



# Matrices related to the Bell polynomials

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## Abstract

In this paper, we study the matrices related to the partial exponential Bell polynomials and those related to the Bell polynomials with respect to  $\Omega$ . As a result, the factorizations of these matrices are obtained, which give unified approaches to the factorizations of many lower triangular matrices. Moreover, some combinatorial identities are also derived from the corresponding matrix representations.

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## 1. Introduction

Recently, the lower triangular matrices have catalyzed many investigations. The Pascal matrix and several generalized Pascal matrices first received wide concern [2–4,9,14–17], and some other lower triangular matrices were also studied systematically, for example, the Stirling matrices of the first kind and of the second kind [5,6], the Lah matrix [11], as well as the matrices related to the idempotent numbers and the numbers of planted forests [12]. In the papers referred to above, we can see not only the various properties satisfied by the corresponding matrices, especially the factorizations of them, but also some combinatorial identities.

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By the impetus of these works, we find it will be instructive and interesting to do some researches on the matrices related to the Bell polynomials, where the Bell polynomials, or more explicitly, the exponential partial Bell polynomials, are defined as follows [7, p. 133]:

**Definition 1.1.** The exponential partial Bell polynomials are the polynomials

$$B_{n,k} = B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

in an infinite number of variables  $x_1, x_2, \dots$ , defined by the formal double series expansion

$$\begin{aligned} \Phi &= \Phi(t, u) := \exp\left(u \sum_{m \geq 1} x_m \frac{t^m}{m!}\right) = \sum_{n,k \geq 0} B_{n,k} \frac{t^n}{n!} u^k \\ &= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \left\{ \sum_{k=1}^n u^k B_{n,k}(x_1, x_2, \dots) \right\}, \end{aligned}$$

or by the series expansion

$$\Phi_k(t) := \frac{1}{k!} \left( \sum_{m \geq 1} x_m \frac{t^m}{m!} \right)^k = \sum_{n \geq k} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \dots \tag{1.1}$$

It is well known that many combinatorial sequences, for instance, the Stirling numbers and the Lah numbers, are special cases of the Bell polynomials (see, e.g., [7, p. 135]). Thus, by means of the study of the matrices related to the Bell polynomials, we will have a unified approach to various lower triangular matrices. In addition to these, if we generalize the partial exponential Bell polynomials to the Bell polynomials with respect to  $\Omega$  [7, p. 137, 145], we will get a more powerful tool this time: even the Pascal matrix is included as a specialization of the corresponding matrix of these Bell polynomials.

The main contribution of this article is giving the factorizations of the matrices related to the two kinds of Bell polynomials, the partial exponential Bell polynomials as well as the Bell polynomials with respect to  $\Omega$ . Some interesting combinatorial identities are also obtained.

This article is organized as follows. In Section 2, we will consider the factorization of the matrix related to the partial exponential Bell polynomials. As applications, the factorizations of some special matrices will also be demonstrated there. Section 3 is devoted to the matrix related to the Bell polynomials with respect to  $\Omega$ . Some generalizations of the matrices can be found at the end of these two sections, respectively.

**2. Matrix related to the partial exponential Bell polynomials**

**Lemma 2.1.** *The partial exponential Bell polynomials  $B_{n,k}$  satisfy the vertical recurrence relation:*

$$B_{n,k} = \sum_{l=k-1}^{n-1} \binom{n-1}{l} x_{n-l} B_{l,k-1} = \sum_{l=k}^n \binom{n-1}{l-1} x_{n-l+1} B_{l-1,k-1}. \tag{2.1}$$

**Proof.** Differentiate (1.1) with respect to  $t$  and identify the coefficients of  $t^{n-1}/(n-1)!$  in the first and last member of

$$\sum_{n \geq k} B_{n,k} \frac{t^{n-1}}{(n-1)!} = \frac{d\Phi_k}{dt} = \Phi_{k-1} \sum_{j \geq 1} x_j \frac{t^{j-1}}{(j-1)!} = \sum_{n \geq k-1} \sum_{l=k-1}^n \binom{n}{l} B_{l,k-1} x_{n-l+1} \frac{t^n}{n!},$$

and we will get the result finally.  $\square$

Now, defining  $B_n$  and  $P_n$  to be the  $n \times n$  matrices by

$$(B_n)_{i,j} = B_{i,j}, \quad (P_n)_{i,j} = x_{i-j+1} \binom{i-1}{j-1} \quad \text{for } i, j = 1, 2, \dots, n,$$

and using the notation  $\oplus$  for the direct sum of two matrices, we can obtain the factorization of the matrix  $B_n$  from Lemma 2.1.

**Theorem 2.2.** *The matrix  $B_n$  related to the partial exponential Bell polynomials can be factorized as*

$$B_n = P_n([1] \oplus B_{n-1}). \tag{2.2}$$

For example, if  $n = 4$ , we have

$$B_4 = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1^2 & 0 & 0 \\ x_3 & 3x_1x_2 & x_1^3 & 0 \\ x_4 & 4x_1x_3 + 3x_2^2 & 6x_1^2x_2 & x_1^4 \end{pmatrix} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ x_3 & 2x_2 & x_1 & 0 \\ x_4 & 3x_3 & 3x_2 & x_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & x_2 & x_1^2 & 0 \\ 0 & x_3 & 3x_1x_2 & x_1^3 \end{pmatrix}.$$

Analogous to [3,5,14,15], for any  $k \times k$  matrix  $P_k$ , if we define the  $n \times n$  matrix  $\bar{P}_k$  by

$$\bar{P}_k = \begin{pmatrix} I_{n-k} & O \\ O & P_k \end{pmatrix},$$

we can further factorize the matrix  $B_n$ . It is obvious that  $\bar{P}_n = P_n$ .

**Theorem 2.3.** *The matrix  $B_n$  can be factorized by the matrices  $\bar{P}_k$ 's:*

$$B_n = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_2 \bar{P}_1. \tag{2.3}$$

By virtue of Theorems 2.2 and 2.3, the factorizations of some special lower triangular matrices can be obtained directly.

**Example 2.1.** The Stirling number of the second kind  $S(n, k)$  and the unsigned Stirling number of the first kind  $s(n, k)$  satisfy the following equations, respectively [7, p. 135, Eqs. (3g) and (3i)]:

$$B_{n,k}(1, 1, 1, \dots) = S(n, k), \quad B_{n,k}(0!, 1!, 2!, \dots) = s(n, k).$$

If we define the  $n \times n$  Stirling matrices of the first kind and of the second kind by

$$(s_n)_{i,j} = s(i, j), \quad (S_n)_{i,j} = S(i, j) \quad \text{for } i, j = 1, 2, \dots, n,$$

we can get the factorizations of them from (2.2) and (2.3). In fact,

$$S_n = P_n([1] \oplus S_{n-1}) = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_2 \bar{P}_1, \tag{2.4}$$

$$s_n = Q_n([1] \oplus s_{n-1}) = \bar{Q}_n \bar{Q}_{n-1} \cdots \bar{Q}_2 \bar{Q}_1, \tag{2.5}$$

where the matrices  $P_n$  and  $Q_n$  are defined by

$$(P_n)_{i,j} = \binom{i-1}{j-1}, \quad (Q_n)_{i,j} = \binom{i-1}{j-1} (i-j)! = \frac{(i-1)!}{(j-1)!} \quad \text{for } i, j = 1, 2, \dots, n.$$

For instance, we demonstrate the matrix factorization for  $s_4$ :

$$s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 6 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{pmatrix}.$$

Note that (2.4) has already been given in [5, Lemma 2.1, Theorem 2.2], while (2.5) has not been presented before. Instead of the factorization (2.5), the following one was obtained [5, Eq. (3.2)]:

$$s_n = ([1] \oplus s_{n-1})P_n = \bar{P}_1 \bar{P}_2 \cdots \bar{P}_{n-1} \bar{P}_n, \tag{2.6}$$

where  $P_n$  is the same as that appeared in (2.4), which, in fact, is the famous Pascal matrix. Then, by combining (2.5) and (2.6), we obtain an identity related to  $s(n, k)$ , that is,

$$\sum_{l=k-1}^{n-1} \frac{(n-1)!}{l!} s(l, k-1) = \sum_{l=k-1}^{n-1} \binom{l}{k-1} s(n-1, l). \tag{2.7}$$

On the other hand, it follows from (2.5) that

$$s_n^{-1} = ([1] \oplus s_{n-1})^{-1} Q_n^{-1}. \tag{2.8}$$

With a simple computation, we can derive the values of  $(Q_n^{-1})_{i,j}$ , i.e.,

$$(Q_n^{-1})_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -j, & \text{if } i = j + 1, \\ 0, & \text{else.} \end{cases}$$

Thus, by noticing the relationship between the Stirling numbers of both kinds, we can translate (2.8) to the well known recurrence:

$$S(i, j) = S(i-1, j-1) + jS(i-1, j).$$

**Remark.** In the referee’s report, the combinatorial proof of (2.7) was presented. In fact, if we rewrite it in the more natural form

$$\sum_{l=k}^n \binom{n}{l} (n-l)!s(l, k) = \sum_{l=k}^n \binom{l}{k} s(n, l),$$

then both sides count the number of permutations of  $\{1, 2, \dots, n\}$  with  $k$  cycles where all the numbers are red, and possibly some other cycles where all the numbers are blue.

**Example 2.2.** In [1, Remark 5], the following equation was obtained:

$$L^{(a)}(n, k) := B_{n,k}(1!a^0, 2!a^1, 3!a^2, \dots) = a^{n-k} \binom{n-1}{k-1} \frac{n!}{k!}, \tag{2.9}$$

which, in light of (2.1)–(2.3), leads us at once to the next two equations:

$$L^{(a)}(n, k) = \sum_{l=k}^n \binom{n-1}{l-1} (n-l+1)!a^{n-l} L^{(a)}(l-1, k-1), \tag{2.10}$$

$$L_n^{(a)} = P_n([1] \oplus L_{n-1}^{(a)}) = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_2 \bar{P}_1, \tag{2.11}$$

where  $(L_n^{(a)})_{i,j} = L^{(a)}(i, j)$  and  $(P_n)_{i,j} = \binom{i-1}{j-1}(i-j+1)!a^{i-j}$  for  $i, j = 1, 2, \dots, n$ . For example, if  $n = 4$ , we have

$$L_4^{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 6a^2 & 6a & 1 & 0 \\ 24a^3 & 36a^2 & 12a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 6a^2 & 4a & 1 & 0 \\ 24a^3 & 18a^2 & 6a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2a & 1 & 0 \\ 0 & 6a^2 & 6a & 1 \end{pmatrix}.$$

Particularly, by setting  $a = 1$  in (2.10) and (2.11), we achieve the corresponding recurrence relation and matrix factorization of the Lah numbers  $L(i, j)$ . Moreover, we get (see [7, p. 156] and [11])

$$L_n = s_n S_n = P_n([1] \oplus L_{n-1}),$$

where  $L_n$  is the Lah matrix with  $(L_n)_{i,j} = L^{(1)}(i, j) = L(i, j)$ , and  $(P_n)_{i,j} = \binom{i-1}{j-1}(i-j+1)!$ , from which we have

$$\begin{aligned} \sum_{l=j}^i s(i, l)S(l, j) &= \sum_{l=j}^i \binom{i-1}{l-1} (i-l+1)!L(l-1, j-1) \\ &= \binom{i-1}{j-1} \sum_{l=j}^i \binom{i-j}{l-j} \frac{(i-l+1)!(l-2)!}{(j-2)!}. \end{aligned}$$

**Example 2.3.** Making use of the equation [13, Eq. (22)]

$$I^{(a)}(n, k) := B_{n,k}(1, 2a, 3a^2, \dots) = \binom{n}{k} (ka)^{n-k},$$

we find that

$$I^{(a)}(n, k) = \sum_{l=k}^n \binom{n-1}{l-1} (n-l+1)a^{n-l} I^{(a)}(l-1, k-1), \tag{2.12}$$

$$IP_n^{(a)} = P_n([1] \oplus IP_{n-1}^{(a)}) = \overline{P}_n \overline{P}_{n-1} \cdots \overline{P}_2 \overline{P}_1, \tag{2.13}$$

where  $(IP_n^{(a)})_{i,j} = I^{(a)}(i, j)$  and  $(P_n)_{i,j} = \binom{i-1}{j-1}(i-j+1)a^{i-j}$  for  $i, j = 1, 2, \dots, n$ . For example, if  $n = 4$ , we have

$$IP_4^{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 3a^2 & 6a & 1 & 0 \\ 4a^3 & 24a^2 & 12a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2a & 1 & 0 & 0 \\ 3a^2 & 4a & 1 & 0 \\ 4a^3 & 9a^2 & 6a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2a & 1 & 0 \\ 0 & 3a^2 & 6a & 1 \end{pmatrix}.$$

Again, by setting  $a = 1$  in (2.12) and (2.13), we obtain the recurrence relation and matrix factorization associated with the idempotent numbers  $I(i, j) = I^{(1)}(i, j) = \binom{i}{j} j^{i-j}$ , which was studied in [12].

**Example 2.4.** Let us consider the following equation [13, Eq. (17)]:

$$B_{n,k}(B_0(x), 2B_1(x), 3B_2(x), \dots) = \binom{n}{k} B_{n-k}(kx),$$

where  $B_n(x) := \sum_{j=0}^n S(n, j)x^j$  is the single variable Bell polynomial. It should be noticed that  $B_n(1) = \sum_{j=0}^n S(n, j)$  is the Bell number [7, p. 210].

From (2.1)–(2.3), we get

$$\binom{n}{k} B_{n-k}(kx) = \sum_{l=k}^n \binom{n-1}{l-1} (n-l+1)B_{n-l}(x) \binom{l-1}{k-1} B_{l-k}((k-1)x), \tag{2.14}$$

$$\overline{B}_n = P_n([1] \oplus \overline{B}_{n-1}) = \overline{P}_n \overline{P}_{n-1} \cdots \overline{P}_2 \overline{P}_1, \tag{2.15}$$

where  $(\overline{B}_n)_{i,j} = \binom{i}{j} B_{i-j}(jx)$  and  $(P_n)_{i,j} = \binom{i-1}{j-1} (i-j+1)B_{i-j}(x)$  for  $i, j = 1, 2, \dots, n$ . For example, if  $n = 4$ , we have

$$\begin{aligned} \overline{B}_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2x & 1 & 0 & 0 \\ 3(x+x^2) & 6x & 1 & 0 \\ 4(x+3x^2+x^3) & 12(x+2x^2) & 12x & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2x & 1 & 0 & 0 \\ 3(x+x^2) & 4x & 1 & 0 \\ 4(x+3x^2+x^3) & 9(x+x^2) & 6x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2x & 1 & 0 \\ 0 & 3(x+x^2) & 6x & 1 \end{pmatrix}. \end{aligned}$$

After some computation, we derive from (2.14) the identity below:

$$\frac{k+m}{k} B_m(kx) = \sum_{j=0}^m \binom{m}{j} (m-j+1)B_{m-j}(x)B_j((k-1)x).$$

Hence, in view of the definition of  $B_n(x)$ , we have

$$\frac{k+m}{k} \sum_{n=0}^m S(m, n)(kx)^n = \sum_{j=0}^m \sum_{i=0}^{m-j} \sum_{l=0}^j \binom{m}{j} (m-j+1)(k-1)^l S(m-j, i)S(j, l)x^{i+l}.$$

By equating the coefficients of  $x^n$ , we obtain an identity associated with the Stirling numbers of the second kind:

$$\begin{aligned} (k+m)S(m, n)k^{n-1} \\ = \sum_{j=0}^m \sum_{l=0}^n \binom{m}{j} (m-j+1)(k-1)^l S(m-j, n-l)S(j, l). \end{aligned} \tag{2.16}$$

**Remark.** As pointed by the referee, (2.16) is a linear combination of the two simpler identities

$$S(m, n)k^n = \sum_{j=0}^m \sum_{l=0}^n \binom{m}{j} (k-1)^l S(m-j, n-l)S(j, l), \tag{2.17}$$

$$S(m, n)k^{n-1} = \sum_{j=0}^m \sum_{l=0}^n \binom{m-1}{j} (k-1)^l S(m-j, n-l)S(j, l), \tag{2.18}$$

which may be proved by coloring set partitions. In (2.17) we are putting  $\{1, 2, \dots, m\}$  into  $n$  blocks and then coloring each block with one of  $k$  colors; in (2.18) we are doing the same thing except that the block containing the largest element  $m$  (or some specific element) must be colored red. On the right sides of (2.17) and (2.18),  $l$  is the number of blocks that are not red, and  $j$  is the number of elements in those blocks. If we multiply (2.18) by  $m$  and then add it to (2.17) we get (2.16).

In [1], the authors proposed two methods for the determination of new identities for Bell polynomials, and Yang [13] generalized one of the methods recently. We can see, with the help of their works, various lower triangular matrices can be factorized, just as the examples above.

And at the end of this section, we will consider some generalizations of the Bell matrix. For any nonzero real numbers  $y$  and  $z$ , let us define the  $n \times n$  matrices  $B_n[y], \bar{B}_n[y], B_n[y, z]$  by

$$\begin{aligned} (B_n[y])_{i,j} &= y^{i-j} B_{i,j}, & (\bar{B}_n[y])_{i,j} &= y^{i+j-2} B_{i,j}, \\ (B_n[y, z])_{i,j} &= y^{i-j} z^{i+j-2} B_{i,j}. \end{aligned} \tag{2.19}$$

Correspondingly, the  $n \times n$  matrices  $P_n[y], Q_n[y], R_n[y, z]$  are defined by

$$\begin{aligned} (P_n[y])_{i,j} &= y^{i-j} \binom{i-1}{j-1} x_{i-j+1}, & (Q_n[y])_{i,j} &= y^{i+j-2} \binom{i-1}{j-1} x_{i-j+1}, \\ (R_n[y, z])_{i,j} &= y^{i-j} z^{i+j-2} \binom{i-1}{j-1} x_{i-j+1}. \end{aligned} \tag{2.20}$$

Then, analogous to [5,11,14,15], we give the following theorem.

**Theorem 2.4.** *For any nonzero real numbers  $y$  and  $z$ , we have*

$$\begin{aligned} B_n[y] &= P_n[y]([1] \oplus B_{n-1}[y]) = \bar{P}_n[y] \bar{P}_{n-1}[y] \cdots \bar{P}_2[y] \bar{P}_1[y], \\ \bar{B}_n[y] &= Q_n[y] \left( [1] \oplus B_{n-1} \begin{bmatrix} 1 \\ y \end{bmatrix} \right) = Q_n[y] \bar{P}_{n-1} \begin{bmatrix} 1 \\ y \end{bmatrix} \cdots \bar{P}_2 \begin{bmatrix} 1 \\ y \end{bmatrix} \bar{P}_1 \begin{bmatrix} 1 \\ y \end{bmatrix}, \\ B_n[y, z] &= R_n[y, z] \left( [1] \oplus B_{n-1} \begin{bmatrix} y \\ z \end{bmatrix} \right) = R_n[y, z] \bar{P}_{n-1} \begin{bmatrix} y \\ z \end{bmatrix} \cdots \bar{P}_2 \begin{bmatrix} y \\ z \end{bmatrix} \bar{P}_1 \begin{bmatrix} y \\ z \end{bmatrix}. \end{aligned}$$

**Example 2.5.** If the sequence  $\{x_n\}$  in  $B_{n,k} = B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  is defined by  $x_n = (n - 1)!$  then we will get the following factorizations for the generalized Stirling matrices of the first kind, which have not been presented before (see [5]).

$$\begin{aligned} s_n[y] &= P_n[y]([1] \oplus s_{n-1}[y]) = \bar{P}_n[y] \bar{P}_{n-1}[y] \cdots \bar{P}_2[y] \bar{P}_1[y], \\ \bar{s}_n[y] &= Q_n[y] \left( [1] \oplus s_{n-1} \begin{bmatrix} 1 \\ y \end{bmatrix} \right) = Q_n[y] \bar{P}_{n-1} \begin{bmatrix} 1 \\ y \end{bmatrix} \cdots \bar{P}_2 \begin{bmatrix} 1 \\ y \end{bmatrix} \bar{P}_1 \begin{bmatrix} 1 \\ y \end{bmatrix}, \\ s_n[y, z] &= R_n[y, z] \left( [1] \oplus s_{n-1} \begin{bmatrix} y \\ z \end{bmatrix} \right) = R_n[y, z] \bar{P}_{n-1} \begin{bmatrix} y \\ z \end{bmatrix} \cdots \bar{P}_2 \begin{bmatrix} y \\ z \end{bmatrix} \bar{P}_1 \begin{bmatrix} y \\ z \end{bmatrix}, \end{aligned}$$

where the definitions of the matrices referred to above can be obtained from (2.19) and (2.20) immediately. And alternately, if we define  $\{x_n\}$  by  $x_n = n!$  we will derive some new factorizations of the generalized Lah matrices (see [11]).

**3. Matrix related to the Bell polynomials with respect to  $\Omega$**

In this section every formal series  $f$  is supposed to be of the form:

$$f = \sum_{n \geq 1} \Omega_n f_n t^n, \tag{3.1}$$

where  $\Omega_1, \Omega_2, \dots$  is a reference sequence,  $\Omega_1 = 1$  and  $\Omega_n \neq 0$ . In this way we treat at the same time the case of ordinary coefficients of  $f$  ( $\Leftrightarrow \Omega_n = 1$ ), and the case of Taylor coefficients ( $\Leftrightarrow \Omega_n = 1/n!$ ).

With every series  $f$  we associate the infinite lower iteration matrix with respect to  $\Omega$ :

$$B = B(f) := \begin{pmatrix} B_{1,1} & 0 & 0 & \dots \\ B_{2,1} & B_{2,2} & 0 & \dots \\ B_{3,1} & B_{3,2} & B_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where  $B_{n,k} = B_{n,k}^\Omega(f_1, f_2, \dots)$  is the Bell polynomial with respect to  $\Omega$  [7, p. 137, 145], defined as follows:

$$\Omega_k f^k = \sum_{n \geq k} B_{n,k} \Omega_n t^n. \tag{3.2}$$

Now, we shall study how to factorize the  $n \times n$  matrix  $B_n = B_n(f)$ . In fact, similarly to what we have done in the preceding section, we have the following lemma.

**Lemma 3.1.** *The Bell polynomials  $B_{n,k}$  with respect to  $\Omega$  satisfy the vertical recurrence relation:*

$$\begin{aligned} B_{n,k} &= \sum_{i=k-1}^{n-1} \frac{\Omega_i \Omega_{n-i}}{\Omega_n} \frac{n-i}{n} f_{n-i} \frac{k \Omega_k}{\Omega_{k-1}} B_{i,k-1} \\ &= \sum_{l=k}^n \frac{\Omega_{l-1} \Omega_{n-l+1}}{\Omega_n} \frac{n-l+1}{n} f_{n-l+1} \frac{k \Omega_k}{\Omega_{k-1}} B_{l-1,k-1}. \end{aligned} \tag{3.3}$$

**Proof.** Differentiate (3.2) with respect to  $t$ ,

$$\begin{aligned} \sum_{n \geq k} B_{n,k} \Omega_n n t^{n-1} &= \Omega_k \cdot k f^{k-1} f' = \frac{k \Omega_k}{\Omega_{k-1}} \sum_{i \geq k-1} B_{i,k-1} \Omega_i t^i \cdot \sum_{j \geq 1} \Omega_j j f_j t^{j-1} \\ &= \frac{k \Omega_k}{\Omega_{k-1}} \sum_{n \geq k-1} \left( \sum_{i=k-1}^n \frac{\Omega_i \Omega_{n-i+1}}{\Omega_n} (n-i+1) f_{n-i+1} B_{i,k-1} \right) \Omega_n t^n, \end{aligned}$$

and identify the coefficients of  $\Omega_{n-1} t^{n-1}$  in the equation above, then we get

$$\frac{n \Omega_n B_{n,k}}{\Omega_{n-1}} = \frac{k \Omega_k}{\Omega_{k-1}} \sum_{i=k-1}^{n-1} \frac{\Omega_i \Omega_{n-i}}{\Omega_{n-1}} (n-i) f_{n-i} B_{i,k-1},$$

which, after some transformations, leads us to (3.3) at once.  $\square$

It should be noticed that, if  $\Omega_n = 1/n!$  (3.3) gives us



$$B_{n,k} = \sum_{i=k-1}^{n-1} \binom{n}{i} \frac{n-i}{n} f_{n-i} B_{i,k-1} = \sum_{i=k-1}^{n-1} \binom{n-1}{i} f_{n-i} B_{i,k-1},$$

which is just the assertion (2.1) of Lemma 2.1; while by setting  $\Omega_n = 1$ , we have

$$B_{n,k} = \sum_{i=k-1}^{n-1} \frac{n-i}{n} f_{n-i} k B_{i,k-1} = \sum_{l=k}^n \frac{n-l+1}{n} f_{n-l+1} k B_{l-1,k-1}. \tag{3.4}$$

Once we get the recurrence, we can derive the matrix factorization immediately. Let us define the  $n \times n$  matrices  $B_n, P_n$  and  $D_n$  by

$$(B_n)_{i,j} = B_{i,j}, \quad (P_n)_{i,j} = \frac{\Omega_{j-1} \Omega_{i-j+1}}{\Omega_i} \frac{i-j+1}{i} f_{i-j+1} := P_{i,j},$$

$$D_n = \text{diag} \left\{ \frac{1\Omega_1}{\Omega_0}, \frac{2\Omega_2}{\Omega_1}, \dots, \frac{n\Omega_n}{\Omega_{n-1}} \right\}, \tag{3.5}$$

for  $i, j = 1, 2, \dots, n$ , where  $\Omega_0 = 1$ . Then the following theorem holds.

**Theorem 3.2.** *The matrix  $B_n$  related to the Bell polynomials with respect to  $\Omega$  can be factorized as*

$$B_n = P_n ([1] \oplus B_{n-1}) D_n = \overline{P}_n \overline{P}_{n-1} \cdots \overline{P}_1 \overline{D}_1 \cdots \overline{D}_{n-1} \overline{D}_n. \tag{3.6}$$

**Example 3.1.** For  $f = t(1-t)^{-1}$  and  $\Omega_n = 1$ , we have  $f = \sum_{n \geq 1} t^n = \sum_{n \geq 1} f_n t^n$ , then  $f_n = 1$ . Moreover, according to (3.2),

$$\sum_{n \geq k} B_{n,k} t^n = t^k (1-t)^{-k} = t^k \sum_{m \geq 0} \binom{-k}{m} (-1)^m t^m = \sum_{n \geq k} \binom{n-1}{k-1} t^n,$$

which implies that  $B_{n,k} = \binom{n-1}{k-1}$  and the iteration matrix  $B_n(f)$  is the Pascal matrix. Hence, Lemma 3.1, or equivalently, (3.4) shows us the following recurrence for binomial coefficients:

$$\binom{n-1}{k-1} = \sum_{l=k}^n \frac{n-l+1}{n} k \binom{l-2}{k-2}, \tag{3.7}$$

and Theorem 3.2 gives us new factorizations for the Pascal matrix:

$$P_n = P'_n ([1] \oplus P_{n-1}) D_n = \overline{P}'_n \overline{P}'_{n-1} \cdots \overline{P}'_1 \overline{D}_1 \cdots \overline{D}_{n-1} \overline{D}_n,$$

where  $(P_n)_{i,j} = \binom{i-1}{j-1}$ ,  $(P'_n)_{i,j} = (i-j+1)/i$  and  $D_n = \text{diag}\{1, 2, \dots, n\}$ . We would like to illustrate how to factorize the corresponding matrix when  $n = 3$ .

$$P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

**Example 3.2.** The generating function for the harmonic numbers  $H_n$  is

$$f = \sum_{n \geq 1} H_n t^n = \frac{1}{1-t} \log \frac{1}{1-t},$$

from which we have

$$\begin{aligned} f^k &= \left( \frac{1}{1-t} \log \frac{1}{1-t} \right)^k = k! \sum_{i \geq 0} \binom{-k}{i} (-1)^i t^i \sum_{j \geq k} s(j, k) \frac{t^j}{j!} \\ &= \sum_{n \geq k} \left( \sum_{i=0}^{n-k} \binom{k+i-1}{i} \frac{k!}{(n-i)!} s(n-i, k) \right) t^n. \end{aligned}$$

Thus, the iteration matrix  $B(f)$  is defined by

$$(B(f))_{n,k} = \sum_{i=0}^{n-k} \binom{k+i-1}{i} \frac{k!}{(n-i)!} s(n-i, k) = \sum_{i=k}^n \binom{n-i+k-1}{n-i} \frac{k!}{i!} s(i, k),$$

and (3.4) implies the following identity:

$$\begin{aligned} &\sum_{i=k}^n \binom{n-i+k-1}{n-i} \frac{k!}{i!} s(i, k) \\ &= \sum_{i=k}^n \frac{k!}{(i-1)!} s(i-1, k-1) \sum_{l=i}^n \binom{l+k-i-2}{l-i} \frac{n-l+1}{n} H_{n-l+1}. \end{aligned}$$

More generally, if we start with the generating function [8, p. 351, Eq. (7.43)]

$$g = \sum_{n \geq 1} (H_{m+n} - H_m) \binom{m+n}{m} t^n = \frac{1}{(1-t)^{m+1}} \log \frac{1}{1-t},$$

we get

$$\begin{aligned} (B(g))_{n,k} &= \sum_{i=k}^n (-1)^{n-i} \binom{-k(m+1)}{n-i} \frac{k!}{i!} s(i, k), \\ &\sum_{i=k}^n (-1)^{n-i} \binom{-k(m+1)}{n-i} \frac{k!}{i!} s(i, k) \\ &= \sum_{i=k}^n \frac{k!}{(i-1)!} s(i-1, k-1) \sum_{l=i}^n (-1)^{l-i} \binom{-(k-1)(m+1)}{l-i} \binom{m+n-l+1}{m} \\ &\quad \times \frac{n-l+1}{n} (H_{m+n-l+1} - H_m). \end{aligned}$$

The factorizations of the matrices referred to above can be obtained immediately from Theorem 3.2. However, the matrix defined below may be more interesting.

$$(\tilde{B})_{n,k} = \frac{n!}{k!} (B(f))_{n,k} = \sum_{i=k}^n \binom{n-i+k-1}{n-i} \frac{n!}{i!} s(i, k).$$

For instance,

$$\tilde{B}_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 11 & 9 & 1 & 0 & 0 & 0 & 0 \\ 50 & 71 & 18 & 1 & 0 & 0 & 0 \\ 274 & 580 & 245 & 30 & 1 & 0 & 0 \\ 1764 & 5104 & 3135 & 625 & 45 & 1 & 0 \\ 13068 & 48860 & 40369 & 11515 & 1330 & 63 & 1 \end{bmatrix}.$$

We notice that the sequences in the first four columns can be found in [10, A000254, A001706, A001713, A001719]. Therefore, an identity can be derived by means of the formulae on the web. That is

$$\sum_{i=k}^n \binom{n-i+k-1}{n-i} \frac{n!}{i!} s(i, k) = \sum_{i=k}^n \binom{i}{k} k^{i-k} s(n, i),$$

which can also be verified by the computation of the generating functions. In fact, for the left side, we have

$$\sum_{k \geq 0} \sum_{n \geq k} \frac{n!}{k!} (B(f))_{n,k} \frac{t^n}{n!} u^k = \sum_{k \geq 0} \left( \frac{1}{1-t} \log \frac{1}{1-t} \right)^k \frac{u^k}{k!} = \exp \left( \frac{u}{1-t} \log \frac{1}{1-t} \right);$$

while the right side gives

$$\begin{aligned} \sum_{k \geq 0} \sum_{n \geq k} \sum_{i=k}^n \binom{i}{k} k^{i-k} s(n, i) \frac{t^n}{n!} u^k &= \sum_{k \geq 0} \sum_{i \geq k} \left( \sum_{n \geq i} s(n, i) \frac{t^n}{n!} \right) \binom{i}{k} k^{i-k} u^k \\ &= \sum_{k \geq 0} \sum_{i \geq k} \frac{1}{i!} \log^i \frac{1}{1-t} \binom{i}{k} k^{i-k} u^k \\ &= \sum_{k \geq 0} \sum_{i \geq k} \binom{i}{k} k^{i-k} \frac{1}{i!} \log^i \frac{1}{1-t} u^k \\ &= \exp \left( \frac{u}{1-t} \log \frac{1}{1-t} \right), \end{aligned}$$

where we make use of the generating function for the idempotent numbers [7,12] in the last step.

Furthermore, by appealing to (3.4), we have

$$\begin{aligned} (\tilde{B})_{n,k} &= \frac{n!}{k!} (B(f))_{n,k} = \frac{n!}{k!} \sum_{l=k}^n \frac{n-l+1}{n} H_{n-l+1} k B_{l-1,k-1} \\ &= \sum_{l=k}^n (n-l+1) H_{n-l+1} \frac{(n-1)! (l-1)!}{(l-1)! (k-1)!} (B(f))_{l-1,k-1}, \end{aligned}$$

which leads us to the factorization of the matrix  $\tilde{B}_n$ :

$$\tilde{B}_n = \tilde{P}_n ([1] \oplus \tilde{B}_{n-1}),$$

where  $(\tilde{P}_n)_{i,j} = (i-j+1) \frac{(i-1)!}{(j-1)!} H_{i-j+1}$  for  $i, j = 1, 2, \dots, n$ . For instance,

$$\tilde{B}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 11 & 9 & 1 & 0 \\ 50 & 71 & 18 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 11 & 6 & 1 & 0 \\ 50 & 33 & 9 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 11 & 9 & 1 \end{pmatrix}.$$

Actually, we can do more than these. Let us introduce a lemma [7, p. 145, Theorem A]:

**Lemma 3.3.** For three sequences  $f, g, h$  (written as in (3.1)),  $h = f \circ g$  is equivalent to the matrix equality:

$$B(h) = B(g) \cdot B(f). \tag{3.8}$$

Then, Theorem 3.2 and Lemma 3.3 lead us to the result below.

**Corollary 3.4.** For three formal series  $g_0, g_1, g_2$  written as in (3.1), if  $g_0 = g_2 \circ g_1$ , and the matrices associated with  $g_i$  can be factorized as  $B_n(g_i) = G_n^{(i)}([1] \oplus B_{n-1}(g_i))D_n$ , for  $i = 1, 2, 3$ , then

$$\begin{aligned} B_n(g_0) &= G_n^{(0)}([1] \oplus B_{n-1}(g_0))D_n \\ &= G_n^{(1)}([1] \oplus B_{n-1}(g_1))D_n G_n^{(2)}([1] \oplus B_{n-1}(g_2))D_n. \end{aligned} \tag{3.9}$$

More generally, if we have  $k + 1$  series  $g_0, g_1, g_2, \dots, g_k$  with  $g_0 = g_k \circ \dots \circ g_2 \circ g_1$ , then (3.8) gives the matrix equality  $B(g_0) = B(g_1)B(g_2) \cdots B(g_k)$ . And, Corollary 3.4 indicates how to factorize  $B_n(g_0)$  in two different ways.

**Example 3.3.** We have already shown that the iteration matrix of the series  $f = t(1 - t)^{-1}$  is the Pascal matrix  $P$  (see Example 3.1). And in this example, we will go on to study this series.

We have

$$f^{(2)} = f \circ f = \frac{t}{1 - 2t} = \sum_{n \geq 1} 2^{n-1} t^n,$$

then  $f_n^{(2)} = 2^{n-1}$ . In view of (3.2),

$$\sum_{n \geq k} B_{n,k} t^n = t^k (1 - 2t)^{-k} = \sum_{n \geq k} \binom{n-1}{k-1} 2^{n-k} t^n,$$

from which we get  $B_{n,k} = \binom{n-1}{k-1} 2^{n-k}$ . Hence, Corollary 3.4 indicates that  $B_n(f^{(2)})$  can be factorized in the following ways:

$$B_n(f^{(2)}) = P_n^2 = P_n''([1] \oplus B_{n-1}(f^{(2)}))D_n = P_n'([1] \oplus P_{n-1})D_n P_n'([1] \oplus P_{n-1})D_n,$$

where  $(B_n(f^{(2)}))_{i,j} = \binom{i-1}{j-1} 2^{i-j}$ ,  $(P_n'')_{i,j} = (i - j + 1)2^{i-j}/i$ , and, as in Example 3.1,  $(P_n)_{i,j} = \binom{i-1}{j-1}$ ,  $(P_n')_{i,j} = (i - j + 1)/i$ ,  $D_n = \text{diag}\{1, 2, \dots, n\}$ . For instance,

$$B_3(f^{(2)}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & \frac{1}{2} & 0 \\ 4 & \frac{4}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{aligned}
 &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \\
 &\quad \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.
 \end{aligned}$$

In addition to these, according to the definition given in [7, p. 147], for each complex number  $x$ , we can get the  $x$ th order fractionary iterate  $f^{(x)}$  and the corresponding iteration matrix  $P^x$ . Let us define  $\mathcal{B} := P - I$ , where  $I$  is the identity matrix. By induction, we have  $(\mathcal{B}^j)_{n,k} = \binom{n-1}{k-1} j! S(n-k, j)$ . Thus,  $(\mathcal{B}^j)_{n,1} = j! S(n-1, j)$ . By [7, p. 174, Eq. (7j)],

$$f_n^{(x)} = \sum_{j=1}^{n-1} \binom{x}{j} j! S(n-1, j) = \sum_{j=1}^{n-1} S(n-1, j) (x)_j = x^{n-1}, \quad n \geq 2, \quad f_1^{(x)} = 1,$$

and we get

$$\begin{aligned}
 f^{(x)} &= \sum_{n \geq 1} f_n^{(x)} t^n = \frac{t}{1 - xt}, \\
 \sum_{n \geq k} B_{n,k} t^n &= t^k (1 - xt)^{-k} = \sum_{n \geq k} \binom{n-1}{k-1} x^{n-k} t^n.
 \end{aligned}$$

Therefore,  $(P_n^x)_{i,j} = \binom{i-1}{j-1} x^{i-j}$ , which means that the iteration matrix  $P_n^x$  is the generalized Pascal matrix studied in [14], and we denote it by  $P_n[x]$ . Additionally, Theorem 3.2 gives the following factorizations:

$$P_n[x] = P'_n[x] ([1] \oplus P_{n-1}[x]) D_n = \overline{P}'_n[x] \overline{P}'_{n-1}[x] \cdots \overline{P}'_1[x] \overline{D}_1 \cdots \overline{D}_{n-1} \overline{D}_n,$$

where  $(P'_n[x])_{i,j} = (i-j+1)x^{i-j}/i$ ,  $D_n = \text{diag}\{1, 2, \dots, n\}$ . For instance,

$$\begin{aligned}
 P_4[x] &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & \frac{1}{2} & 0 & 0 \\ x^2 & \frac{2}{3}x & \frac{1}{3} & 0 \\ x^3 & \frac{3}{4}x^2 & \frac{1}{2}x & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x^2 & 2x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.
 \end{aligned}$$

Now, we consider the inverse of the Bell matrix with respect to  $\Omega$ .

**Theorem 3.5.** For two formal series  $f, g$  written as in (3.1), if  $h = f \circ g = g \circ f = t$ , then

$$B(h) = B(g)B(f) = I, \tag{3.10}$$

where  $I$  is the infinite identity matrix.

**Proof.** Since  $\Omega_k h^k = \Omega_k t^k = \sum_{n \geq k} B_{n,k} \Omega_n t^n$ , we have  $B_{k,k} = 1$  and  $B_{n,k} = 0$  for  $n > k$ , which implies that  $B(h) = I$ , and then the relationship (3.10) holds.  $\square$

By virtue of Theorem 3.5, we can get another factorization of the inverse of the Bell matrix besides the one indicated in Theorem 3.2. In fact, if  $f \circ g = g \circ f = t$ , and, by Theorem 3.2,  $B_n(f) = F_n([1] \oplus B_{n-1}(f))D_n$ , then, making use of Theorem 3.5, we find that

$$B_n(g) = B_n^{-1}(f) = D_n^{-1}([1] \oplus B_{n-1}^{-1}(f))F_n^{-1} = D_n^{-1}([1] \oplus B_{n-1}(g))F_n^{-1}. \tag{3.11}$$

**Example 3.4.** If  $f = e^t - 1, g = \log(1 + t)$ , and  $\Omega_n = 1/n!$  then from the fact that

$$\begin{aligned} \frac{1}{k!} (e^t - 1)^k &= \sum_{n \geq k} S(n, k) \frac{t^n}{n!}, \\ \frac{1}{k!} \log^k(1 + t) &= \sum_{n \geq k} (-1)^{n-k} s(n, k) \frac{t^n}{n!}, \end{aligned}$$

we get  $(B_n(f))_{i,j} = S(i, j), (B_n(g))_{i,j} = (-1)^{i-j} s(i, j)$ . And Theorem 3.2 gives us the factorizations of  $B_n(f)$  and  $B_n(g)$ , i.e.,

$$B_n(f) = F_n([1] \oplus B_{n-1}(f)), \quad B_n(g) = G_n([1] \oplus B_{n-1}(g)),$$

where according to (3.5),  $(F_n)_{i,j} = \binom{i-1}{j-1}, (G_n)_{i,j} = (-1)^{i-j} \frac{(i-1)!}{(j-1)!}$  for  $i, j = 1, 2, \dots, n$ . Since  $(F_n^{-1})_{i,j} = (-1)^{i-j} \binom{i-1}{j-1}$ , (3.11) tells us the matrix  $B_n(g)$  can also be factorized as follows:

$$B_n(g) = ([1] \oplus B_{n-1}(g))F_n^{-1}. \tag{3.12}$$

For example, if  $n = 4$ , we have

$$B_4(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$

By multiplying (3.12) with  $\bar{I}_n = \text{diag}\{1, -1, \dots, (-1)^{n-1}\}$ ,

$$\bar{I}_n B_n(g) \bar{I}_n = \bar{I}_n ([1] \oplus B_{n-1}(g)) F_n^{-1} \bar{I}_n = \bar{I}_n ([1] \oplus B_{n-1}(g)) \bar{I}_n \bar{I}_n F_n^{-1} \bar{I}_n,$$

we get  $s_n = ([1] \oplus s_{n-1})P_n$ , the factorization of the Stirling matrix of the first kind (see Eq. (2.6) and [5, Eq. (3.2)]).

Finally, we will present some generalizations of the matrix related to the Bell polynomials with respect to  $\Omega$ . For any nonzero real numbers  $y$  and  $z$ , we define the  $n \times n$  matrices  $B_n[y], \bar{B}_n[y], B_n[y, z]$  by

$$\begin{aligned} (B_n[y])_{i,j} &= y^{i-j} B_{i,j}, & (\bar{B}_n[y])_{i,j} &= y^{i+j-2} B_{i,j}, \\ (B_n[y, z])_{i,j} &= y^{i-j} z^{i+j-2} B_{i,j}, \end{aligned} \tag{3.13}$$

which have the same forms as those defined at the end of Section 2, but, of course, with different meanings. Correspondingly, the  $n \times n$  matrices  $P_n[y], Q_n[y], R_n[y, z]$  are defined by

$$\begin{aligned} (P_n[y])_{i,j} &= y^{i-j} P_{i,j}, & (Q_n[y])_{i,j} &= y^{i+j-2} P_{i,j}, \\ (R_n[y, z])_{i,j} &= y^{i-j} z^{i+j-2} P_{i,j}, \end{aligned} \tag{3.14}$$

where  $P_{i,j}$  has already been defined in Eq. (3.5).

Then the following theorems hold.

**Theorem 3.6.** For any nonzero real numbers  $y$  and  $z$ , if  $f \circ g = g \circ f = t$ , we have

$$B_n(f)[y]B_n(g)[y] = I_n, \quad \overline{B}_n(f)[y]\overline{B}_n(g)\left[\frac{1}{y}\right] = I_n, \quad B_n(f)[y, z]B_n(g)\left[y, \frac{1}{z}\right] = I_n,$$

where  $I_n$  is the  $n \times n$  identity matrix.

**Theorem 3.7.** For any nonzero real numbers  $y$  and  $z$ , we have

$$\begin{aligned} B_n[y] &= P_n[y]([1] \oplus B_{n-1}[y])D_n = \overline{P}_n[y]\overline{P}_{n-1}[y] \cdots \overline{P}_2[y]\overline{P}_1[y]\overline{D}_1\overline{D}_2 \cdots \overline{D}_{n-1}\overline{D}_n, \\ \overline{B}_n[y] &= Q_n[y]\left([1] \oplus B_{n-1}\left[\frac{1}{y}\right]\right)D_n \\ &= Q_n[y]\overline{P}_{n-1}\left[\frac{1}{y}\right] \cdots \overline{P}_2\left[\frac{1}{y}\right]\overline{P}_1\left[\frac{1}{y}\right]\overline{D}_1\overline{D}_2 \cdots \overline{D}_{n-1}\overline{D}_n, \\ B_n[y, z] &= R_n[y, z]\left([1] \oplus B_{n-1}\left[\frac{y}{z}\right]\right)D_n \\ &= R_n[y, z]\overline{P}_{n-1}\left[\frac{y}{z}\right] \cdots \overline{P}_2\left[\frac{y}{z}\right]\overline{P}_1\left[\frac{y}{z}\right]\overline{D}_1\overline{D}_2 \cdots \overline{D}_{n-1}\overline{D}_n. \end{aligned}$$

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