Bernoulli-like polynomials associated with Stirling Numbers

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The Stirling numbers of the first kind can be represented in terms of a new class of polynomials that are closely related to the Bernoulli polynomials. Recursion relations for these polynomials are given.

The Stirling numbers have wide application in science and engineering. For example, the generating function of the Stirling numbers of the first kind appears in the inversion of a Vandermonde matrix [1], which is used in curve fitting, coding theory, and signal processing [2].

In this paper we show that there is a connection between the Stirling numbers of the first kind and the Bernoulli polynomials. The unsigned Stirling numbers of the first kind $(-1)^{n-m}S_n^{(m)}$ are defined as the number of permutations of n symbols which have exactly m permutation cycles [3, 4]. Let us construct the positive integers $T_{n,k}$ by

$$T_{n,k} = (-1)^{k-1}S_n^{(n-k+1)}. \tag{1}$$

Here is a table of the numbers $T_{n,k}$:

$T_{1,k}$:	1	0	0	0	0	0	0
$T_{2,k}$:	1	1	0	0	0	0	0
$T_{3,k}$:	1	3	2	0	0	0	0
$T_{4,k}$:	1	6	11	6	0	0	0
$T_{5,k}$:	1	10	35	50	24	0	0
$T_{6,k}$:	1	15	85	225	274	120	0
$T_{7,k}$:	1	21	175	735	1624	1764	720

Note that the horizontal sums of the numbers in this table are factorials. That is, $1+1=2!$, $1+3+2=3!$, $1+6+11+6=4!$, and so on.

Let us construct a new set of polynomials $P_k(n)$ that reproduces these Stirling numbers. For example, the numbers $(0,\ 1,\ 3,\ 6,\ 10,\ \ldots)$ in the first column in this array as a function of the row label n are given by the polynomial

$$\frac{1}{2^1 1!}n(n-1)P_0(n),$$

where

$$P_0(n) = 1. \tag{2}$$

The numbers $(0,\ 0,\ 2,\ 11,\ 35,\ \ldots)$ in the next column in this array as a function of n are given by

$$\frac{1}{2^2 2!}n(n-1)(n-2)P_1(n),$$

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where

$$P_1(n) = n - \frac{1}{3}. \tag{3}$$

The numbers in the next two columns in this array are given by

$$\frac{1}{2^3 3!} n(n-1)(n-2)(n-3)P_2(n),$$

and

$$\frac{1}{2^4 4!} n(n-1)(n-2)(n-3)(n-4)P_3(n),$$

where

$$P_2(n) = n^2 - n, \tag{4}$$

and

$$P_3(n) = n^3 - 2n^2 + \frac{1}{3}n + \frac{2}{15}. \tag{5}$$

In general, the numbers along the kth column are expressible in the form

$$\frac{1}{2^k k!} n(n-1)\cdots(n-k)P_{k-1}(n).$$

The next seven polynomials $P_k(n)$ are

$$\begin{aligned}
P_4(n) &= n^4 - \tfrac{10}{3}n^3 + \tfrac{5}{3}n^2 + \tfrac{2}{3}n,\\
P_5(n) &= n^5 - 5n^4 + 5n^3 + \tfrac{13}{9}n^2 - \tfrac{2}{3}n - \tfrac{16}{63},\\
P_6(n) &= n^6 - 7n^5 + \tfrac{35}{3}n^4 + \tfrac{7}{9}n^3 - \tfrac{14}{3}n^2 - \tfrac{16}{9}n,\\
P_7(n) &= n^7 - \tfrac{28}{3}n^6 + \tfrac{70}{3}n^5 - \tfrac{56}{9}n^4 - \tfrac{469}{27}n^3 - 4n^2 + \tfrac{404}{135}n + \tfrac{16}{15},\\
P_8(n) &= n^8 - 12n^7 + 42n^6 - \tfrac{448}{15}n^5 - \tfrac{133}{3}n^4 + \tfrac{20}{3}n^3 + \tfrac{404}{15}n^2 + \tfrac{48}{5}n,\\
P_9(n) &= n^9 - 15n^8 + 70n^7 - \tfrac{266}{3}n^6 - \tfrac{245}{3}n^5 + \tfrac{745}{9}n^4 + \tfrac{1072}{9}n^3 + \tfrac{188}{9}n^2 - \tfrac{208}{9}n - \tfrac{256}{33},\\
P_{10}(n) &= n^{10} - \tfrac{55}{3}n^9 + 110n^8 - \tfrac{638}{3}n^7 - \tfrac{847}{9}n^6 + \tfrac{3179}{9}n^5 + \tfrac{968}{3}n^4 - \tfrac{1100}{9}n^3 - \tfrac{2288}{9}n^2 - \tfrac{256}{3}n.
\end{aligned} \tag{6}$$

The structure of these monic polynomials strongly resembles that of the Bernoulli polynomials $B_m(x)$ [4]. First, like the Bernoulli polynomials, every other polynomial in (2) – (6) has a constant term. Second, recall that the constant terms in the Bernoulli polynomials are Bernoulli numbers, $B_{2m}(0) = B_{2m}$, and observe that the constant terms in the polynomials $P_{2m-1}(n)$ are expressible in terms of Bernoulli numbers:

$$P_{2m-1}(0) = -\frac{1}{2m}4^m B_{2m}. \tag{7}$$

Furthermore, both the polynomials $P_m(n)$ and the Bernoulli polynomials $B_m(x)$ satisfy very similar recursion relations. The polynomials $P_m(n)$ satisfy

$$P_{2m}(n) = (2m+1)nP_{2m-1}(n) + \sum_{k=1}^{m}(-2)^k \frac{m!(2m-k+1)}{(k+1)!(m-k)!} n^{k+1} P_{2m-k-1}(n), \tag{8}$$

while the Bernoulli polynomials obey the same recursion relation with an inhomogeneous term:

$$
\begin{aligned}
B_{2m+1}(x) \;=\; & (2m+1)xB_{2m}(x)+\sum_{k=1}^{m}(-2)^k\frac{m!(2m-k+1)}{(k+1)!(m-k)!}x^{k+1}B_{2m-k}(x) \\
& +\tfrac{1}{2}(-1)^{m+1}x^{2m}.
\end{aligned}
\tag{9}
$$

Finally, we note that $P_m(n)$ satisfies a recursion relation in terms of the coefficients of the Bernoulli polynomials:

$$
\begin{aligned}
P_{m+1}(n) \;=\; & nP_m(n)-\frac{n}{m+2}\sum_{k=1}^{[m/2]}4^kP_{m+1-2k}(n)\left[\text{coefficient of } x^{m+2-2k} \text{ in } B_{m+2}(x)\right] \\
& -2^m\left[\frac{4}{m+2}B_{m+2}(0)-2nB_{m+1}(0)\right].
\end{aligned}
\tag{10}
$$

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