The partial r-Bell polynomials

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Abstract. In this paper, we show that the r-Stirling numbers of both kinds, the r-Whitney numbers of both kinds, the r-Lah numbers and the r-Whitney-Lah numbers form particular cases of family of polynomials forming a generalization of the partial Bell polynomials. We deduce the generating functions of several restrictions of these numbers. In addition, a new combinatorial interpretations is presented for the r-Whitney numbers and the r-Whitney-Lah numbers.

Keywords. The partial Bell and r-Bell polynomials, recurrence relations, r-Stirling numbers and r-Lah numbers, r-Whitney numbers, probabilistic interpretation.

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1 Introduction

The exponential partial Bell polynomials $B_{n,k}(x_1, x_2, \ldots) := B_{n,k}(x_j)$ in an infinite number of variables x_j, $(j \geq 1)$, introduced by Bell [1], as a mathematical tool for representing the n-th derivative of composite function. These polynomials are often used in combinatorics, statistics and also mathematical applications. They are defined by their generating function

$$\sum_{n \geq k} B_{n,k}(x_j) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{m \geq 1} x_m \frac{t^m}{m!} \right)^k,$$

and are given explicitly by the formula

$$B_{n,k}(a_1, a_2, \ldots) = \sum_{\pi(n,k)} \frac{n!}{k_1! \cdots k_n!} \left(\frac{a_1}{1!}\right)^{k_1} \left(\frac{a_2}{2!}\right)^{k_2} \cdots \left(\frac{a_n}{n!}\right)^{k_n}, \tag{1}$$

where

$$\pi(n,k) = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n : k_1 + k_2 + \cdots + k_n = k, \quad k_1 + 2k_2 + \cdots + nk_n = n\}.$$

It is well-known that for appropriate choices of the variables x_j, the exponential partial Bell polynomials reduce to some special combinatorial sequences. We mention the following special cases:

$$\begin{bmatrix} n \\ k \end{bmatrix} = B_{n,k}(0!, 1!, 2!, \cdots), \text{ unsigned Stirling numbers of the first kind,}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = B_{n,k}(1, 1, 1, \ldots), \text{ Stirling numbers of the second kind,}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = B_{n,k}(1!, 2!, 3!, \cdots), \text{ Lah numbers,}$$

$$\binom{n}{k} k^{n-k} = B_{n,k}(1, 2, 3, \cdots), \text{ idempotent numbers.}$$

For more details on these numbers, one can see [1, 4, 7, 8, 10].

In 1984, Broder [2] generalized the Stirling numbers of both kinds to the so-called r-Stirling numbers. In this paper, after recalling the partition polynomials, we give a unified method for obtaining a class of special combinatorial sequences, called the exponential partial r-Bell polynomials for which the r-Stirling numbers and other known numbers appear as special cases. In addition, these polynomials generalize the exponential partial Bell polynomials and posses some combinatorial interpretations in terms of set partitions.

2 The partial r-Bell polynomials

First of all, to introduce the partial r-Bell polynomials, we may give some combinatorial interpretations of the partial Bell polynomials. Below, for $B_{n,k}(a_1, a_2, a_3, \ldots)$, we use $B_{n,k}(a_l)$ and sometimes we use $B_{n,k}(a_1, a_2, a_3, \ldots)$ and for $B_{n,k}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots)$, we use $B_{n,k}^{(r)}(a_l; b_l)$ and sometimes we use $B_{n,k}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots)$.

Theorem 1 *Let $(a_n; n \geq 1)$ be a sequence of nonnegative integers. Then, we have*

- *the number $B_{n,k}(a_l)$ counts the number of partitions of a n-set into k blocks such that the blocks of the same cardinality i can be colored with a_i colors,*
- *the number $B_{n,k}((l-1)!a_l)$ counts the number of permutations of a n-set into k cycles such that any cycle of length i can be colored with a_i colors, and,*
- *the number $B_{n,k}(l!a_l)$ counts the number of partitions of a n-set into k ordered blocks such that the blocks of cardinality i can be colored with a_i colors.*

Proof. For a partition of a finite n-set that is decomposed into k blocks, let k_i be the number of blocks of the same cardinality i, $i = 1, \ldots, n$. Then, the number to choice such partition is

$$\frac{n!}{k_1!(1!)^{k_1} k_2!(2!)^{k_2} \cdots k_n!(n!)^{k_n}}, \quad \mathbf{k} = (k_1, \ldots, k_n) \in \pi(n,k),$$

and, the number to choice such partition for which the blocks of the same cardinality i can be colored with a_i colors is

$$\frac{n!}{k_1!(1!)^{k_1} k_2!(2!)^{k_2} \cdots k_n!(n!)^{k_n}} (a_1)^{k_1} (a_2)^{k_2} \cdots (a_n)^{k_n}, \quad \mathbf{k} = (k_1, \ldots, k_n) \in \pi(n,k),$$

Then, the number of partitions of a n-set into k blocks of cardinalities $k_1, k_2, \ldots, k_n$ such that the blocks of the same length i can be colored with a_i colors is

$$\sum_{\mathbf{k} \in \pi(n,k)} \frac{n!}{k_1!(1)^{k_1} \cdots k_n!(n)^{k_n}} (a_1)^{k_1} (a_2)^{k_2} \cdots (a_n)^{k_n} = B_{n,k}(a_l).$$

For the combinatorial interpretations of $B_{n,k}((l-1)!a_l)$ and $B_{n,k}(l!a_l)$, we can proceed similarly as above. □

Definition 2 *Let $(a_n; n \geq 1)$ and $(b_n; n \geq 1)$ be two sequences of nonnegative integers. The number $B_{n+r,k+r}^{(r)}(a_l; b_l)$ counts the number of partitions of a $(n+r)$-set into $(k+r)$ blocks such that:*

- *the r first elements are in different blocks,*
- *any block of the length i with no elements of the r first elements, can be colored with a_i colors,*
- *any block of the length i with one element of the r first elements, can be colored with b_i colors. We assume that any block with 0 color does not appear in partitions.*

On using this definition, the following theorem gives an interesting relation which help us to find a family of polynomials generalize the above numbers.

On using combinatorial arguments, the partial r-Bell polynomials admit the following expression.

Theorem 3 *For $n \geq k \geq r \geq 1$, the partial r-Bell polynomials can be written as*

$$B_{n,k}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots) = \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k=n+r-k} \frac{b_{n_1+1} \cdots b_{n_r+1}}{n_1! \cdots n_r!} \frac{a_{n_{r+1}+1} \cdots a_{n_k+1}}{(n_{r+1}+1)! \cdots (n_k+1)!}.$$

Proof. Consider the $(n+r)$-set as union of two sets $\mathbf{R}$ which contains the r first elements and $\mathbf{N}$ which contains the n last elements. To partition a $(n+r)$-set into $k+r$ blocks $B_1, \ldots, B_{k+r}$ given as in Definition 2, let the elements of $\mathbf{R}$ be in different r blocks $B_1, \ldots, B_r$.
There is $\frac{1}{k!}\binom{n}{n_1,\ldots,n_{k+r}} b_{n_1+1} \cdots b_{n_r+1} a_{n_{r+1}} \cdots a_{n_{r+k}}$ ways to choose $n_1, \ldots, n_{k+r}$ in $\mathbf{N}$ on using colors, such that
- $n_1 \geq 0, \ldots, n_r \geq 0$: $n_1, \ldots, n_r$ to be, respectively, in $B_1, \ldots, B_r$ with $b_{n_1+1} \cdots b_{n_r+1}$ ways to color these blocks,
- $n_{r+1} \geq 1, \ldots, n_{k+r} \geq 1$: $n_{r+1}, \ldots, n_{k+r}$ to be, respectively, in $B_{r+1}, \ldots, B_{k+r}$ with $\frac{1}{k!} a_{n_{r+1}} \cdots a_{n_{r+k}}$ ways to color these blocks.

Then, the total number of colored partitions is

$$B_{n+r,k+r}^{(r)}(a_1, a_2, \ldots; b_1, b_2, \ldots) = \frac{1}{k!} \sum_{(n_1,\ldots,n_{k+r}) \in M_{n+r,k+r}} \binom{n}{n_1, \ldots, n_{k+r}} b_{n_1+1} \cdots b_{n_r+1} a_{n_{r+1}} \cdots a_{n_{r+k}},$$

where $M_{n,k} = \left\{(n_1, \ldots, n_k) : n_1 + \cdots + n_k = n,\ (n_1, \ldots, n_r, n_{r+1}-1, \ldots, n_k - 1) \in \mathbb{N}^k\right\}$. □

On using Theorem 3, we may state that:

Corollary 4 *We have*

$$\sum_{n \geq k} B_{n+r,k+r}^{(r)}(a_l; b_l) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{j \geq 1} a_j \frac{t^j}{j!} \right)^k \left(\sum_{j \geq 0} b_{j+1} \frac{t^j}{j!} \right)^r. \tag{2}$$

Proof. From Theorem 3 we get

$$\begin{aligned}
&\sum_{n \geq k} B_{n+r,k+r}^{(r)}(a_l; b_l) \frac{t^n}{n!} \\
&= \sum_{n \geq k} \left(\frac{1}{k!} \sum_{n_1+\cdots+n_{r+k}=n+r-k} \frac{b_{n_1+1} \cdots b_{n_r+1}}{n_1! \cdots n_r!} \frac{a_{n_{r+1}+1} \cdots a_{n_{r+k}+1}}{(n_{r+1}+1)! \cdots (n_{r+k}+1)!} \right) t^n \\
&= \frac{1}{k!} \sum_{n_1 \geq 0, \ldots, n_r \geq 0,\ n_{r+1} \geq 1, \ldots, n_{r+k} \geq 1} \frac{b_{n_1+1} \cdots b_{n_r+1}}{n_1! \cdots n_r!} \frac{a_{n_{r+1}} \cdots a_{n_{r+k}}}{n_{r+1}! \cdots n_{r+k}!} t^{n_1+\cdots+n_{r+k}} \\
&= \frac{1}{k!} \left(\sum_{j \geq 1} a_j \frac{t^j}{j!} \right)^k \left(\sum_{j \geq 0} b_{j+1} \frac{t^j}{j!} \right)^r.
\end{aligned}$$

□

To give an explicit expression of the number $B_{n+r,k+r}^{(r)}(a_l;b_l)$ generalizing the formula (1), we use the Touchard polynomials defined in [3] as follows. Let $(x_i; i\geq 1)$ and $(y_i; i\geq 1)$ be two sequences of indeterminates, the Touchard polynomials

$$T_{n,k}(x_j,y_j)\equiv T_{n,k}(x_1,\ldots,x_n;y_1,\ldots,y_n),\ n=k,k+1,\cdots,$$

are defined by $T_{0,0}=1$ and the sum

$$T_{n,k}(x_1,x_2,\ldots;y_1,y_2,\ldots)=\sum_{\Lambda(n,k)}\left[\frac{n!}{k_1!k_2!\cdots}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\right]\left[\frac{1}{r_1!r_2!\cdots}\left(\frac{y_1}{1!}\right)^{r_1}\left(\frac{y_2}{2!}\right)^{r_2}\cdots\right],$$

where

$$\Lambda(n,k)=\left\{\mathbf{k}=(k_1,k_2,\ldots):k_i\in\mathbb{N},\ i\geq 1,\ \sum_{i\geq 1}k_i=k,\ \sum_{i\geq 1}i(k_i+r_i)=n\right\},$$

and admits a vertical generating function given by

$$\sum_{n=k}^{\infty}T_{n,k}(x_1,x_2,\ldots;y_1,y_2,\ldots)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{i\geq 1}x_i\frac{t^i}{i!}\right)^k\exp\left(\sum_{i\geq 1}y_i\frac{t^i}{i!}\right),\ k=0,1,\ldots. \tag{3}$$

Theorem 5 *We have*

$$B_{n+r,k+r}^{(r)}(a_l;b_l)=\sum_{\Lambda(n,k,r)}\left[\frac{n!}{k_1!k_2!\cdots}\left(\frac{a_1}{1!}\right)^{k_1}\left(\frac{a_2}{2!}\right)^{k_2}\cdots\right]\left[\frac{r!}{r_0!r_1!\cdots}\left(\frac{b_1}{0!}\right)^{r_0}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\right],$$

where

$$\Lambda(n,k,r)=\left\{\begin{array}{c}(\mathbf{k},\mathbf{r})=((k_i:i\geq 1);(r_i:i\geq 0)):\\ k_i\in\mathbb{N},\ r_i\in\mathbb{N},\ \sum_{i\geq 1}k_i=k,\ \sum_{i\geq 0}r_i=r,\ \sum_{i\geq 1}i(k_i+r_i)=n\end{array}\right\}.$$

Proof. Setting

$$\pi(n,k,j)=\left\{\mathbf{k}=(k_1,\ldots,k_n;r_1,\ldots,r_n):\sum_{i=1}^{n}k_i=k,\ \sum_{i=1}^{n}r_i=j,\ \sum_{i=1}^{n}i(k_i+r_i)=n\right\},$$

$$\Pi(n,k,r)=\left\{\mathbf{k}=(k_1,\ldots,k_n;r_0,\ldots,r_n):\sum_{i=1}^{n}k_i=k,\ \sum_{i=0}^{n}r_i=r,\ \sum_{i=1}^{n}i(k_i+r_i)=n\right\},$$

$$T_{n,k,s}(a_l;b_{l+1})=\sum_{\pi(n,k,s)}\frac{n!}{k_1!\cdots k_n!r_1!\cdots r_n!}\left(\frac{a_1}{1!}\right)^{k_1}\cdots\left(\frac{a_n}{n!}\right)^{k_n}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\left(\frac{b_{n+1}}{n!}\right)^{r_n}.$$

On using Corollary 4, we obtain

$$\sum_{n\geq k}\left(\exp(-b_1u)\sum_{r\geq 0}B_{n+r,k+r}^{(r)}(a_l;b_l)\frac{u^r}{r!}\right)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{j\geq 1}a_j\frac{t^j}{j!}\right)^k\exp\left(u\sum_{j\geq 1}b_{j+1}\frac{t^j}{j!}\right).$$

Upon using (3), the last expression shows that

$$\begin{aligned}
&\exp\left(-b_1 u\right) \sum_{r\geq 0} B_{n+r,k+r}^{(r)}\left(a_l;b_l\right)\frac{u^r}{r!}\\
&= T_{n,k}\left(a_1,\ldots,a_n;ub_2,\ldots,ub_{n+1}\right)\\
&= \sum_{\pi(n,k)} \frac{n!}{k_1!\cdots k_n! r_1!\cdots r_n!}\left(\frac{a_1}{1!}\right)^{k_1}\cdots\left(\frac{a_n}{n!}\right)^{k_n}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\left(\frac{b_{n+1}}{n!}\right)^{r_n} u^{r_1+\cdots+r_n}\\
&= \sum_{s\geq 0} u^s \sum_{\pi(n,k,s)} \frac{n!s!}{k_1!\cdots k_n! r_1!\cdots r_n!}\left(\frac{a_1}{1!}\right)^{k_1}\cdots\left(\frac{a_n}{n!}\right)^{k_n}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\left(\frac{b_{n+1}}{n!}\right)^{r_n}\\
&= \sum_{s\geq 0} s! T_{n,k,s}\left(a_l;b_{l+1}\right)\frac{u^s}{s!}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\sum_{r\geq 0} B_{n+r,k+r}^{(r)}\left(a_l;b_l\right)\frac{u^r}{r!} &= \exp\left(b_1 u\right)\sum_{r\geq 0} s! T_{n,k,s}\left(a_l;b_{l+1}\right)\frac{u^s}{s!}\\
&= \sum_{r\geq 0}\frac{u^r}{r!}\sum_{j=0}^{r}\binom{r}{j} j! b_1^{r-j} T_{n,k,j}\left(a_l;b_{l+1}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&B_{n+r,k+r}^{(r)}\left(a_l;b_l\right)\\
&= \sum_{j=0}^{r}\binom{r}{j} b_1^{r-j} j! T_{n,k,j}\left(a_l;b_{l+1}\right)\\
&= \sum_{r_0=0}^{r}\frac{b_1^{r_0}}{r_0!}\sum_{\pi(n,k,r_0-j)}\frac{n!r!}{k_1!\cdots k_n! r_1!\cdots r_n!}\left(\frac{a_1}{1!}\right)^{k_1}\cdots\left(\frac{a_n}{n!}\right)^{k_n}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\left(\frac{b_{n+1}}{n!}\right)^{r_n}\\
&= \sum_{\Pi(n,k,r)}\frac{n!r!}{k_1!\cdots k_n! r_0! r_1!\cdots r_n!}\left(\frac{a_1}{1!}\right)^{k_1}\cdots\left(\frac{a_n}{n!}\right)^{k_n}\left(\frac{b_1}{0!}\right)^{r_0}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\left(\frac{b_{n+1}}{n!}\right)^{r_n}.
\end{aligned}$$

The elements of $\Lambda\left(n,k,r\right)$ can be reduced to those of $\Pi\left(n,k,r\right)$ because we get necessarily $k_j = r_{j+1} = 0$ for $j \geq n+1$. Thus, the expression of $B_{n+r,k+r}^{(r)}\left(a_l;b_l\right)$ results. $\square$

3 Some properties of the partial r-Bell polynomials

Other combinatorial processes give the following identity.

Proposition 6 *We have*

$$B_{n+r,k+r}^{(r)}\left(a_1,a_2,\ldots;b_1,b_2,\ldots\right) = \sum_{i=0}^{r}\sum_{j=0}^{k}\binom{r}{i}\binom{n}{j} b_1^i a_1^j B_{n-j+r-i,k-j+r-i}^{(r-i)}\left(0,a_2,a_3,\ldots;0,b_2,b_3,\ldots\right). \tag{4}$$

Proof. Consider the $(n+r)$-set as union of two sets $\mathbf{R}$ which contains the r first elements and $\mathbf{N}$ which contains the n last elements. Choice i elements in $\mathbf{R}$ and j elements in $\mathbf{N}$ to form $i+j$ singletons.

Because each singleton can be colored with b_1 colors if it is in $\mathbf{R}$ and a_1 colors if it is in $\mathbf{N}$, then, the number of the colored singletons is $\binom{r}{i}\binom{n}{j}b_1^i a_1^j$. The elements not really used is of number $r-i+n-j$ which can be partitioned into $r-i+k-j$ colored partitions with non singletons (such that the $r-i$ first elements are in different blocks) in $B_{n-j+r-i,k-j+r-i}^{(r-i)}(0,a_2,a_3,\ldots;0,b_2,b_3,\ldots)$ ways. Then, for a fixed i and a fixed j, there are $\binom{r}{i}\binom{n}{j}b_1^i a_1^j B_{n-j+r-i,k-j+r-i}^{(r-i)}(0,a_2,a_3,\ldots;0,b_2,b_3,\ldots)$ colored partitions. So, the number of all colored partitions is

$$\sum_{i=0}^{r}\sum_{j=0}^{k}\binom{r}{i}\binom{n}{j}b_1^i a_1^j B_{n-j+r-i,k-j+r-i}^{(r-i)}(0,a_2,a_3,\ldots;0,b_2,b_3,\ldots)=B_{n+r,k+r}^{(r)}(a_1,a_2,\ldots;b_1,b_2,\ldots).$$

□

On using Corollary 4 or Theorem 5, we can verity that

Proposition 7 *We have*

$$B_{n+r,k+r}^{(r)}(xa_l;yb_l)=x^k y^r B_{n+r,k+r}^{(r)}(a_l;b_l), \tag{5}$$

$$B_{n+r,k+r}^{(r)}\left(x^l a_l;x^l b_l\right)=x^{n+r}B_{n+r,k+r}^{(r)}(a_l;b_l), \tag{6}$$

$$B_{n+r,k+r}^{(r)}\left(x^{l-1}a_l;x^{l-1}b_l\right)=x^{n-k}B_{n+r,k+r}^{(r)}(a_l;b_l). \tag{7}$$

The relations of the following proposition generalize some of the known relations on partial Bell polynomials.

Proposition 8 *We have*

$$\sum_{j=1}^{n}\binom{n}{j}a_j B_{n+r-j,k+r-1}^{(r)}(a_l;b_l)=kB_{n+r,k+r}^{(r)}(a_l;b_l),$$

$$\sum_{j=1}^{n}\binom{n}{j-1}b_j B_{n-j+r-1,k+r-1}^{(r-1)}(a_l;b_l)=rB_{n+r,k+r}^{(r)}(a_l;b_l)$$

and

$$\sum_{j=1}^{n}ja_j\binom{n}{j}B_{n+r-j,k+r-1}^{(r)}(a_l;b_l)+r\sum_{j=1}^{n}jb_j\binom{n}{j-1}B_{n-j+r-1,k+r-1}^{(r-1)}(a_l;b_l)=(n+r)B_{n+r,k+r}^{(r)}(a_l;b_l).$$

Proof. On using Corollary 4, we deduce that

$$\frac{\partial}{\partial a_j}B_{n+r,k+r}^{(r)}(a_l;b_l)=\binom{n}{j}B_{n-j+r,k-1+r}^{(r)}(a_l;b_l),$$

$$\frac{\partial}{\partial b_j}B_{n+r,k+r}^{(r)}(a_l;b_l)=\binom{n}{j-1}B_{n-j+r-1,k+r-1}^{(r-1)}(a_l;b_l).$$

Then, by derivation the two sides of (5) in first time respect to x and in second time respect to y, we obtain

$$\sum_{j=1}^{n}\binom{n}{j}a_j B_{n+r-j,k+r-1}^{(r)}(a_l x;yb_l)=kx^{k-1}y^r B_{n+r,k+r}^{(r)}(a_l;b_l),$$

$$\sum_{j=1}^{n}\binom{n}{j-1}b_j B_{n-j+r-1,k+r-1}^{(r-1)}(a_l x;yb_l)=rx^k y^{r-1}B_{n+r,k+r}^{(r)}(a_l;b_l),$$

and by derivation the two sides of (6) respect to x, we obtain

$$\sum_{j=1}^{n} jx^{j-1}a_j\binom{n}{j}B_{n+r-j,k+r-1}^{(r)}\left(a_lx^l;b_ly^l\right)+r\sum_{j=1}^{n} jx^{j-1}b_j\binom{n}{j-1}B_{n-j+r-1,k+r-1}^{(r-1)}\left(a_lx^l;b_ly^l\right)$$
$$=(n+r)\,x^{n+r-1}B_{n+r,k+r}^{(r)}\left(a_l;b_l\right).$$

The three relations of the proposition follow by taking $x=y=1$. □

The partial r-Bell polynomials can be expressed by the partial bell polynomials as follows.

Proposition 9 *We have*

$$B_{n+r,k+r}^{(r)}\left(a_l;b_l\right)=\binom{n+r}{r}^{-1}\sum_{j=k}^{n}\binom{n+r}{j}B_{j,k}\left(a_l\right)B_{n+r-j,r}\left(lb_l\right).$$

Proof. This proposition follows from the expansion

$$\begin{aligned}
t^r\sum_{n\geq k}B_{n+r,k+r}^{(r)}\left(a_l;b_l\right)\frac{t^n}{n!} &= \frac{t^r}{k!}\left(\sum_{j\geq 1}a_j\frac{t^j}{j!}\right)^k\left(\sum_{j\geq 0}b_{j+1}\frac{t^j}{j!}\right)^r\\
&=\frac{1}{k!}\left(\sum_{j\geq 1}a_j\frac{t^j}{j!}\right)^k\left(\sum_{j\geq 1}jb_j\frac{t^j}{j!}\right)^r\\
&=r!\left(\sum_{i\geq k}B_{i,k}\left(a_l\right)\frac{t^i}{i!}\right)\left(\sum_{j\geq r}B_{j,r}\left(lb_l\right)\frac{t^j}{j!}\right).
\end{aligned}$$

□

Proposition 10 *We have*

$$\binom{n}{r}B_{n+k-r,k+r}^{(r)}\left(la_l;b_l\right)=\binom{n}{k}B_{n-k+r,k+r}^{(k)}\left(lb_l;a_l\right),\quad n\geq 2\max\left(k,r\right).$$

Proof. From Corollary 4 we get

$$\frac{t^r}{r!}\sum_{n\geq k}B_{n+k,k+r}^{(r)}\left(la_l;b_l\right)\frac{t^n}{n!}=\frac{1}{k!r!}\left(\sum_{j\geq 1}ja_j\frac{t^j}{j!}\right)^k\left(\sum_{j\geq 1}jb_j\frac{t^j}{j!}\right)^r$$

and by the symmetry respect to $(k,(a_j))$ and $(r,(b_j))$ in the last expression we get

$$\frac{t^r}{r!}\sum_{n\geq k}B_{n+k,k+r}^{(r)}\left(la_l;b_l\right)\frac{t^n}{n!}=\frac{t^k}{k!}\sum_{n\geq r}B_{n+r,k+r}^{(k)}\left(lb_l;a_l\right)\frac{t^n}{n!}.$$

So, we obtain the desired identity. □

4 New combinatorial interpretations of the r-Whitney numbers

The r-Whitney numbers of both kinds $w_{m,r}(n,k)$ and $W_{m,r}(n,k)$ are introduced by Mező [6] and the r-Whitney-Lah numbers $L_{m,r}(n,k)$ are introduced by Cheon and Jung [5, 9]. Some of the properties of these numbers are given in [5]. In this paragraph we use the combinatorial interpretation of the partial r-Bell polynomials given above to deduce a new combinatorial interpretations for the numbers $|w_{m,r}(n,k)|$, $W_{m,r}(n,k)$ and $L_{m,r}(n,k)$.

The r-Whitney numbers of the first kind $w_{m,r}(n,k)$ are given by their generating function

$$\sum_{n\geq k} w_{m,r}(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\ln(1+mt)\right)^k\left((1+mt)^{-\frac{1}{m}}\right)^r.$$

So that

$$w_{m,r}(n,k) = (-1)^{n-k+r} B_{n+r,k+r}^{(r)}\left((l-1)!m^{l-1}; (m+1)(2m+1)\cdots((l-1)m+1)\right).$$

This means that the absolute r-Whitney number of the first kind $|w_{m,r}(n,k)|$ counts the number of partitions of a n-set into k blocks such that
- the r first elements are in different blocks,
- any block of cardinality i and no contain an element of the r first elements can be colored with $(i-1)!m^{i-1}$ colors, and,
- any block of cardinality i and contain one element of the r first elements can be colored with $(m+1)(2m+1)\cdots((i-1)m+1)$ colors.

The r-Whitney numbers of the second kind $W_{m,r}(n,k)$ are given by their generating function

$$\sum_{n\geq k} W_{m,r}(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\frac{\exp(mt)-1}{m}\right)^k \exp(rt).$$

So that

$$W_{m,r}(n,k) = B_{n+r,k+r}^{(r)}\left(m^{l-1}; 1\right).$$

This means that the r-Whitney number of the second kind $W_{m,r}(n,k)$ counts the number of partitions of a n-set into k blocks such that
- the r first elements are in different blocks,
- any block of cardinality i and no contain any element of the r first elements can be colored with m^{i-1} colors, and,
- any block of cardinality i and contain one element of the r first elements can be colored with one color.

The r-Whitney-Lah numbers $L_{m,r}(n,k)$ are given by their generating function

$$\sum_{n\geq k} L_{m,r}(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(t(1-mt)^{-1}\right)^k\left((1-mt)^{-\frac{2}{m}}\right)^r.$$

So that

$$L_{m,r}(n,k) = B_{n+r,k+r}^{(r)}\left(l!m^{l-1}; 2(m+2)\cdots((l-1)m+2)\right).$$

This means that the r-Whitney number of the second kind $L_{m,r}(n,k)$ counts the number of partitions of a n-set into k blocks such that
- the r first elements are in different blocks,
- any block no contain an element of the r first elements and is of length i can be colored with $i!m^{i-1}$ colors, and,
- any block of cardinality i and contain one element of the r first elements can be colored with $2(m+2)\cdots((i-1)m+2)$ colors.

5 Application to the r-Stirling numbers of the second kind

From Definition 2 we may state that the number $B_{n,k}^{(r)}(a_l) := B_{n,k}^{(r)}(a_l, a_l)$ counts the number of partitions of n-set into k blocks such that the blocks of the same cardinality i can be colored with a_i colors (such cycles with $a_i = 0$ does not exist) and the r first elements are in different blocks.

For $a_n = 1$, $n \geq 1$, we get the know r-Stirling numbers of the second kind

$$\left\{ {n \atop k} \right\}_r = B_{n,k}^{(r)}(1, 1, \cdots)$$

which counts the number of partitions of a n-set into k blocks such that the r first elements are in different blocks.

For $a_n = 1$, $n \geq m$ and $a_n = 0$, $n \leq m - 1$, we get the m-associated r-Stirling numbers of the second kind

$$\left\{ {n \atop k} \right\}_r^{m\uparrow} = B_{n,k}^{(r)}\left(\overbrace{0, \cdots, 0}^{m-1}, 1, 1, \cdots\right)$$

which counts the number of partitions of a n-set into k blocks such that the cardinality of any block is at least m elements and the r first elements are in different blocks.

For $a_n = 0$, $n \geq m + 1$ and $a_n = 1$, $n \leq m$, we get the m-truncated r-Stirling numbers of the second kind

$$\left\{ {n \atop k} \right\}_r^{m\downarrow} = B_{n,k}^{(r)}\left(\overbrace{1, \cdots, 1}^{m}, 0, 0, \cdots\right)$$

which counts the number of partitions of a n-set into k blocks such that the cardinality of any block is $\leq m$ elements and the r first elements are in different blocks.

For $a_{2n-1} = 0$ and $a_{2n} = 1$, $n \geq 1$, we get the r-Stirling numbers of the second kind in even parts

$$\left\{ {n \atop k} \right\}_r^{even} = B_{n,k}^{(r)}(0, 1, 0, 1, 0, \cdots)$$

which counts the number of partitions of a n-set into k blocks such that the cardinality of any block is even and the r first elements are in different blocks.

For $a_{2n-1} = 1$ and $a_{2n} = 0$, $n \geq 1$, we get the r-Stirling numbers of the second kind in odd parts

$$\left\{ {n \atop k} \right\}_r^{odd} = B_{n,k}^{(r)}(1, 0, 1, 0, \cdots)$$

which counts the number of partitions of a n-set into k blocks such that the cardinality of any block is odd and the r first elements are in different blocks.

6 Application to the r-Stirling numbers of the first kind

We start this application by giving a second combinatorial interpretation of the partial r-Bell polynomials.

Proposition 11 *The number $B_{n,k}^{(r)}((l-1)!a_l) := B_{n,k}^{(r)}((l-1)!a_l, (l-1)!a_l)$ counts the number of permutations of a n-set into k cycles such that the cycles of the same length i can be colored with a_i colors (such cycles with $a_i = 0$ does not exist) and the r first elements are in different cycles.*

Proof. Let $\Pi_{r,k}$ be the set of partitions π of the set n-set into k blocks such that the blocks of the same cardinality i posses $(i-1)!a_i$ colors and the r first elements are in different blocks, and, $P_{r,k}$ be the set of permutations P of the elements of the set n-set into k cycles such that the cycles of the same length i can be colored with a_i colors the r first elements are in different cycles. The application $\varphi : \Pi_{r,k} \to P_{r,k}$ which associate any (colored) partition π of $\Pi_{r,k}$, $\pi = S_1 \cup \cdots \cup S_k$, $1 \leq k \leq n$, a (colored) permutation P of $P_{r,k}$, $P = C_1 \cup \cdots \cup C_k$, such that the elements of C_i are exactly those of S_i. It is obvious that the application φ is bijective. $\square$

For $a_n = (n-1)!$, $n \geq 1$, we get the known r-Stirling numbers of the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = B_{n,k}^{(r)}\left(0!, 1!, 2!, \cdots\right)$$

which counts the number of permutations of a n-set into k cycles such that the r first elements are in different cycles.

For $a_n = (n-1)!$, $n \geq m$ and $a_n = 0$, $n \leq m-1$, we get the m-associated r-Stirling numbers of the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{m\uparrow} = B_{n,k}^{(r)}\left(\overbrace{0, \cdots, 0}^{m-1}, (m-1)!, m!, \cdots\right)$$

which counts the number of permutations of a n-set into k cycles such that the length of any cycle is equal at least m and the r first elements are in different cycles.

For $a_n = (n-1)!$, $n \leq m$ and $a_n = 0$, $n \geq m+1$, we get the m-truncated r-Stirling numbers of the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{m\downarrow} = B_{n,k}^{(r)}\left(\overbrace{0!, \cdots, (m-1)!}^{m}, 0, 0, \cdots\right)$$

which counts the number of permutations of a n-set into k cycles such that the length of any cycle is $\leq m$ and the r first elements are in different cycles.

For $a_{2n-1} = 0$ and $a_{2n} = (2n-1)!$, $n \geq 1$, we get the r-Stirling numbers of the first kind in cycles of even lengths

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{even} = B_{n,k}^{(r)}\left(0, 1!, 0, 3!, 0, 5!, 0, \cdots\right)$$

which counts the number of permutations of a n-set into k cycles such that the length of any cycle is even and the r first elements are in different cycles.

For $a_{2n-1} = (2n-2)!$ and $a_{2n} = 0$, $n \geq 1$, we get the r-Stirling numbers of the first kind in cycles of odd lengths

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{odd} = B_{n,k}^{(r)}\left(0!, 0, 2!, 0, 4!, 0, \cdots\right)$$

which counts the number of permutations of a n-set into k cycles such that the length of any cycle is odd and the r first elements are in different cycles.

7 Application to the r-Lah numbers

Then, similarly to the partial Bell polynomials, we establish the following (third) combinatorial interpretation of the partial r-Bell polynomials.

Proposition 12 *The number* $B_{n,k}^{(r)}(l!a_l) := B_{n,k}^{(r)}(l!a_l, l!a_l)$ *counts the number of partitions of a* n*-set into* k *ordered blocks such that the blocks of the same cardinality* i *can be colored with* a_i *colors (such block with* $a_i = 0$ *does not exist) and the* r *first elements are in different blocks.*

Proof. Let $\Pi'_{r,k}$ be the set of partitions π of the a n-set into k blocks such that the blocks of the same cardinality i can be colored with $i!a_i$ colors and the r first elements are in different blocks, and, $\Pi_{r,k}^{ord}$ be the set of partitions π^{ord} of the a n-set into k ordered blocks such that the blocks of the same length i can be colored with a_i the r first elements are in different blocks. The application $\varphi : \Pi'_{r,k} \to \Pi_{r,k}^{ord}$ which associate a (colored) partition π of $\Pi'_{r,k}$, $\pi = S_1 \cup \cdots \cup S_k$, $1 \leq k \leq n$, a (colored) partition of ordered blocks π^{ord} of Π_r^{ord}, $\pi^{ord} = P_1 \cup \cdots \cup P_k$, such that the elements of P_i are exactly those of S_i. It is obvious that the application φ is bijective. □

For $a_n = n!$, $n \geq 1$, we get the r-Lah numbers

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = B_{n,k}^{(r)}(1!, 2!, 3!, \cdots).$$

The r-Lah number $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r$ counts the number of partitions of a n-set into k ordered blocks such that the r first elements are in different blocks.

For $a_n = 0$, $n \leq m-1$, and $a_n = n!$, $n \geq m$, we get the m-degenerate r-Lah numbers

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{m\uparrow} = B_{n,k}^{(r)}\left(\overbrace{0, \cdots, 0}^{m-1}, m!, (m+1)!, \cdots\right)$$

which counts the number of partitions of a n-set into k ordered blocks such that the cardinality of any block is $\geq m$ and the r first elements are in different blocks.

For $a_n = n!$, $n \leq m$, and $a_n = 0$, $n \geq m+1$, we get the m-truncated r-Lah numbers

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{m\downarrow} = B_{n,k}^{(r)}\left(\overbrace{1!, \cdots, m!}^{m}, 0, 0, \cdots\right)$$

which counts the number of partitions of a n-set into k ordered blocks such that the cardinality of any block is $\leq m$ and the r first elements are in different blocks.

For $a_{2n-1} = 0$ and $a_{2n} = (2n)!$, $n \geq 1$, we get the r-Lah numbers in blocks of even cardinalities

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{even} = B_{n,k}^{(r)}(0, 2!, 0, 4!, 0, 6!, 0, \cdots),$$

which represents the number of partitions of a n-set into k ordered blocks such that the cardinality of any block is even and the r first elements are in different blocks.

For $a_{2n-1} = (2n-1)!$ and $a_{2n} = 0$, $n \geq 1$, we get the r-Lah numbers in blocks of even cardinalities

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{odd} = B_{n,k}^{(r)}(1!, 0, 3!, 0, 5!, 0, \cdots)$$

which represents the number of partitions of a n-set into k ordered blocks such that the cardinality of any block is odd and the r first elements are in different blocks.

8 Application to sum of independent random variables

It is known that for a sequence of independent random variables $\{X_n\}$ with all its moments exist and are the same, $\mu_n = \mathrm{E}(X^n)$ we have $\mathrm{E}(S_p^n) = \binom{n+p}{p}^{-1} B_{n+p,p}(l\mu_{l-1})$. The following theorem generalize this result.

Theorem 13 *Let $\{X_n\}$ and $\{Y_n\}$ be two independent sequences of independent random variables with all their moments exist and are the same, $\mu_n = \mathrm{E}(X^n)$, $\nu_n = \mathrm{E}(Y^n)$ and let*

$$S_{p,q} = X_1 + \cdots + X_p + Y_1 + \cdots + Y_q.$$

Then we have

$$\mathrm{E}\left(S_{p,q}^n\right) = \binom{n+p}{p}^{-1} B_{n+p+q,p+q}^{(q)}\left(l\mu_{l-1}, \nu_{l-1}\right).$$

Proof. Let $\varphi_X(t)$ be the common generating function of moments for X_n, $n \geq 1$, $\varphi_Y(t)$ be the common generating function of moments for Y_n, $n \geq 1$, and, $\varphi_{S_{p,q}}(t)$ be the generating function of moments of $S_{p,q}$. Then, in first part, we get

$$\begin{aligned} t^p \varphi_{S_{p,q}}(t) &= \mathrm{E}\left(t^p \exp\left(tS_{p,q}\right)\right) \\ &= \left(\mathrm{E}\left(t\exp\left(tX_1\right)\right)\right)^p \left(\mathrm{E}\left(\exp\left(tY_1\right)\right)\right)^q \\ &= \left(\sum_{j\geq 1} j\mu_{j-1}\frac{t^j}{j!}\right)^p \left(\sum_{j\geq 0} \nu_j \frac{t^j}{j!}\right)^q \\ &= p!\sum_{n\geq p} B_{n+q,p+q}^{(q)}\left(l\mu_{l-1}, \nu_{l-1}\right)\frac{t^n}{n!}, \end{aligned}$$

and, in second part, we have

$$t^p \varphi_{S_{p,q}}(t) = \sum_{j\geq 0} \mathrm{E}\left(S_{p,q}^j\right)\frac{t^{j+p}}{j!} = \sum_{n\geq p} \frac{n!}{(n-p)!} \mathrm{E}\left(S_{p,q}^{n-p}\right)\frac{t^n}{n!}.$$

□

For the choice $q_0 = 1$ and $q_j = 0$ if $j \geq 1$ in the last theorem, we may state that:

Corollary 14 *Let $\{X_n\}$ be a sequence of independent random variables with all their moments exist and are the same, $\mu_n = \mathrm{E}(X^n)$ and*

$$S_{p,q} = X_1 + \cdots + X_p + q.$$

Then we have

$$\mathrm{E}\left(S_{p,q}^n\right) = \binom{n+p}{p}^{-1} B_{n+p+q,p+q}^{(q)}\left(l\mu_{l-1}, 1\right).$$

Example 1 *Let $\{X_n\}$ be a sequence of independent random variables with the same law of probability $\mathcal{U}(0,1)$.*

$$S_{p,q} = X_1 + \cdots + X_p + r.$$

Then we have

$$\left\{ {n+p+r \atop p+r} \right\}_r = \binom{n+p}{p} \mathrm{E}\left(S_{p,r}^n\right).$$

It is also known that for a sequence of independent discrete random variables $\{X_n\}$ with the same law of probability $p_j := P(X_1 = j)$, $j \geq 0$ we have $P(S_p = n) = \frac{p!}{(n+p)!} B_{n+p,p}(l! p_{l-1})$. The following theorem generalize this result.

Theorem 15 *Let $\{X_n\}$ and $\{Y_n\}$ be two independent sequences of independent random variables with $p_j := P(X_n = j)$, $q_j := P(Y_n = j)$ and let*

$$S_{p,q} = X_1 + \cdots + X_p + Y_1 + \cdots + Y_q.$$

Then we have

$$P(S_{p,q} = n) = \frac{p!}{(n+p)!} B^{(q)}_{n+p+q,p+q}(l! p_{l-1}, (l-1)! q_{l-1}).$$

Proof. It suffices to take in the last theorem $q_0 = 1$ and $q_j = 0$ if $j \geq 1$.

$$\begin{aligned} \sum_{n \geq p} P(S_{p,q} = n - p) t^n &= t^p \sum_{s \geq 0} P(S_{p,q} = s) t^s \\ &= t^p \operatorname{E}\left(t^{S_{p,q}}\right) \\ &= \left(\sum_{j \geq 1} p_{j-1} t^j\right)^p \left(\sum_{j \geq 0} q_j t^j\right)^q \\ &= p! \sum_{n \geq p} B^{(q)}_{n+q,p+q}(l! p_{l-1}, (l-1)! q_{l-1}) \frac{t^n}{n!}. \end{aligned}$$

This gives $P(S_{p,q} = n) = \frac{p!}{(n+p)!} B^{(q)}_{n+p+q,p+q}(l! p_{l-1}, (l-1)! q_{j-1})$. □

For the choice $q_0 = 1$ and $q_j = 0$ if $j \geq 1$ in the last theorem, we may state that:

Corollary 16 *Let $\{X_n\}$ be a sequence of independent discrete random variables with the same law of probability $p_j := P(X_1 = j)$ and*

$$S_{p,q} = X_1 + \cdots + X_p + q.$$

Then we have

$$P(S_{p,q} = n) = \frac{p!}{(n+p)!} B_{n+p+q,p+q}(l! p_{l-1}, l! p_{l-1}).$$

9 Application on the successive derivatives of a function

Let $F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!} \in C^\infty(0)$ and $G(x) = \sum_{n \geq 1} g_n \frac{(x-a)^n}{n!} \in C^\infty(a)$. It is shown in [4] that

$$\frac{d^n}{dx^n}(F(G(x)))\bigg|_{x=a} = \sum_{k=0}^{n} f_k B_{n,k}(g_j).$$

The following theorem gives a similar result on using the partial r-Bell polynomials.

Theorem 17 *Let F, G be as above and $H(x) = \sum_{n \geq 1} h_n \frac{(x-a)^n}{n!} \in C^\infty(a)$. Then, we have*

$$\frac{d^n}{dx^n}\left(\left(\frac{d}{dx} H(x)\right)^r F(G(x))\right)\bigg|_{x=a} = \sum_{k=0}^{n} f_k B^{(r)}_{n+r,k+r}(g_j, h_j).$$

Proof. This follows from

$$\begin{aligned}
&\sum_{n\geq 0}\left(\sum_{k=0}^{n} f_k B_{n+r,k+r}^{(r)}\left(g_j,h_j\right)\right)\frac{(x-a)^n}{n!}\\
&=\sum_{k\geq 0} f_k \sum_{n\geq k} B_{n+r,k+r}^{(r)}\left(g_j,h_j\right)\frac{(x-a)^n}{n!}\\
&=\left(\sum_{j\geq 0} h_{j+1}\frac{(x-a)^j}{j!}\right)^r \sum_{k\geq 0}\frac{f_k}{k!}\left(\sum_{j\geq 1} g_j \frac{(x-a)^j}{j!}\right)^k\\
&=\left(\frac{d}{dx}H\left(x\right)\right)^r F\left(G\left(x\right)\right).
\end{aligned}$$

□

For the choice $F\left(x\right)=\exp\left(x\right)$, we obtain:

Corollary 18 *For $G,H\in C^{\infty}\left(a\right)$ with $G\left(0\right)=0$, we have*

$$\frac{d^n}{da^n}\left(\left(\frac{d}{da}H\left(a\right)\right)^r \exp\left(G\left(a\right)\right)\right)=\exp\left(G\left(a\right)\right)\sum_{k=0}^{n} B_{n+r,k+r}^{(r)}\left(\frac{d^j}{da^j}G\left(a\right),\frac{d^j}{da^j}H\left(a\right)\right).$$

Example 2 *Let $G\left(a\right)=\frac{\exp(ma)-1}{m}$ and $H\left(a\right)=\exp\left(a\right)$. On using Corollary 18 and the generating function of the numbers $W_{m,r}\left(n,k\right)$ given above, we get*

$$\frac{d^n}{da^n}\left(\exp\left(\frac{\exp\left(ma\right)}{m}+ra\right)\right)=\exp\left(\frac{\exp\left(ma\right)}{m}+ra\right)\sum_{k=0}^{n} W_{m,r}\left(n,k\right)\exp\left(mak\right).$$

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