theorem 4 we find that

$$\begin{bmatrix}w\\ i\end{bmatrix}\quad\text{and}\quad\binom{w}{i}$$

are divisible by exactly the same power of p. But if $i<p^j$, it is clear from the identity

$$\binom{w}{i}=\frac{w}{i}\binom{w-1}{i-1}$$

that

$$\binom{w}{i}\equiv 0 \pmod{p^{e-j}}\qquad (j\leqq e).$$

Consequently if $p^{j-1} \leqq k < p^j$ and $j < e$, we see that (11.5) implies

$$A_{m+w,k} \equiv A_{mk} \pmod{p^m, p^{e-j}}. \tag{11.7}$$

This proves

THEOREM 6. *Let* $p \geqq 3$, $q = a \equiv 1 \pmod{p}$, $p^{e-1}(p-1) \mid w$, *and* $p^{j-1} \leqq k < p^j$, *where* $j < e$. *Then* (11.7) *holds.*

When $a \not\equiv 1 \pmod{p}$ let a belong to the exponent $t \pmod{p}$. Then it is clear from (11.6) that we need only consider those factors in the right member with exponents divisible by t. Thus if i_0 is the greatest integer $\leqq i/t$ we need only examine

$$\begin{bmatrix} w/t \\ i_0 \end{bmatrix}$$

with a replaced by a^t. The preceding discussion therefore applies and we obtain the following theorem which includes Theorem 6.

THEOREM 7. *Let* $p \geqq 3$, *let* $q = a$ *belong to the exponent* $t \pmod{p}$, $p^{e-1}(p-1) \mid w$ *and* $k < tp^j$, *where* $j < e$. *Then* (11.7) *holds.*

The case $k = w$ is not covered by the theorem. We find for example that if $w = t = p - 1$ (so that a is a primitive root $\pmod{p}$), then

$$A_{m+p-1,k} \equiv \begin{cases} A_{m,k} \pmod{p} & (k < p-1), \\ A_{m,p-1} + \left(\dfrac{a}{p}\right) A_{m,0} \pmod{p} & (k = p-1), \end{cases}$$

where (a/p) is Legendre's symbol.

Returning to (11.2) we can also consider the case in which x is put equal to an integer $\pmod{p}$, provided the resulting denominators are not divisible by p. Now the least common denominator is evidently

$$\psi_{m+rw}(x) = \prod_{s=1}^{m+rw} (x - a^s).$$

It will therefore suffice to assume that $x \not\equiv a^s \pmod{p}$ for any s. We may therefore state

THEOREM 8. *Let a and x be rational numbers that are integral* $\pmod{p}$ *and suppose that* $x \not\equiv a^s \pmod{p}$ *for any s. Let*

$$p^{e-1}(p-1) \mid w \quad \text{and} \quad r \geqq 1.$$

Then

$$H^m(x)(H^m(x) - 1)^r \equiv 0 \pmod{p^m, p^{re}}. \tag{11.8}$$

In particular the theorem may be applied with slight changes to $\epsilon_m(u) = \epsilon_m(u, a)$ defined in (9.2); we have explicitly [2, (8.18)]

$$\epsilon_m(u) = \sum_{s=0}^{m} \frac{(-1)^s a^s}{(a+1)(a^2+1)\cdots(a^{s+1}+1)} \cdot \sum_{r=0}^{s} (-1)^r a^{r(r-1)/2} \begin{bmatrix} s \\ r \end{bmatrix} [u+s-r]^m,$$

which is included in (7.13). If u is an integer we have

$$\epsilon^m(u)(\epsilon^w(u) - 1)^r \equiv 0 \pmod{p^m, p^{re}}$$

provided $p \equiv 3 \pmod 4$ and a is a quadratic residue (mod p). For in this case -1 is a nonresidue (mod p) and therefore $-1 \not\equiv a^s$ for any s.

12. **Congruences involving η_m and β_m.** Let

$$(12.1) \qquad \omega_m = \omega_{m,k,r} = \frac{1}{m}\left\{ k[k]^{m-1}\eta_m\left(\frac{r}{k}, q^k\right) - \eta_m - (-1)^m \frac{k-[k]}{(q-1)^m}\right\}$$

so that by (8.5)

$$(12.2) \qquad \omega_m = \frac{1}{q} \sum_{\zeta \neq 1} \frac{\zeta^{r+1}}{1-\zeta} H_{m-1}(\zeta q^{-1}),$$

where ζ runs through the kth roots of unity distinct from 1. As for the denominators in the right member of (12.2), note that

$$\prod_{\zeta \neq 1} (a^s - \zeta) = \frac{a^{ks}-1}{a^s-1} = a^{s(k-1)} + \cdots + a^s + 1,$$

which is prime to p for all s provided $p \nmid k$. We may therefore state

THEOREM 9. *If a is integral* (mod p) *and $p \nmid k$, then*

$$(12.3) \qquad \omega^m(\omega^w - 1)^r \equiv 0 \pmod{p^{m-1}, p^{re}},$$

where ω_m is defined by (12.1) *and $p^{e-1}(p-1) \mid w$.*

As for β_m we have

$$(12.4) \qquad \begin{aligned} &k[k]^{m-1}q^r\beta_m\left(\frac{r}{k}, q^k\right) - \beta_m \\ &= \frac{(m+1)(q-1)}{q} \sum_{\zeta \neq 1} \frac{\zeta^{r+1}}{1-\zeta} H_m(\zeta q^{-1}) + \frac{m}{q} \sum_{\zeta \neq 1} \frac{\zeta^{r+1}}{1-\zeta} H_{m-1}(\zeta q^{-1}) \end{aligned}$$

analogous to (8.5). In much the same way as above (12.4) implies

THEOREM 10. *If a is integral* (mod p) *and $p \nmid k$ then*

$$(12.5) \qquad \Omega^m(\Omega^w - 1)^r \equiv 0 \pmod{p^{m-1}, p^{(r-1)e}},$$

where Ω_m *stands for the left member of* (12.4) *and* $p^{e-1}(p-1)\mid w$.

Unfortunately we seem unable to obtain simpler congruences for β_m and η_m.

13. Combinatorial interpretation of a_{msr}. Put

$$A^*_{ms} = \sum_{r=0}^{(s-1)(m-s)} a_{msr_0}q^r \qquad (r_0 = s(s-1)/2 + r).$$

The following combinatorial interpretation of the coefficients a_{msr} was kindly suggested by J. Riordan. The number a_{msr} is the number of permutations of m things requiring s *readings* and such that $r = r_2 + 2r_3 + \cdots + (s-1)r_s$, where r_k is the number of elements read on the kth reading. The following numerical illustration for $m = 4$ was also supplied by Riordan.

Permutation	Reading	s	r
1234	1234	1	0
1243 1423 4123	123|4	2	1
1324 1342 3124 3142 3412	12|34	2	2
2134 2314 2341	1|234	2	3
1432 4132 4312	12|3|4	3	3
2134 2413 2431 4213 4231	1|23|4	3	4
3214 3241 3421	1|2|34	3	5
4321	1|2|3|4	4	6

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