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A finite difference approach to degenerate Bernoulli and Stirling polynomials

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Abstract

Starting with divided differences of binomial coefficients, a class of multivalued polynomials (three parameters), which includes Bernoulli and Stirling polynomials and various generalizations, is developed. These carry a natural and convenient combinatorial interpretation. Calculation of particular values of the polynomials yields some binomial identities. Properties of the polynomials are established and several factorization results are proven and conjectured.

1. Introduction

Our study began with a binomial identity involving an alternating sum of vector space dimensions, which arose in the course of proving Bezout's Theorem. The identity led to consideration of a class of polynomials, which are best understood as higher-order divided differences of binomial coefficients. These polynomials are closely related to many of the standard polynomials of combinatorial analysis, in particular to the Stirling polynomials, which determine their coefficients. Although it was not possible to find a closed form in general, which was the original intention, calculation of particular values gave some new binomial identities and some new derivations of old ones. Consideration of 'symmetry' properties of the polynomials led to some interesting factorization questions, resulting in several theorems and conjectures.

Since writing the original version of this paper, it was called to our attention that our polynomials are closely related to arbitrary-order Bernoulli polynomials. Carlitz [4], in particular, defined the degenerate Bernoulli and Stirling polynomials, which can be brought into conformity with ours by a modification of the parameters, which we will provide in Section 7. His approach, which starts with generating functions, is quite different from ours. Our approach ends with generating functions, which are mainly used for computation. It has a more combinatorial flavor and shows considerably more of the landscape along the way.

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It is clear that these polynomials and their relatives play an important role in many combinatorial situations where inclusion–exclusion is involved [cf. 5, 11, 13]. The connection with Bernoulli polynomials caused us to shift our emphasis from binomial identities to investigating the algebraic and arithmetic properties of the polynomials themselves. Some strong factorization results are proven in Section 9, which was added to an earlier version of the paper. Further arithmetic results can be found in [1, 2].

2. Difference operators and the polynomials

Let k be a field of characteristic 0 and let $\Delta_y : k[x,y] \to k[x,y]$ by $\Delta_y(f(x,y)) = f(x+y,y) - f(x,y)$ be the *difference with increment* y.

Clearly Δ_y is linear over $k[y]$, $k[y] \subseteq \ker(\Delta_y)$, and $\mathrm{Im}(\Delta_y) \subseteq (y)$ since $\Delta_y(f(x,0)) = f(x+0,0) - f(x,0) = 0$.

Let $\nabla_y : k[x,y] \to k[x,y]$ be the *divided difference*, defined by

$$\nabla_y(f(x,y)) = \frac{\Delta_y(f(x,y))}{y} = \frac{f(x+y,y)-f(x,y)}{y} \in k[x,y]. \tag{2.1}$$

If $F(x,y) = \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} P(x+ky)$ and $A(x,y) = F(x,y)/y^s$, then

$$F(x,y) = \Delta_y^s(P(x)) \quad \text{and} \quad A(x,y) = \nabla_y^s(P(x)). \tag{2.2}$$

(It is crucial that $\Delta_y(P(x)) = P(x+y) - P(x)$ and $\nabla_y(P(x)) = (P(x+y) - P(x))/y$.)

If $c \in k$, $\Delta_c(P(x)) = P(x+c) - P(x)$ and $\nabla_c(P(x)) = (P(x+c) - P(x))/c$ establishes conformity with the standard difference and divided difference with increment c, but we are particularly interested in polynomial algebra.

Thus $\Delta_1 = \Delta$ is the standard forward difference operator, and $F(x,1) = A(x,1) = \Delta^s(P(x)) = \nabla^s(P(x))$ is the sth forward difference of $P(x)$.

Similarly $\Delta_{-1}(P(x)) = P(x-1) - P(x)$ and $\nabla_{-1}(P(x)) = P(x) - P(x-1)$ is the backward difference, so $A(x,-1) = \nabla_{-1}^s(P(x)) =$ sth backward difference of $P(x)$.

$\nabla_0(P(x))$, which is obtained by setting $y=0$ in the polynomial $\nabla_y(f(x,y))$, can be written as $(P(x+0) - P(x))/0$, and is just $P'(x)$; similarly, $A(x,0) = P^{(s)}(x)$.

These are polynomial derivatives, so they do not necessarily involve limits. Since the characteristic is 0, $\deg(P(x)) = r \Rightarrow \deg(P^{(s)}(x)) = r-s$, so $\deg(A(x,y)) = r-s$ and $\deg(F(x,y)) = r$ if $r \geqslant s$, while $A(x,y) = F(x,y) = 0$ if $r < s$.

The remainder of this paper is devoted to the special case $P(x) = \binom{x}{r} = (x)_r/r!$.

Remark 2.1. The polynomial

$$F(x,y) = \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} \binom{x+ky}{r}$$

has an important combinatorial interpretation: Let a set S be partitioned into a subset X and subset $Y_1, \ldots, Y_s$ with $|X|=x$ and $|Y_i|=y$ for each i, including the possibilities $x=0$ or $y=0$. Then $F(x, y)$ is the number of r-subsets of S which meet each of the Y_i. The proof is an immediate application of the inclusion–exclusion principle, where subsets are classified by how many of the Y_i that they meet.

Let

$$A_{n,s}(x,y)=\nabla_y^s\binom{x}{s+n}=\left(\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}\binom{x+ky}{s+n}\right)\Big/y^s$$

$$\text{for } n,\ s=0,1,2,\ldots\ . \tag{2.3}$$

($A_{n,s}=0$ for $n=-1,-2,\ldots$ is consistent with the preceding analysis.)

Then $A_{n,s}(x,y)$ is a polynomial of degree n.

Since $\binom{x+1}{r}-\binom{x}{r}=\binom{x}{r-1}$ is well known,

$$A_{n,s}(x,1)=\Delta^s\tbinom{x}{s+n}=\tbinom{x}{n}, \tag{2.4}$$

$$A_{n,s}(x,-1)=\binom{x-s}{n}, \tag{2.5}$$

and

$$A_{n,s}(x,0)=\binom{x}{s+n}^{(s)}. \tag{2.6}$$

Let

$$B_{n,s}(y)=A_{n,s}(0,y)=\left(\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}\binom{ky}{s+n}\right)\Big/y^s. \tag{2.7}$$

Then

$$B_{n,s}(1)=\binom{0}{n}=\delta_{0n}, \tag{2.8}$$

$$B_{n,s}(-1)=\binom{-s}{n}=(-1)^n\binom{s+n-1}{n}, \tag{2.9}$$

and

$$B_{n,s}(0)=\binom{0}{s+n}^{(s)}=\binom{x}{s+n}^{(s)}_{x=0}. \tag{2.10}$$

Finally,

$$A_{0,s}(x,y)=B_{0,s}(y)=\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}\binom{k}{s}=1, \tag{2.11}$$

$$A_{n,0}(x,y)=\binom{x}{n} \quad \text{and} \quad B_{n,0}(y)=\delta_{n0}. \tag{2.12}$$

3. Formulas related to the Vandermonde convolution

The *Vandermonde Convolution formula* is

$$\binom{x+y}{r}=\sum_{l=0}^{r}\binom{x}{l}\binom{y}{r-l}.$$

This is equivalent to the equations

$$\nabla_y\binom{x}{y}=\sum_{l=0}^{r-1}\binom{x}{l}\binom{y}{r-l}\Big/y=\sum_{l=0}^{r-1}\binom{x}{l}\binom{y-1}{r-l-1}\Big/(r-l).$$

Iterating this last formula gives

$$\nabla_y^s\binom{x}{r}=\sum_{l=0}^{r-s}\binom{x}{l}\sum\frac{\binom{y}{n_1}\cdots\binom{y}{n_s}}{y^s}$$

over all $n_1,\dots,n_s>0$ such that $\sum n_k=r-l$.

We can then deduce the following theorem.

Theorem 3.1.

(i) $$B_{n,s}(y)=\sum\frac{\binom{y}{n_1}\cdots\binom{y}{n_s}}{y^s}=\sum\frac{\binom{y-1}{n_1-1}\cdots\binom{y-1}{n_s-1}}{n_1\cdots n_s}=\sum\frac{\binom{y-1}{m_1}\cdots\binom{y-1}{m_s}}{(m_1+1)\cdots(m_s+1)},$$

where $n_k>0$, $\sum n_k=s+n$ and $m_k=n_k-1$, $\sum m_k=n$, $m_k\geqslant 0$.

(ii) $$A_{n,s}(x,y)=\sum_{l=0}^{n}\binom{x}{l}B_{n-l,s}(y).$$

(iii) $$B_{n,s}(y)=\sum\binom{s}{t_0\dots t_n}\left(\frac{\binom{y-1}{0}}{1}\right)^{t_0}\cdots\left(\frac{\binom{y-1}{n}}{n+1}\right)^{t_n},$$

summed over all $t_i\geqslant 0$ where $\sum it_i=n$ and $\sum t_i=s$.

(iv) $$B_{n,s}(y)=\sum\binom{s}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-1}}{r_n}\left(\frac{y-1}{2}\right)^{r_1}\cdots\left(\frac{y-n}{n+1}\right)^{r_n},$$

summed over all $r_1\geqslant r_2\geqslant\cdots\geqslant r_n\geqslant 0$ where $\sum r_i=n$.

Proof. (i) and (ii) restate the preceding formula with $r=s+n$. Their combinatorial meaning is clear from Remark 2.1, after multiplying by y^s. (iii) follows from (i) by collecting the terms where $\binom{y-1}{m}/(m+1)$ occurs t_m times. (iv) follows from (i) by factoring $\binom{y-1}{m}/(m+1)=((y-1)/2)((y-2)/3)\cdots$ and then collecting terms. □

Corollary 3.2. (i) *The highest coefficient of* $B_{n,s}(y)$ *is*

$$\sum \prod_{k=1}^{s} \frac{1}{n_k!} = \sum \binom{s}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-1}}{r_n}\frac{1}{2^{r_1}\cdots(n+1)^{r_n}},$$

where the indices are as above.

(ii) *The constant term of* $B_{n,s}(y)$ *is*

$$(-1)^n \sum \prod_{k=1}^{s} \frac{1}{n_k} = (-1)^n \sum \binom{s}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-1}}{r_n}\left(\frac{1}{2}\right)^{r_1}\left(\frac{2}{3}\right)^{r_2}\cdots\left(\frac{n}{n+1}\right)^{r_n}.$$

Note 3.3. Formula (Theorem 3.1(iv)) is particularly important, since it shows that $B_{n,s}(y) = B_n(y,s)$ is a polynomial of degree n in s, with highest coefficient $(1/n!)((y-1)/2)^n$, as is $A_{n,s}(x,y) = A_n(x,y,s)$, with the same highest coefficient. This will be exploited later, where it will be important to treat x, y and s as variables.

The following special cases can be easily derived from the preceding theorem.

$(s=1)$:

$$B_{n,1}(y) = \frac{\binom{y}{n+1}}{y} = \frac{\binom{y-1}{n}}{n+1}$$

and

$$A_{n,1}(x,y) = \sum_{l=0}^{n} \binom{x}{l}\binom{y}{n-l+1}\Big/ y = \sum_{l=0}^{n} \frac{\binom{x}{l}\binom{y-1}{n-l}}{n-l+1}. \tag{3.1}$$

$(n=1)$:

$$B_{1,s}(y) = s\left(\frac{y-1}{2}\right) \quad \text{and} \quad A_{1,s}(x,y) = x + \frac{s(y-1)}{2}. \tag{3.2}$$

If m is a positive integer, then

$$B_{n,s}(m) = \sum \binom{s}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-1}}{0}\left(\frac{m-1}{2}\right)^{r_1}\cdots\left(\frac{1}{m}\right)^{r_{m-1}} \quad \text{where } \sum_{i=1}^{m-1} r_i = n. \tag{3.3}$$

We can deduce the following (with $n, s = 0, 1, 2, \ldots$):

$$B_{n,s}(1) = 0 \quad \text{if } n \geqslant 1. \tag{3.4}$$

(In fact $y=1$ is a simple zero of $B_{n,s}(y)$ since it is a simple zero of the term where all $r_i = 1$ and a multiple zero of all other terms.)

$B_{n,s}(y) > 0$ if $y > n$ and $s > 0$; $B_{n,s}(n) > 0$ if $n > 1$ and $s > 1$, and the sign of $B_{n,s}(y)$ is $(-1)^n$ if $y < 1$, so in particular if $s > 0$ then $B_{n,s}(y)$ has no zeros $y < 1$. (3.5)

If m is a positive integer then $B_{n,s}(m) \geqslant 0$ and $B_{n,s}(m) = 0$ iff $s \leqslant \left[\dfrac{n-1}{m-1}\right]$. (3.6)

(Since $s \geqslant r_1 \geqslant \cdots \geqslant r_{m-1}$ and $\sum r_k = n$, there is a nonzero term iff $(m-1)s \geqslant n$.)

$$B_{n,s}(2) = \binom{s}{n} 2^{-n}. \tag{3.7}$$

(This says $\sum(-1)^{s-k}\binom{s}{k}\binom{2k}{s+n} = \binom{s}{n}2^{s-n}$, which is well known (cf. [8, 15]); apparently there is no simple formula for $A_{n,s}(x,2)$.)

$$B_{n,s}(3) = \sum \binom{s}{r}\binom{r}{n-r} 3^{r-n}. \tag{3.8}$$

(This says $\sum(-1)^{s-k}\binom{s}{k}\binom{3k}{s+n} = 3^{s-n} \sum_r \binom{s}{r}\binom{r}{n-r}3^r$, which may be new.)

4. The coefficients

Recall the definition of the Stirling numbers:

$(x)_n = \sum S_1(n,k)x^k$ defines the *Stirling numbers of the first kind*,

$x^n = \sum S_2(n,k)(x)_k$ defines the *Stirling numbers of the second kind.*

Obviously the triangular unit diagonal matrices $(S_1(n,k))$ and $(S_2(n,k))$ are inverse to each other.

Clearly $S_1(n,k) = (-1)^{n-k}\sum r_1 \cdots r_{n-k}$ where $0 < r_1 < \cdots < r_{n-k} < n$. It is well known and easy to establish that $S_2(n,k) = (1/k!)\sum_i (-1)^{k-i}\binom{k}{i} i^n$.

Let $A_{n,s}(x,y) = \sum_{i+j \leqslant n} a_{ij} y^i x^j$ where $a_{ij} = a_{ijn}(s)$, and $B_{n,s}(y) = \sum_{i=0}^n b_i y^i$ where $b_i = b_{in}(s)$.

Theorem 4.1. $a_{ij} = (s!/(s+n)!)\binom{s+i+j}{j} S_2(s+i,s) S_1(s+n, s+i+j)$.

Proof.

$$\begin{aligned}
\sum_k (-1)^{s-k}\binom{s}{k}(x+ky)_r &= \sum_k (-1)^{s-k}\binom{s}{k}\sum_v S_1(r,v)(x+ky)^v \\
&= \sum_k (-1)^{s-k}\binom{s}{k}\sum_v S_1(r,v)\sum_{j+\lambda=v}\binom{v}{j}x^j(ky)^\lambda \\
&= \sum_{j,\lambda} S_1(r,j+\lambda)\binom{j+\lambda}{j}\left(\sum_k (-1)^{s-k}\binom{s}{k}k^\lambda\right)y^\lambda x^j \\
&= s!\sum_{j,\lambda} S_1(r,j+\lambda)\binom{j+\lambda}{j} S_2(\lambda,s) y^\lambda x^j.
\end{aligned}$$

Now divide by $r!\,y^s$ where $s = r-n$ and set $i = \lambda - s$. $\square$

Corollary 4.2. $a_{0j} = S_1(s+n, s+j)/j!(s+n)_{n-j}$.

Since we have previously noted that a_{0j} is a polynomial in s, it follows that $S_1(s+n, s+j)$ is a polynomial in s, divisible by $(s+n)_{n-j}$, which we will return to shortly in our discussion of Stirling polynomials.

Corollary 4.3. $b_i = S_2(s+i, s)S_1(s+n, s+i)/(s+n)_n$.

In particular, the highest coefficient is $b_n = S_2(s+n, s)/(s+n)_n$, so we now see that $S_2(s+n, s)$ is a polynomial in s, divisible by $(s+n)_n$.

Hence by Corollary 3.2

$$S_2(s+n, s) = (s+n)_n \sum \binom{s}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-1}}{r_n}\frac{1}{2^{r_1}3^{r_2}\cdots(n+1)^{r_n}}. \tag{4.1}$$

Finally, the constant coefficients $b_0 = S_1(s+n, s)/(s+n)_n$ so

$$S_1(s+n, n) = (-1)^n (s+n)_n \sum \binom{s}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{n-1}}{r_n}\left(\frac{1}{2}\right)^{r_1}\left(\frac{2}{3}\right)^{r_2}\cdots\left(\frac{n}{n+1}\right)^{r_n}. \tag{4.2}$$

Now, recall the definitions of the *Stirling polynomials* mentioned above (cf. [7, 12])

$$f_n(s) = S_2(s+n, s) \quad \text{and} \quad g_n(s) = (-1)^n S_1(s, s-n). \tag{4.3}$$

It follows from the preceding discussion that $f_n(s)$ is a polynomial of degree $2n$ divisible by $(s+n)_n$ and that $g_n(s)$ is a polynomial of degree $2n$ divisible by $(s)_n$. Since $B_{n,0}(y) = 0$ if $n > 0$, $s \mid a_{n0}$ whence $s \mid f_n(s)$ and $s \mid a_{00}$ whence $s-n \mid g_n(s)$ for $n > 0$.

The standard difference equations, $\Delta f_k(n) = (n+1)f_{k-1}(n+1)$ and $\Delta g_k(n) = n g_{k-1}(n)$ are easy to establish. These, with the initial conditions $f_0(n) = g_0(n) = 1$ and $f_n(0) = g_n(0) = \delta_{n0}$, determine the Stirling polynomials. They imply the duality relation $g_n(-s) = f_n(s)$.

Write $A_{n,s}(x, y) = \sum a_{ijn}(s) y^i x^j$ and $B_{n,s}(y) = \sum b_{in}(s) y^i$ to exhibit explicitly dependence on the parameters. We can restate Theorem 4.1 and its corollaries as

$$b_{in}(s) = (-1)^{n-i}\frac{f_i(s)g_{n-i}(s+n)}{(s+n)_n} = b_{ii}(s)b_{0,n-i}(s+i), \tag{4.4}$$

since

$$b_{nn}(s) = \frac{f_n(s)}{(s+n)_n} \quad \text{and} \quad b_{0n}(s) = (-1)^n \frac{g_n(s+n)}{(s+n)_n}. \tag{4.5}$$

Similarly,

$$a_{ijn}(s) = (-1)^{n-i-j}\frac{\binom{s+i+j}{j}}{(s+n)_n} f_i(s) g_{n-i-j}(s+n) = \frac{b_{ii}(s)b_{0,n-i-j}(s+i+j)}{j!}. \tag{4.6}$$

The following corollary generalizes the duality of the Stirling polynomials.

Corollary 4.4. $s b_{in}(-(s+n)) = (s+n) b_{n-i,n}(s)$.

Proof. The left-hand side equals

$$\frac{s(-1)^{n-i}f_i(-(s+n))g_{n-i}(-s)}{(-s)_n}=\frac{(s+n)(-1)^i g_i(s+n)f_{n-i}(s)}{(s+n)_n},$$

which is the right-hand side. $\square$

Translating the preceding corollary in terms of our original polynomials yields the following.

Corollary 4.5. $(s+n)B_n(\frac{1}{y},s)y^n=sB_n(y,-(s+n))$.

This remarkable corollary deserves a prominent place on the list of symmetries that will be given later. As an important special case, $s=1$ gives

$$B_n(y,-(n+1))=(n+1)y^n\binom{\frac{1}{y}-1}{n}\Big/(n+1)=\frac{(1-y)(1-2y)\cdots(1-ny)}{n!}. \tag{4.7}$$

The standard proofs of this corollary use Lagrange inversion (cf. [6]). This formula will be generalized in Section 5.

There is an interesting skew-symmetry that occurs in $B_{n,s}(y)$ when $s=-n/2$. In this case, $s+n=-s$, so Corollary 4.4 gives

$$b_{n-i,n}(s)=-b_{in}(s) \quad \text{if } s=-n/2. \tag{4.8}$$

In particular, if n is even, then the middle coefficient

$$b_{n/2,n}(-n/2)=0.$$

As an example, calculated with Macsyma,

$$B_8(y,-4)=(-1/725\,760)(199y^8+792y^7-4862y^6-13608y^5$$
$$+13608y^3+4862y^2-792y-199).$$

5. Symmetries

The following properties of the polynomials are useful and easy to verify.

Theorem 5.1.

(Si) $A_{n,s}(x+y,y)=A_{n,s}(x,y)+yA_{n-1,s+1}(x,y)$,

(Sii) $A_{n,s}(x,y)=A_{n,s}(x+sy,-y)$,

(Siii) $A_{n,s}(x,y)=(-1)^n A_{n,s}(-x+s+n-1,-y)$,

(Siv) $A_{n,s}(x+x',y)=\sum_l \binom{x'}{l} A_{n-l,s}(x,y)$,

(Sv) $B_n(y,s+t)=\sum_l B_l(y,s)B_{n-l}(y,t)$,

(Svi) $(s+n)A_n(x,y,s)=xA_{n-1}(x-1,y,s)+sA_n(x+y-1,y,s-1)$.

Proof.

(Si) It says that $\nabla_y(A_{n,s})=A_{n-1,s+1}$ which is true since $A_{n,s}=\nabla_y^s\binom{x}{s+n}$.

(Sii) It is true since the right-hand side equals

$$\sum_{l=0}^{s}(-1)^{s-l}\binom{s}{l}\binom{x+sy-ly}{s+n}\Big/(-y)^s=\sum_{k=0}^{s}(-1)^{s-k}\binom{s}{k}\binom{x+ky}{s+n}\Big/y^s,$$

where $k=s-l$.

(Siii) It is true since

$$(-1)^n\binom{-x+s+n-1-ky}{s+n}\Big/(-y)^s=\binom{x+ky}{s+n}\Big/y^s.$$

(Siv) It comes from the Vandermonde convolution.

(Sv) It follows from (i) of Theorem 3.1.

(Svi) It is true since the left-hand side equals

$$(s+n)\sum_k(-1)^{s-k}\binom{s}{k}\binom{x+ky}{s+n}\Big/y^s=\sum_k(-1)^{s-k}\binom{s}{k}\binom{x+ky-1}{s+n-1}(x+ky)/y^s$$

$$=x\sum(-1)^{s-k}\binom{s}{k}\binom{x+ky-1}{s+n-1}\Big/y^s$$

$$+s\sum(-1)^{s-k}\binom{s-1}{k-1}\binom{x+y-1+(k-1)y}{s-1+n}\Big/y^{s-1},$$

which is the right-hand side. □

Remark 5.2. All but (Sii) and (Siii) have combinatorial interpretations along the lines of Remark 2.1. To illustrate this, multiply symmetry (Svi) by y^s. The left-hand side then counts the number of $(s+n)$-subsets of S in which an element of each subset is distinguished, while the two terms of the right-hand side give the count where the distinguished element is or is not in X respectively.

Many other useful properties may be deduced by judicious substitutions, e.g., put $x=0$ and $x=-y$ in (Si) respectively to get

(Svii) $A_{n,s}(y,y)=B_{n,s}(y)+yB_{n-1,s+1}(y)$ and

(Sviii) $B_{n,s}(y)=A_{n,s}(-y,y)+yA_{n-1,s+1}(-y,y)$.

Put $x=0$ in (Sii) and (Siii) respectively to get

(Six) $B_{n,s}(y)=A_{n,s}(sy,-y)$ and

(Sx) $B_{n,s}(y)=(-1)^nA_{n,s}(s+n-1,-y)$.

Put $s=-n$ in (Svi) to get

(Sxi) $xA_{n-1}(x-1,y,-n)-nA_n(x+y-1,y,-n-1)=0$.

Put $x=0$, 1, and $1-y$ into (Svi) and (Sxi), respectively to get

(Sxii) (a) $(s+n)B_n(y,s)=sA_n(y-1,y,s-1)$ and $A_n(y-1,y,-n-1)=0$ if $n>0$,
(b) $B_{n-1}(y,s)=(s+n)A_n(1,y,s)-sA_n(y,y,s-1)$
and $B_{n-1}(y,-n)=nA_n(y,y,-n-1)$,
(c) $sB_n(y,s-1)=(s+n)A_n(1-y,y,s)-(1-y)A_{n-1}(-y,y,s)$
and $nB_n(y,-n-1)=(1-y)A_{n-1}(-y,y,-n)$.

For completeness, we include the symmetry noted in Section 4, the duality formula

(Sxiii) $(s+n)y^nB_n(\frac{1}{y},s)=sB_n(y,-(s+n))$.

We can then deduce from (Sxii(a)) and (Sxiii)

(Sxiv) $B_n(y,-(s+n))=y^nA_n((1/y)-1,1/y,s-1)$.

Finally, we can generalize Corollary 4.5 to give

(Sxv) $A_n(x,y,-(s+n))=y^nA_n((x+1)/y)-1,1/y,s-1)$.

Proof. Since (Sxiv) is the special case $x=0$, it suffices to assume true for x and deduce for $x+1$. But

$$\begin{aligned}A_n(x+1,y,-(s+n))&=A_n(x,y,-(s+n))+A_{n-1}(x,y,-(s+n)) \quad \text{by (Siv)}\\&=y^nA_n\left(\frac{x+1}{y}-1,\frac{1}{y},s-1\right)+y^{n-1}A_{n-1}\left(\frac{x+1}{y}-1,\frac{1}{y},s\right)\\&=y^nA_n\left(\frac{x+2}{y}-1,\frac{1}{y},s-1\right) \quad \text{by (Si).} \qquad \square\end{aligned}$$

Corollary 5.3. *If s is a positive integer, then*

$$A_n(x,y,-(s+n))=y^{s+n-1}\sum_k(-1)^{s-1-k}\binom{s-1}{k}\binom{(x+1+k-y)/y}{s+n-1}.$$

As a special case of this corollary, take $s=1$ to get the following corollary.

Corollary 5.4.

$$A_n(x,y,-n-1)=(-y)^n\binom{n-(1+x)/y}{n}=\frac{(1+x-y)(1+x-2y)\cdots(1+x-ny)}{n!}.$$

6. Further consideration of the coefficients and some factoring results

Lemma 6.1. *If n is odd then $B_{n,s}((s+n-1)/s)=B_{n,s}((s+n)/(s+1))=0$.*

Proof. If $sy=s+n-1$, then by symmetries (Six) and (Sx)

$$B_{n,s}(y)=A_{n,s}(sy,-y)=A_{n,s}(s+n-1,-y)=(-1)^nB_{n,s}(y)=-B_{n,s}(y),$$

which implies that $B_{n,s}(y)=0$. Similarly, if $(s+1)y=s+n$, use symmetries (Sxii a), (Sii), and (Siii) as well to get

$$\begin{aligned}B_{n,s}(y)&=\frac{s}{s+n}A_{n,s-1}(y-1,y)\\&=\frac{s}{s+n}A_{n,s-1}((s-1)y+y-1,-y)\\&=\frac{s}{s+n}A_{n,s-1}(s+n-2+(1-y),-y)\\&=(-1)^n\frac{s}{s+n}A_{n,s-1}(y-1,y)=(-1)^nB_{n,s}(y),\end{aligned}$$

which again implies that $B_{n,s}(y)=0$. □

But $B_{n,s}(y)=B_n(y,s)$ is a polynomial in s and y and $s\,|\,B_{n,s}(y)$ if $n>0$, so if n is odd, $sy-(s+n-1)\,|\,B_{n,s}(y)$ and if n is odd >1, then $(s+1)y-(s+n)\,|\,B_{n,s}(y)$ whence

$$s(y-1)(sy-(s+n-1))((s+1)y-(s+n))\,|\,B_{n,s}(y) \quad \text{if } n \text{ is odd} > 1. \tag{6.1}$$

Thus if n is odd >1, then

$$s^2(s+1)\,|\,b_{nn}(s) \tag{6.2}$$

and

$$s(s+n-1)(s+n)\,|\,b_{0n}(s). \tag{6.3}$$

Since $b_{nn}(s)=f_n(s)/(s+n)_n$, this says that $s=-1$ and $s=0$ are at least double roots of $f_n(s)$ for odd $n>1$; similarly $b_{0n}(s)=(-1)^n g_n(s+n)/(s+n)_n$, so $s=1$ and $s=0$ are multiple roots of $g_n(s)$ for odd $n>1$, which is clearly an equivalent statement.

Furthermore, since $b_{in}(s)=b_{ii}(s)b_{0,n-i}(s+i)$ and $f_n(-n)=0=g_n(n)$ for $n>0$, we get the following list of factoring rules.

Theorem 6.2. *Let* $b_i=b_{in}(s)$ *as usual.*

(i) *If* $n>0$, *then* $s\,|\,b_i$.
(ii) *If* $i<n$, *then* $s+i\,|\,b_i$.
(iii) *If* i *is odd* >1, *then* $s^2(s+1)\,|\,b_i$ (*while if* $i=1$ *and* $n>1$, *only* $s(s+1)\,|\,b_i$).
(iv) *If* $n-i$ *is odd* >1, *then* $(s+n-1)(s+n)\,|\,b_i$.
(v) $3s+1\,|\,b_2$ *and* $3s+3n-1\,|\,b_{n-2}$.

Proof. The first four results follow from (4.4) and Corollary 4.4. For (v), observe that $b_{22}(s)=\frac{1}{24}s(3s+1)$ and $b_{02}(s)=\frac{1}{24}s(3s+5)$, by direct calculation. □

Linear factors of the b_n's correspond to rational roots of the '*normalized*' Stirling polynomials $f_n(s)/(s+n)_n$. We have used Macsyma to calculate and factor these polynomials up to $n=16$. Up to that point the above factoring rules and obvious manipulations account for all rational factoring. It is tempting to conjecture that this is always the case, i.e., that the only rational roots of the normalized Stirling polynomials are $s=0$ for $n>0$, $s=-1$ for n odd >1, $s=0$ double for n odd >1, and $s=-\frac{1}{3}$ when $n=2$, and that dividing by the first degree factors leaves an irreducible polynomials over the rationals.

Observe that $sy+2x-(s+n-1)\,|\,A_n(x,y,s)$ for n odd. This follows from symmetries (Sii) and (Siii) with $sy=s+n-1-2x$.

A useful theorem concerning rational factorization is the following.

Theorem 6.3. *Let $n+1$ be an odd prime p. Then the following rational polynomials are irreducible*:

(i) $b_{nn}(s)/s=f_n(s)/(s+n)_{n+1}$ *and* $b_{0n}(s)/s=(-1)^n g_n(s+n)/(s+n)_{n+1}$,

(ii) $B_n(y,s)/s(y-1)$ *and* $B_n(a,s)/s$ *if* $a\in\mathbb{Z}$ *and* $p\,|\,a$,

(iii) $A_n(x,y,s)$ *and* $A_n(x,a,b)$ *if* $a,b\in\mathbb{Z}$ *and* $p\,|\,a$ *and* $p\nmid b$,

(iv) $A_n(-1,0,s)/(s+n+1)$.

Proof. Use Eisenstein's criterion with $p=n+1$, concentrating on the single term $s(y-1)\cdots(y-n)/(n+1)!$ in each part. For (i) and (ii), recall the formulas for $B_{n,s}(y)$ and $b_{nn}(s)$ given in Theorem 3.1(iv) and in Corollary 3.2. Observe that s^n occurs only in the term where $r_1=n$ and p occurs only in the term where $r_1=1$. Hence if we clear denominators we get, in each case, a polynomial whose top coefficient is divisible by p but not by p^2, bottom coefficient is prime to p, and all other coefficients are divisible by p. Hence $b_{nn}(s)/s$ and $B_n(y,s)/s(y-1)$ are p-Eisenstein. (Observe for the latter that the coefficients, which are in $Z[y]$, are relatively prime, since the top coefficient is a power of $y-1$ whereas the constant coefficient is prime to $y-1$.) For (iii), use Theorem 3.1(ii). Observe that x^n occurs only when $l=n$ and that p occurs only when $l=0$. Hence if we clear denominators, the resulting polynomial in x with coefficients in $Z[s,y]$ is again p-Eisenstein. For the special values $y=a$ and $s=b$, the assumptions guarantee that $p\nmid b(a-1)\cdots(a-n)$, so again p-Eisenstein considerations apply. Similarly for (iv), $A_n(-1,0,s)$ is divisible by $s+n+1$ by Corollary 5.4 and has an irreducible factor of degree $\geqslant n-1$. □

Carlitz has a proof of a variant of (iv) in [3]. His method, which relies on the Clausen–Staudt Theorem, appears to be less elementary than ours. An analysis of p-Eisenstein occurrences of the polynomials $A_n(x,0,b)$ will be found in [1].

Remark 6.4. The preceding theorem refers to *rational* irreducibility. We have recently discovered proofs of the *absolute* irreducibility of some multi-variable polynomials related to $A_n(x,y,s)$. See Section 9.

7. The generating functions

The various product representations that have appeared in formulas for $A_{n,s}(x,y)$ and $B_{n,s}(y)$ indicate that there is a *generating function* which is power function. That is indeed the case:

Let $B_s(y;t)=\sum_{n=0}^{\infty}B_{n,s}(y)t^n$ be the generating function for $B_{n,s}(y)$ and let $A_s(x,y;t)=\sum_{n=0}^{\infty}A_{n,s}(x,y)t^n$ be the generating function for $A_{n,s}(x,y)$, each with respect to n. Then we have the following theorem.

Theorem 7.1.
(i) $[B_s(y;t)=(\frac{(1+t)^y-1}{ty})^s,]$
(ii) $[A_s(x,y;t)=(1+t)^x(\frac{(1+t)^y-1}{ty})^s=(1+t)^x B_s(y;t).]$

Proof.

$$\frac{(1+t)^y-1}{ty}=\frac{\sum_{n=0}^{\infty}\binom{y}{n}t^n-1}{ty}=\sum_{n=0}^{\infty}\frac{\binom{y}{n+1}}{y}t^n=\sum_{n=0}^{\infty}\frac{\binom{y-1}{n}}{n+1}t^n.$$

Hence the assertions are true if s is a nonnegative integer by Theorem 3.1, and since we are dealing with polynomial functions of s, they remain true in general. □

The generating function is particularly useful for negative integer and fractional values of s. The results established thus far which are of polynomial character in s remain true if s is a variable by the principle of 'prolonging algebraic identities'.

Almost without exception, the properties and computations done to this point can be handled at least as easily using generating functions.

Consider some sample calculations and derivations. For example, when $y=0$ we have

$$B_s(0;t)=\left(\sum_{n=0}^{\infty}\frac{\binom{-1}{n}}{n+1}t^n\right)^s=\left(\sum_n(-1)^n\frac{t^n}{n+1}\right)^s=\left(\frac{\ln(1+t)}{t}\right)^s, \tag{7.1}$$

and the generating function

$$A_s(x,0;t)=(1+t)^x\left(\frac{\ln(1+t)}{t}\right)^s, \tag{7.2}$$

the latter being particularly important for the study of arbitrary-order Bernoulli polynomials.

As another example, $y=-1$ gives

$$B_s(-1;t)=\left(\frac{(1+t)^{-1}-1}{-t}\right)^s=(1+t)^{-s},$$

which implies that $B_{n,s}(-1)=\binom{-s}{n}$ and $A_s(x,-1;t)=(1+t)^{x-s}$, which in turn implies that $A_{n,s}(x,-1)=\binom{x-s}{n}$.

Symmetries: To drive symmetry (Svi), use logarithmic differentiation of the generating function to obtain

$$\frac{\partial A_s(x,y;t)}{\partial t}=xA_s(x-1,y;t)+\frac{s}{t}[A_{s-1}(x+y-1,y;t)-A_s(x,y;t)].$$

Equating coefficients of t^{n-1} yields

$$nA_n(x,y,s)=xA_{n-1}(x-1,y,s)+s[A_n(x+y-1,y,s-1)-A_n(x,y,s)]. \qquad \square$$

We can also use the generating functions to find new symmetries. For example, continuing the numbering of Theorem 5.1 we get the following formula for $A_{n,s}(x,y+1)$ if s is a nonnegative integer:

(Sxvi) $(y+1)^s A_{n,s}(x,y+1)=\sum_{l=0}^{s}\binom{s}{l}y^l A_{n,l}(x+l,y)$.

In order to prove this, consider

$$\frac{(1+t)^{y+1}-1}{t}=y(1+t)\left(\frac{(1+t)^y-1}{yt}\right)+1.$$

Raise this equation to the sth power and multiply by $(1+t)^x$ to get a generating function identity. Now equate coefficients of t^n.

Properties of the coefficient polynomials can also be rederived. For example, it is easy to show that

$$B_{-1}(y;t)-B_{1,-1}(y)t=\left(\frac{(1+t)^y-1)}{ty}\right)^{-1}+\frac{(y-1)t}{2}$$

is an even function of y. Thus if $n\neq 1$, then $B_{n,-1}(y)$ is even and $s+1\,|\,b_{in}(s)$ for i odd.

Remark 7.2. We can now place our polynomials in the literature. In [4, Section 6], Carlitz defines *degenerate Bernoulli polynomials of arbiterary order* by

$$\sum_{n=0}^{\infty}\beta_n^{(\omega)}(\lambda,x)\frac{t^n}{n!}=\left(\frac{t}{(1+\lambda t)^{\mu}-1}\right)^{\omega}(1+\lambda t)^{\mu x}\quad \text{where } \lambda\mu=1.$$

Using (Sxv) we get

$$\beta_n^{(\omega)}(\lambda,x)=\lambda^n n!\,A_n\left(\frac{x}{\lambda},\frac{1}{\lambda},-\omega\right)=n!\,A_n(x+\lambda-1,\lambda,\omega-n-1). \tag{7.3}$$

Carlitz then defines *degenerate Stirling numbers* of the first and second kinds $S_1(n,k\,|\,\lambda)$ and $S_2(n,k\,|\,\lambda)$ respectively, which specialize to the ordinary Stirling numbers for $\lambda=0$, and expresses them as values of certain degenerate Bernoulli polynomials (formulas (6.15) and (6.16) of [4]).

Howard [10, Section 1] defines *weighted degenerate Stirling numbers* $S_1(n,k,\lambda|\theta)$ and $S_2(n,k,\lambda|\theta)$ which generalize the above, by his power series expansions (1.1) and (1.2). He makes use of the binomial theorem to get formulas which translate to

$$S_1(n,k,\lambda|\theta)=(-1)^{n-k}(n)_{n-k}A_{n-k}(\theta-\lambda,\theta,k) \tag{7.4}$$

and

$$S_2(n,k,\lambda|\theta)=(n)_{n-k}\theta^{n-k}A_{n-k}\left(\frac{\lambda}{\theta},\frac{1}{\theta},k\right). \tag{7.5}$$

He then gives combinatorial interpretations of his numbers, which are precisely special cases of our combinatorial interpretations for the polynomials $A_n(x,y,s)$.

Put $\lambda=0$ in $\beta_n^{(\omega)}(\lambda,x)$ to get the *arbitrary-order Bernoulli* polynomials $B_n^{(\omega)}(x)=\beta_n^{(\omega)}(0,x)$, with $\omega=1$ giving the *first-order Bernoulli* polynomials $B_n(x)$.

The exponential generating function for $B_n^{(\omega)}(x)$ is (cf. [14, 16])

$$\sum_n B_n^{(\omega)}(x)\frac{t^n}{n!}=\left(\frac{t}{e^t-1}\right)^{\omega}e^{xt}. \tag{7.6}$$

Put $\lambda=0$ in (7.3) to get

$$A_n(x,0,s)=(1/n!)B_n^{(n+1+s)}(x+1). \tag{7.7}$$

Thus the arbitrary-order Bernoulli polynomials are expressible in terms of our polynomials, and vice versa, with first order (classical) given by

$$A_n(x,0,-n)=(1/n!)B_n(x+1). \tag{7.8}$$

From symmetry (Siii), if n is odd then $2x-(s+n-1)|A_n(x,0,s)$, which is equivalent to $2x-\omega|B_n^{(\omega)}(x)$ if n is odd.

8. Additional computations

In this section we continue to look for rational roots of $B_n(y,s)$ and $B_n(1/y,s)$ where y is a positive integer, primarily $y=2$ or 3.

First observe that since it has been shown that

$$B_n(2,s)=\binom{s}{n}2^{-n}, \tag{8.1}$$

it follows by (Sxiii) that

$$\begin{aligned}B_n\left(\frac{1}{2},s\right)&=2^{-n}\frac{s}{s+n}B_n(2,-(s+n))\\&=(-1)^n4^{-n}\frac{s}{s+n}\binom{s+2n-1}{n}\\&=\left(-\frac{1}{4}\right)^n\left(\binom{2s+n-1}{n}-\binom{2s+n-1}{n-1}\right).\end{aligned} \tag{8.2}$$

From this formula, we can derive two interesting binomial identities that are to be found in Gould's collection [8, 3.163 and 3.164].

This first, attributed to Carlitz, can be written

$$\sum_{k=0}^{s}\binom{s}{k}\binom{k/2}{n}=(-1)^n\left(\frac{1}{2}\right)^{2n-s}\left(\binom{2n-1-s}{n}-\binom{2n-1-s}{n-1}\right). \tag{8.3}$$

To prove this one, observe that if s is a nonnegative integer

$$\left(\frac{(1+t)^{1/2}-1}{\frac{1}{2}t}\right)^{-s}=2^{-s}((1+t)^{1/2}+1)^s=2^{-s}\sum_n\sum_{k=0}^{s}\binom{s}{k}\binom{k/2}{n}t^n,$$

that is,

$$\sum_{k=0}^{s}\binom{s}{k}\binom{k/2}{n}=2^sB_n\left(\frac{1}{2},-s\right). \qquad \square \tag{8.4}$$

The other identity, attributable to Gray–Rosenstock–Riordan, can be written as

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}\binom{k/2}{s+n}=(-1)^{s+n}\left(\frac{1}{2}\right)^{s+2n}\left(\binom{s+2n-1}{n}-\binom{s+2n-1}{n-1}\right). \tag{8.5}$$

To prove this one, observe that the left-hand side equals $(-\frac{1}{2})^sB_n(\frac{1}{2},s)$.

Next, recall that by (3.6), $B_n(m,s)=0$ if $s=0,1,\ldots,[(n-1)/(m-1)]$, so

$$B_n(m,s)=\sum_{k=\lceil\frac{n}{m-1}\rceil}^{n}a_k\binom{s}{k}, \tag{8.6a}$$

where all $a_k>0$, $a_n=((m-1)/2)^n$ and if $m-1\,|\,n$ then

$$a_{n/(m-1)}=B_n\left(m,\frac{n}{m-1}\right)=\left(\frac{1}{m}\right)^{n/(m-1)}. \tag{8.6b}$$

It follows from symmetry (Sxiii) that

$$B_n\left(\frac{1}{m},s\right)=0 \quad \text{if } s=-(n+j) \text{ where } j=1,2,\ldots,\left[\frac{n-1}{m-1}\right]. \tag{8.7}$$

Thus

$$B_n\left(\frac{1}{m},s\right)=\sum_{k=[\frac{n-1}{m-1}]}^{n}c_k\binom{s+n+[\frac{n-1}{m-1}]}{k} \quad \text{where } c_n=\left(\frac{1-m}{2m}\right)^n, \tag{8.8}$$

and

$$c_{[(n-1)/(m-1)]}=B_n\left(\frac{1}{m},-n\right). \tag{8.9}$$

For the remainder of this section take $m=3$. We are looking for rational roots of $B_n(3,s)$ and $B_n(\frac{1}{3},s)$. The following proposition gives such roots.

Proposition 8.1. *Rational roots (linear factors) occur as follows:*

(i) $s^2 \mid B_n(3,s)$ *if and only if* $n \equiv 3 \pmod 6$.

(i)′ $s^2 \mid B_n(\frac{1}{3}, -n) = 0$ *if and only if* $n \equiv 3 \pmod 6$.

(ii) $B_n(3,-1) = 0$ *if and only if* $n \equiv 5 \pmod 6$.

(ii)′ $B_n(\frac{1}{3}, -(n-1)) = 0$ *if and only if* $n \equiv 5 \pmod 6$.

(iii) $3s - n + 1 \mid B_n(3,s)$ *if and only if* $3 \nmid n$.

(iii)′ $3s + 4n - 1 \mid B_n(\frac{1}{3}, s)$ *if and only if* $3 \nmid n$.

Proof. Observe that the equivalence of each pair of formulas follows immediately from the duality symmetry (Sxiii).

(i) Since $(s+n)B_n(y,s) = sA_n(y-1,y,s-1)$ by (Sxii), we must show that $s \mid A_n(2,3,s-1)$, i.e., that $A_n(2,3,-1) = 0$, iff $n \equiv 3 \pmod 6$.

The generating function here is

$$(1+t)^2\left(\frac{(1+t)^3-1}{3t}\right)^{-1} = 3 - \frac{3(t+2)}{3+3t+t^2}.$$

Consider the partial fraction decomposition

$$-\frac{3(t+2)}{3+3t+t^2} = \frac{a}{t-t_1} + \frac{b}{t-t_2},$$

where t_1 and t_2 are the roots of $t^2+3t+3=0$. It follows that $a=t_1$ and $b=t_2$, and since $t_1t_2=3$, that $-A_n(2,3,-1) = (t_1^n + t_2^n)/3^n$ for $n>0$.

But an easy computation gives $t_1^3 = -t_2^3$, whence $t_1^n = -t_2^n$ iff $n \equiv 3 \pmod 6$, so that $A_n(2,3,-1)=0$ iff $n \equiv 3 \pmod 6$.

(ii) The generating function for $B_n(3,-1)$ is

$$\frac{3}{3+3t+t^2} = \frac{a}{t-t_1} + \frac{b}{t-t_2},$$

where t_1, t_2 are as above but now $a = -i\sqrt{3}$ and $b = i\sqrt{3}$. Hence the generating function is

$$i\sqrt{3}\left(\frac{1}{t_1-t} - \frac{1}{t_2-t}\right) = i\sqrt{3}\sum \frac{t_2^{n+1} - t_1^{n+1}}{3^{n+1}} t^n.$$

But as noted above, $t_1^3 = -t_2^3$, so $t_1^{n+1} = t_2^{n+1}$ iff $n \equiv 5 \pmod 6$, from which the result follows immediately.

Before proving (iii), we prove a lemma.

Lemma 8.2. $B_n(3,s-1) = (1/n)((s-n)B_{n-1}(3,s-1) + ((2s-n)/3)B_{n-2}(3,s-1))$ if $n>0$.

Proof. From (Sv) and (3.1) we have

$$B_n(3,s) = \sum_l B_l(3,s-1)\frac{\binom{2}{n-l}}{n-l+1} = B_n(3,s-1) + B_{n-1}(3,s-1) + \frac{1}{3}B_{n-2}(3,s-1),$$

while from (Sxii) and Theorem 3.1(ii) we have

$$(s+n)B_n(3,s)=sA_{n,s-1}(2,3)=s(B_n(3,s-1)+2B_{n-1}(3,s-1)+B_{n-2}(3,s-1)).$$

Eliminating $B_n(3,s)$ gives the desired result. $\square$

We now return to the proof of Proposition 8.1.

(iii) If $n\equiv 1 \pmod 3$ then the result follows immediately from (3.6), since $s=(n-1)/3$ is an integer $\leqslant (n-1)/2$. Thus only the cases $n\equiv 2 \pmod 3$ and $n\equiv 0 \pmod 3$ need work.

Applying the recursion of the preceding lemma twice to the right-hand side of the equation

$$B_{n+3}(3,s+1)=B_{n+3}(3,s)+B_{n+2}(3,s)+\tfrac{1}{3}B_{n+1}(3,s),$$

for $n>-2$ we get

$$B_{n+3}(3,s+1)=\frac{s+1}{3(n+3)(n+2)}((3s-n+1)B_{n+1}(3,s)+(2s-n)B_n(3,s)). \qquad (8.10)$$

It follows that if $s=(n-1)/3$ and $n>-2$, then

$$B_{n+3}(3,s+1)=-\frac{(n+2)}{27(n+3)}B_n(3,s). \qquad (11)$$

The result now follows immediately by induction on n (with $B_2(3,\tfrac{1}{3})=0$ from $n=-1$ or by direct verification). $\square$

Note 8.3. Using (3.8) we can restate (ii) and (iii) of the proposition as binomial identities, namely

(ii) gives

$$\sum_k \binom{-1}{n-k}\binom{n-k}{k}3^{-k}=0$$

if and only if $n\equiv 5 \pmod 6$, and

(iii) gives

$$\sum_k \binom{(n-1)/3}{n-k}\binom{n-k}{k}3^{-k}=0$$

if and only if $3\nmid n$.

Similarly (8.11) yields, for nonnegative integers l, an interesting binomial identity

$$\sum_k \binom{(3l-1)/3}{3l-k}\binom{3l-k}{k}3^{-k}=\frac{(3l-1)(3l-4)\cdots 2}{l!}\left(-\frac{1}{81}\right)^l.$$

9. Addendum — Further factorization results

We have recently found proofs for the absolute factorization (over $\mathbb{C}$) of the A_n polynomials. For ease of application, we will give alternate formulations of the statements, but for convenience we will use Bernoulli notations for the proofs so as to use standard formulas, all of which follow readily from our symmetries.

If $A_n(x,s)=n!\,A_n(x,0,s)$ and $B_n^{(\omega)}=B_n^{(\omega)}(0)$ is the *Bernoulli number of order* ω *and degree* n, we have seen that $A_n(x,s)=B_n^{(s+n+1)}(x+1)$, so that $B_n^{(\omega)}=A_n(-1,\omega-n-1)$, which is a degree n rational polynomial in ω, *called the Nörlund polynomial.* ($B_n=B_n^{(1)}$ is an *ordinary Bernoulli number.*) Regard $B_n^{(\omega)}(x)$ as a polynomial in variables ω and x.

Theorem 9.1.

(i) *If n is even >0, then $A_n(x,y,s)$, $A_n(x,s)$ and $B_n^{(\omega)}(x)$ are absolutely irreducible.*

(ii) *If n is odd >1, then $A_n(x,y,s)/(sy+2x-(s+n-1))$, $A_n(x,s)/(2x-(s+n-1))$, and $B_n^{(\omega)}(x)/(2x-\omega)$ are absolutely irreducible.*

Proof. It is easy to show (from symmetry (Siii)) that

$$B_n^{(\omega)}(x)=\left(x-\frac{\omega}{2}\right)^n+c_1(\omega)\left(x-\frac{\omega}{2}\right)^{n-2}+\cdots+c_f(\omega)\left(x-\frac{\omega}{2}\right)^{n-2f},$$

where $f=[n/2]$ and where $\omega\,|\,c_i(\omega)$ for all i (from Corollary 5.4).

Thus applying the Eisenstein criterion over the unique factorization domain $\mathbb{C}[\omega]$ with prime element ω, it will suffice to show that $\omega^2 \nmid c_f(\omega)$. Thus the following lemma will conclude the proof.

Lemma 9.2.

(i) *Let n be even >0. Then $\omega=0$ is a simple root of $B_n^{(\omega)}$.*

(ii) *Let n be odd >1. Then $\omega=0$ is a double root of $B_n^{(\omega)}$.*

Proof. Since

$$\sum_n B_n^{(\omega)} t^n/n!=\left(\frac{t}{e^t-1}\right)^{\omega},$$

differentiation gives

$$\sum_n D_\omega(B_n^{(\omega)})t^n/n!=\left(\frac{t}{e^t-1}\right)^{\omega}\ln\left(\frac{t}{e^t-1}\right)$$

and

$$\sum_n D_\omega^2(B_n^{(\omega)})t^n/n!=\left(\frac{t}{e^{t-1}}\right)^{\omega}\left(\ln\left(\frac{t}{e^t-1}\right)\right)^2.$$

Now put $\omega=0$. The first series shows $\omega \mid B_n^{(\omega)}$ if $n>0$. Since

$$D_t\left(\ln\left(\frac{t}{e^t-1}\right)\right)=\frac{1}{t}-\frac{e^t}{e^t-1}=\frac{1}{t}\left(1-t-\frac{t}{e^t-1}\right), \qquad (*)$$

it is readily established that

$$\ln\left(\frac{t}{e^t-1}\right)=-\frac{1}{2}t+\sum_{k=1}^{\infty} a_k t^{2k},$$

where $(-1)^k a_k>0$.

The second series now shows that $\omega=0$ is a simple root if n is even >0 and a multiple root if n is odd >1, while the third series shows that the multiple roots are double. □

Remark 9.3. Since $f_n(\omega)=\binom{\omega+n}{n}B_n^{(-\omega)}$ and $g_n(\omega)=(-1)^n\binom{\omega-1}{n}B_n^{(\omega)}$ express the Stirling polynomials in terms of the Bernoulli numbers, an equivalent formulation is that $\omega=0$ is a simple root of the Stirling polynomials of even positive order and a double root of the Stirling polynomials of odd order >1 (where n is the order of the polynomial).

We conclude with a final lemma, which is clearly relevant to factorization of the $B_n(y,s)$ polynomials, and which has some independent interest. Its proof is similar to that of Lemma 9.2, and so is omitted.

Lemma 9.4. *If n is odd >1, then $\omega=1$ is a simple root of $B_n^{(\omega)}$.*

Remark 9.5. Since it is well known that $B_n \neq 0$ for n even (and follows immediately from $(*)$), the preceding lemma may be restated in terms of Stirling polynomials: $\omega=1$ and $\omega=-1$ are roots of $g_n(\omega)$ and $f_n(\omega)$ respectively, which are simple for even $n>0$ and double for odd $n>1$.

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