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A Generalized Recurrence for Bell Numbers

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Abstract

We show that the two most well-known expressions for Bell numbers, $\varpi_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ and $\varpi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \varpi_k$, are both special cases of a third expression for the Bell numbers, and we give a combinatorial proof of the latter.

1 Introduction

In previous work [2] we derived formulas that have as special cases expressions for the sums $\sum_{j=0}^{n} j^m \binom{n}{j}$ and $\sum_{j=0}^{n} j^m \left[{n \atop j} \right]$, where $\binom{n}{j}$ is a binomial coefficient (A007318) and $\left[{n \atop j} \right]$ is a Stirling cycle number (or Stirling number of the first kind, A008275). However, our approach does not work for the corresponding sum $\sum_{j=0}^{n} j^m \left\{ {n \atop j} \right\}$ involving Stirling subset numbers $\left\{ {n \atop j} \right\}$ (or Stirling numbers of the second kind, A008277). By modifying our approach, though, we realized that $\sum_{j=0}^{n} j^m \left\{ {n \atop j} \right\}$ can be expressed as a linear combination of Bell numbers (A000110). After some algebraic manipulation we found ourselves with a new expression for the Bell number ϖ_{m+n}, one that generalizes the two most common expressions for Bell numbers. The purpose of this short note is to give a simple combinatorial proof of that generalization.

2 Main Result

The *Bell number* ϖ_m is the number of ways to partition a set of m objects. For example, $\varpi_3 = 5$ because there are five ways to partition the set $\{1, 2, 3\}$:

$$\{1, 2, 3\}; \ \{1, 2\} \cup \{3\}; \ \{1, 3\} \cup \{2\}; \ \{2, 3\} \cup \{1\}; \ \{1\} \cup \{2\} \cup \{3\}.$$

There is no known simple closed-form expression for ϖ_m. The two most well-known expressions for the Bell numbers are the following [1, pp. 373, 566]:

$$\varpi_m = \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \tag{1}$$

and

$$\varpi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \varpi_k. \tag{2}$$

The following generalizes both (1) and (2):

$$\varpi_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \left\{ {m \atop j} \right\} \binom{n}{k} \varpi_k. \tag{3}$$

(We take 0^0 to be 1.) Equations (1) and (2) are the special cases $n = 0$ and $m = 1$, respectively.

Proof. Given a set of m objects and a set of n objects, one can count the number of ways to partition these $m + n$ objects in the following manner. Partition the set of size m into exactly j subsets; there are $\left\{ {m \atop j} \right\}$ ways to do this. Choose k of the objects from the set of size n to be partitioned into new subsets, and distribute the remaining $n - k$ objects among the j subsets formed from the set of size m. There are $\binom{n}{k}$ ways to choose the k objects, ϖ_k ways to partition them into new subsets, and j^{n-k} ways to distribute the remaining $n - k$ objects among the j subsets. Thus there are $j^{n-k} \left\{ {m \atop j} \right\} \binom{n}{k} \varpi_k$ partitions if the set of size m is partitioned into j subsets and k objects from the set of size n are chosen to form new subsets. Summing over all possible values of j and k yields all ways to partition the $m + n$ objects. □

References

[1] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, 1994.

[2] Michael Z. Spivey, Counting functions and finite differences, *Discrete Math.* **307** (2007), 3130–3146.

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(Concerned with sequences A000110, A007318, A008275, and A008277.)

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