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The Inverse of Power Series and the Partial Bell Polynomials

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Abstract

Using the Bell polynomials, in this paper we give the explicit compositional inverses and/or the reciprocals of some power series. We illustrate the obtained results by some examples on Stirling numbers.

1 Introduction

The applications of the partial Bell polynomials have attracted the attention of several authors. Comtet [7] studied these polynomials, and Riordan [17] used them in combinatorial analysis and Roman [18] in umbral calculus. Recently, more applications of these polynomials have appeared in different frameworks, including integration [6], inverse relations [13], congruences [14], Dyck paths [11] and Blissard problem [10], all of which motivate us to apply these polynomials to determine the explicit compositional inverses and/or reciprocals of some power series.

Indeed, recall that the study of the existence of the compositional inverses of power series is a well-known result of complex analysis; see Forsyth [9] and Stanley [19, Proposition 5.4.1]. Under some conditions on a function f, to find the compositional inverse $f^{\langle -1\rangle}$ of f around zero, three methods are used to compute the coefficients y_n for which the series

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$f^{\langle -1\rangle}(t)=\sum\limits_{n\geq 0}y_n\frac{t^n}{n!}$ is the compositional inverse of the series $f(t)=\sum\limits_{n\geq 0}x_n\frac{t^n}{n!}$. The first method is based on the solution on $\boldsymbol{y}=(y_1,y_2,\ldots)$ in the equation

$$t=\sum_{k\geq 0}\frac{y_k}{k!}\left(\sum_{j\geq 0}x_j\frac{t^j}{j!}\right)^k.$$

If we set $x_n=y_n=0$ if $n\leq 0$, it was shown by Whittaker [20] that

$$y_1=-\frac{1}{x_1},\quad y_n=-\frac{(-1)^{n-1}}{n!x_1^{2n-1}}\det\Big(((1+i-j)n+j-1)x_{2+i-j}\Big)_{1\leq i,j\leq n-1},\quad n\geq 2.$$

This can be reduced to solve the equation

$$t=\sum_{k\geq 1}\frac{y_k}{k!}\left(\sum_{j\geq 1}x_j\frac{t^j}{j!}\right)^k=\sum_{n\geq 1}\frac{t^n}{n!}\sum_{k=1}^{n}y_kB_{n,k}(x_1,x_2,\ldots),$$

which is equivalent to solving the system

$$\sum_{k=1}^{n}y_kB_{n,k}(x_1,x_2,\ldots)=\delta_{n-1},\quad n\geq 1,$$

where δ_n is the Kronicker's symbol, i.e. $\delta_0=1$ and $\delta_n=0$ if $n\geq 1$, and the polynomials $B_{n,k}(x_1,x_2,\ldots)$ are the (exponential) partial Bell polynomials defined by their generating function

$$\sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,\ldots)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k.$$

The second method is based on Lagrange's inversion formula [1] for which we have

$$y_n=\frac{d^{n-1}}{d\varphi^{n-1}}\left(\frac{\varphi}{f(\varphi)}\right)^n\Bigg|_{\varphi=0}.$$

The third method is based on the n^{th} nested derivative of a function g defined in [8] by

$$\mathcal{D}^0[g](\varphi):=1,\quad \mathcal{D}^n[g](\varphi):=\frac{d}{d\varphi}\Big(f(\varphi)\mathcal{D}^{n-1}[f](\varphi)\Big),\quad n\geq 1,$$

for which Dominici [8] showed that if $f(t)=\int_\alpha^t\frac{1}{g(\varphi)}d\varphi$, with $g(\alpha)\neq 0,\pm\infty$, we have

$$f^{\langle -1\rangle}(t)=\alpha+g(\alpha)\sum_{n\geq 1}\mathcal{D}^n[g](\alpha)\frac{t^n}{n!}.$$

For our contribution, we show that the partial Bell polynomials define two families of power series for which we can obtain explicit compositional inverses. We also give the reciprocals of power series connected to these families. For the applications, we give some examples on power series whose coefficients are related to Stirling and r-Stirling numbers. Indeed, let

r, s, d be integers with $d \geq 1$ and $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a sequence of real numbers with $x_1 = 1$. For $r > \max(s, -2s)$ we consider

$$H_1(t) = t\left(1 - \sum_{n\geq 1} \frac{((d-1)n)!s}{rdn - s} \frac{\binom{(r+1)dn-s}{(d-1)n}}{\binom{(rd+1)n-s}{n}} B_{(rd+1)n-s,rdn-s}(\boldsymbol{x}) \frac{t^{dn}}{n!}\right), \tag{1}$$

and for $r > \max(-s, (d+1)s)$ we consider

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{s}{rn - s} \frac{B_{(r+1)n-s,\ rn-s}(\boldsymbol{x})}{\binom{(r+1)n-s}{rn-s}} \frac{t^{dn}}{n!}\right). \tag{2}$$

We give below the explicit compositional inverses $H_1^{\langle -1\rangle}$ of H_1 (Theorem 3) and $H_2^{\langle -1\rangle}$ of H_2 (Theorem 16). Also, we present the explicit reciprocal power series $\frac{t}{H_1(t)}$ of $\frac{H_1(t)}{t}$.
The mathematical tools used are based on the connection between the partial Bell polynomials and the polynomials of binomial type. For a given real number α and for any sequence of binomial type $(f_n(\varphi))$, in what follows we let $(f_n(\varphi; \alpha))$ denote any sequence of binomial type such that

$$f_n(\varphi; \alpha) := \frac{\varphi}{\alpha n + \varphi} f_n(\alpha n + \varphi), \tag{3}$$

see Comtet [7, pp. 133–175], Aigner [2, pp. 99–116] and Proposition 1 given in [12].

For example, the sequences (φ^n) and $(\varphi(\alpha n + \varphi)^{n-1})$ are sequences of binomial type.

Below, we use the following notation: $B_{n,k}(x_j)$ or $B_{n,k}(\boldsymbol{x})$ with $\boldsymbol{x} = (x_1, x_2, \ldots)$ for the partial Bell polynomial $B_{n,k}(x_1, x_2, \ldots)$,

$D^k_{z=0} f(z)$ for $\frac{d^k f}{dz^k}(0)$, $k \geq 2$, and $D_{z=0} f(z)$ for $\frac{df}{dz}(0)$,

${n \brack k}$ and ${n \brack k}_r$ for the unsigned Stirling and r-Stirling numbers of the first kind, respectively,

${n \brace k}$ and ${n \brace k}_r$ for the Stirling and r-Stirling numbers of the second kind, respectively.

2 The first family of power series and their inverses

We give in this section the reciprocal and/or the compositional inverse of a power series given from a large family of power series which have coefficients can be expressed in terms of partial Bell polynomials.

Proposition 1. *Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a sequence of real numbers with $x_1 = 1$. Then, for r, s integers such that $r > \max(s, -2s)$, the reciprocal and compositional inverse of the power series*

$$H(t) = t\left(1 - \sum_{n\geq 1} \frac{s}{rn - s} \frac{B_{(r+1)n-s,rn-s}(\boldsymbol{x})}{\binom{(r+1)n-s}{n}} \frac{t^n}{n!}\right) \tag{4}$$

are given by

$$H^{\langle -1\rangle}(t) = t\Bigg(1 + \sum_{n\geq 1} \frac{s}{(r+s)n+s} \frac{B_{(r+s+1)n+s,(r+s)n+s}(\boldsymbol{x})}{\binom{(r+s+1)n+s}{n}} \frac{t^n}{n!}\Bigg), \tag{5}$$

$$\frac{t}{H(t)} = 1 + \sum_{n\geq 1} \frac{s}{rn+s} \frac{B_{(r+1)n+s,rn+s}(\boldsymbol{x})}{\binom{(r+1)n+s}{n}} \frac{t^n}{n!}. \tag{6}$$

Proof. We consider only the case $s \geq 0$ (for $s < 0$, we can proceed similarly). Let $(f_n(\varphi))$ be a sequence of binomial-type polynomials such that $f_n(1) = x_{n+1}/(n+1)$ with $x_2 \neq 0$ to ensure that $Df_1(\varphi) \neq 0$. Then, by Proposition 1 given in [12], we get

$$f_n(k) = \binom{n+k}{k}^{-1} B_{n+k,k}(\boldsymbol{x}). \tag{7}$$

In [13, Theorem 1], we proved that

$$H(t) = t\Bigg(1 - \sum_{n\geq 1} \frac{s}{rn-s} f_n(rn-s) \frac{t^n}{n!}\Bigg),$$

has inverse

$$H^{\langle -1\rangle}(t) = t\Bigg(1 + \sum_{n\geq 1} \frac{s}{(r+s)n+s} f_n((r+s)n+s) \frac{t^n}{n!}\Bigg),$$

and because for any binomial-type sequence of polynomials $(f_n(\varphi))$ we have

$$\Bigg(1 + \sum_{n\geq 1} f_n(-s) \frac{t^n}{n!}\Bigg)^{-1} = 1 + \sum_{n\geq 1} f_n(s) \frac{t^n}{n!},$$

then, by replacing in the last identity $f_n(s)$ by $f_n(s; r)$ defined by (3) we get

$$\frac{t}{H(t)} = 1 + \sum_{n\geq 1} \frac{s}{rn+s} f_n(rn+s) \frac{t^n}{n!}.$$

Then, replace $f_n(rn-s)$ and $f_n((r+s)n+s)$ by their expressions obtained from the identity (7) by

$$f_n(rn-s) = \frac{B_{(r+1)n-s,rn-s}(\boldsymbol{x})}{\binom{(r+1)n-s}{n}} \quad \text{and} \quad f_n((r+s)n+s) = \frac{B_{(r+s+1)n+s,(r+s)n+s}(\boldsymbol{x})}{\binom{(r+s+1)n+s}{n}}.$$

The proposition remains true for the case $x_2 = 0$ by continuity. □

Lemma 2. *Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a sequence of real numbers with $x_1 = 1$, $x_n = 0$ if $d \nmid n-1$, and $\boldsymbol{y} = (y_1, y_2, \ldots)$ with $y_j = j!x_{d(j-1)+1}/(d(j-1)+1)!$. Then, we have*

$$\frac{B_{dn+k,k}(\boldsymbol{x})}{(dn+k)!} = \frac{B_{n+k,k}(\boldsymbol{y})}{(n+k)!} \quad \text{and} \quad B_{dn+k+l,k}(\boldsymbol{x}) = 0 \text{ if } 1 \leq l \leq d-1.$$

Proof. Setting $\mathbf{z} = (z_1, z_2, \ldots)$ with $z_j = y_{j+1}/(j+1)$.
From the definition of partial Bell polynomials we have

$$\sum_{n\geq k} B_{n,k}(\boldsymbol{x})\frac{t^n}{n!} = \frac{t^k}{k!}\left(1+\sum_{j\geq 1} z_j\frac{(t^d)^j}{j!}\right)^k = \frac{t^k}{k!}\sum_{l=0}^{k}\frac{(k)_l}{l!}\left(\sum_{j\geq 1} z_j\frac{(t^d)^j}{j!}\right)^l$$

i.e.

$$\sum_{n\geq k} B_{n,k}(\boldsymbol{x})\frac{t^n}{n!} = \frac{t^k}{k!}\sum_{l=0}^{k}(k)_l\sum_{n\geq l} B_{n,l}(\boldsymbol{z})\frac{t^{dn}}{n!} = \frac{t^k}{k!}\sum_{n\geq 0}\frac{t^{dn}}{n!}\sum_{l=0}^{\min(n,k)}(k)_l B_{n,l}(\boldsymbol{z}).$$

Now, from the identity [3l] given in Comtet [7, pp. 136], we have

$$\sum_{l=0}^{\min(n,k)}(k)_l B_{n,l}(\boldsymbol{z}) = \binom{n+k}{k}^{-1} B_{n+k,k}(\boldsymbol{y})$$

and the above expansion becomes

$$\sum_{n\geq k} B_{n,k}(\boldsymbol{x})\frac{t^n}{n!} = \frac{1}{k!}\sum_{n\geq 0}\frac{t^{dn+k}}{n!}\binom{n+k}{k}^{-1} B_{n+k,k}(\boldsymbol{y}) = \sum_{n\geq 0}\frac{(dn+k)!}{(n+k)!}B_{n+k,k}(\boldsymbol{y})\frac{t^{dn+k}}{(dn+k)!}.$$

This gives $B_{dn+k,k}(\boldsymbol{x}) = \frac{(dn+k)!}{(n+k)!}B_{n+k,k}(\boldsymbol{y})$ and $B_{dn+k+l,k}(\boldsymbol{x}) = 0$ if $1 \leq l \leq d-1$. $\square$

Theorem 3. *Let $\boldsymbol{y} = (y_1, y_2, \ldots)$ be a sequence of real numbers with $y_1 = 1$, r, s, d be integers such that $d \geq 1$, $r > \max(s, -2s)$ and let*

$$U_n(r,s;d) = \frac{((d-1)n)!s}{rdn+s}\frac{\binom{(r+1)dn+s}{(d-1)n}}{\binom{(rd+1)n+s}{n}}B_{(rd+1)n+s,rdn+s}(\boldsymbol{y}), \quad n\geq 1, \quad U_0(r,s;d) = 1. \tag{8}$$

Then for

$$H_1(t) = t\left(1+\sum_{n\geq 1} U_n(r,-s;d)\frac{t^{dn}}{n!}\right) \tag{9}$$

we have

$$H_1^{\langle -1\rangle}(t) = t\left(1+\sum_{n\geq 1} U_n(r+s,s;d)\frac{t^{dn}}{n!}\right), \tag{10}$$

$$\frac{t}{H_1(t)} = 1+\sum_{n\geq 1} U_n(r,s;d)\frac{t^{dn}}{n!}, \tag{11}$$

Proof. In Proposition 1, choice $\boldsymbol{x} = (x_1, x_2, \ldots)$ such that $x_n = 0$, if $d \nmid n-1$ and $x_{dn+1} = (dn+1)!y_{n+1}/(n+1)!$, after that, apply Lemma 2. $\square$

Example 4. For $\mathbf{y} = (1!, 2!, \ldots, (q+1)!, 0, \ldots)$ in Theorem 3, the identity

$$B_{n,k}(\mathbf{y}) = \frac{n!}{k!}\binom{k}{n-k}_q, \tag{12}$$

see [3], gives

$$H_1(t) = t\left(1 - s\sum_{n\geq 1}\frac{((d-1)n)!}{rdn-s}\frac{\binom{(r+1)dn-s}{(d-1)n}}{\binom{(rd+1)n-s}{n}}\binom{rdn-s}{n}_q\frac{t^{dn}}{n!}\right),$$

$$H_1^{\langle -1\rangle}(t) = t\left(1 + s\sum_{n\geq 1}\frac{((d-1)n)!}{(r+s)dn+s}\frac{\binom{(r+s+1)dn+s}{(d-1)n}}{\binom{((r+s)d+1)n+s}{n}}\binom{(r+s)dn+s}{n}_q\frac{t^{dn}}{n!}\right),$$

$$\frac{t}{H_1(t)} = 1 + s\sum_{n\geq 1}\frac{((d-1)n)!}{rdn+s}\binom{(r+1)dn+s}{(d-1)n}\binom{rdn+s}{n}_q t^{dn},$$

where $\binom{k}{n}_q$ is the coefficients defined by $(1+\varphi+\varphi^2+\cdots+\varphi^q)^k = \sum\limits_{n\geq 0}\binom{k}{n}_q\varphi^n$.

Example 5. For $\mathbf{y} = (0!, 1!, \ldots)$ in Theorem 16, the identity $B_{n,k}(\mathbf{y}) = \left[{n \atop k}\right]$ gives

$$H_1(t) = t\left(1 - \sum_{n\geq 1}\frac{((d-1)n)!s}{rdn-s}\frac{\binom{(r+1)dn-s}{(d-1)n}}{\binom{(rd+1)n-s}{rdn-s}}\left[{(rd+1)n-s \atop rdn-s}\right]\frac{t^{dn}}{n!}\right),$$

$$H_1^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1}\frac{((d-1)n)!s}{(r+s)dn+s}\frac{\binom{(r+s+1)dn+s}{(d-1)n}}{\binom{((r+s)d+1)n+s}{(r+s)dn+s}}\left[{((r+s)d+1)n+s \atop (r+s)dn+s}\right]\frac{t^{dn}}{n!}\right),$$

$$\frac{t}{H_1(t)} = 1 + \sum_{n\geq 1}\frac{((d-1)n)!s}{rdn+s}\frac{\binom{(r+1)dn+s}{(d-1)n}}{\binom{(rd+1)n+s}{rdn+s}}\left[{(rd+1)n+s \atop rdn+s}\right]\frac{t^{dn}}{n!}$$

Example 6. For $\mathbf{y} = (1, 1, \ldots)$ in Theorem 16, the identity $B_{n,k}(\mathbf{y}) = \left\{{n \atop k}\right\}$ gives

$$H_1(t) = t\left(1 - \sum_{n\geq 1}\frac{((d-1)n)!s}{rdn-s}\frac{\binom{(r+1)dn-s}{(d-1)n}}{\binom{(rd+1)n-s}{rdn-s}}\left\{{(rd+1)n-s \atop rdn-s}\right\}\frac{t^{dn}}{n!}\right),$$

$$H_1^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1}\frac{((d-1)n)!s}{(r+s)dn+s}\frac{\binom{(r+s+1)dn+s}{(d-1)n}}{\binom{((r+s)d+1)n+s}{(r+s)dn+s}}\left\{{((r+s)d+1)n+s \atop (r+s)dn+s}\right\}\frac{t^{dn}}{n!}\right),$$

$$\frac{t}{H_1(t)} = 1 + \sum_{n\geq 1}\frac{((d-1)n)!s}{rdn+s}\frac{\binom{(r+1)dn+s}{(d-1)n}}{\binom{(rd+1)n+s}{rdn+s}}\left\{{(rd+1)n+s \atop rdn+s}\right\}\frac{t^{dn}}{n!},$$

Proposition 7. *Let r, s, d be integers with $s \neq 0$, $d \geq 1$ and $(f_n(\varphi))$ be binomial-type* *polynomials. Then, for* $r > \max(s, -2s)$ *we get*

$$H_1(t) = t\left(1 - \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+1)dn - s}{(d-1)n}\frac{f_n(\alpha n - \varphi)}{\alpha n - \varphi}\frac{t^{dn}}{n!}\right),$$

$$H_1^{(-1)}(t) = t\left(1 + \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+s+1)dn + s}{(d-1)n}\frac{f_n((\alpha + sd)n + \varphi)}{(\alpha + sd)n + \varphi}\frac{t^{dn}}{n!}\right),$$

$$\frac{t}{H_1(t)} = 1 + \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+1)dn + s}{(d-1)n}\frac{f_n(\alpha n + \varphi)}{\alpha n + \varphi}\frac{t^{dn}}{n!}.$$

Proof. For H_1, take $y_n = f_n(\varphi;\alpha) = \frac{\varphi}{\alpha n+\varphi}f_n(\alpha n + \varphi)$ in Theorem 3 and use Proposition 1 given in [12], after that, replace α by $\alpha - rd$ and φ by $\frac{\varphi}{s}$. □

Example 8. For $f_n(\varphi) = \varphi^n$ in Proposition 7 we get

$$H_1(t) = t\left(1 - \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+1)dn - s}{(d-1)n}(\alpha n - \varphi)^{n-1}\frac{t^{dn}}{n!}\right),$$

$$H_1^{(-1)}(t) = t\left(1 + \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+s+1)dn + s}{(d-1)n}((\alpha + sd)n + \varphi)^{n-1}\frac{t^{dn}}{n!}\right),$$

$$\frac{t}{H_1(t)} = 1 + \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+1)dn + s}{(d-1)n}(\alpha n + \varphi)^{n-1}\frac{t^{dn}}{n!},$$

and for $f_n(\varphi) = n!\binom{\varphi}{n}$ in Proposition 7 we get

$$H_1(t) = t\left(1 - \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+1)dn - s}{(d-1)n}\binom{\alpha n - \varphi}{n}t^{dn}\right),$$

$$H_1^{(-1)}(t) = t\left(1 + \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+s+1)dn + s}{(d-1)n}\binom{(\alpha + sd)n + \varphi}{n}t^{dn}\right),$$

$$\frac{t}{H_1(t)} = 1 + \varphi\sum_{n\geq 1}((d-1)n)!\binom{(r+1)dn + s}{(d-1)n}\binom{\alpha n + \varphi}{n}t^{dn}.$$

The following theorem generalizes Theorem 3.

Theorem 9. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots)$ *be a sequence of real numbers with* $x_1 = 1$ *and* r, s, u, v, d *be integers such that* $d \geq 1$, $r > \max(s, -2s)$ *and* $u > \max(|v|, -sd - v)$. *Then for*

$$H_1(t) = t\left(1 - \sum_{n\geq 1}\frac{((d-1)n)!v}{un - v}\binom{(u+1)n - v}{un - v}^{-1}\binom{(r+1)dn - s}{(d-1)n}B_{(u+1)n-v,un-v}(\boldsymbol{x})\frac{t^{dn}}{n!}\right)$$

we have

$$H_1^{(-1)}(t) = t\left(1 + \sum_{n\geq 1}\frac{((d-1)n)!v}{(u+sd)n + v}\frac{\binom{(r+s+1)dn+s}{(d-1)n}}{\binom{(u+sd+1)n+v}{(u+sd)n+v}}B_{(u+sd+1)n+v,(u+sd)n+v}(\boldsymbol{x})\frac{t^{dn}}{n!}\right),$$

$$\frac{t}{H_1(t)} = 1 + \sum_{n\geq 1}\frac{((d-1)n)!v}{un + v}\binom{(u+1)n + v}{un + v}^{-1}\binom{(r+1)dn + s}{(d-1)n}B_{(u+1)n+v,un+v}(\boldsymbol{x})\frac{t^{dn}}{n!}.$$

Proof. Let $(f_n(\varphi))$ be a sequence of of binomial type of polynomials such that $f_n(1) = x_{n+1}/(n+1)$ with $x_2 \neq 0$ to ensure that $Df_1(\varphi) \neq 0$. Then, $f_n(\varphi)$ satisfies (7). Set $\alpha = u$ and $\varphi = v$ in Proposition 7, after that, use the identity (7) in the three power series of Proposition 7, respectively, for $k = un - v$, $k = (u + sd)n + v$ and $k = un + v$.
The theorem remains true for the case $x_2 = 0$ by continuity. □

For $u = rd$ and $v = s$ in Theorem 9 we obtain Theorem 3.

Example 10. Set $\mathbf{x} = (x_1, x_2, \ldots)$ with $x_n = n!\binom{p+n-2}{p-1}$, $p \geq 1$, in Theorem 9. The identity [16, Example 13]

$$B_{n,k}(\mathbf{x}) = \frac{n!}{k!}\binom{n + k(p-1) - 1}{kp - 1} \tag{13}$$

implies for the power series

$$H_1(t) = t\left(1 - \sum_{n\geq 1} \frac{((d-1)n)!v}{un - v}\binom{(r+1)dn - s}{(d-1)n}\binom{(up+1)n - vp - 1}{upn - vp - 1} t^{dn}\right)$$

we have

$$H_1^{(-1)}(t) = t\left(1 + \sum_{n\geq 1} \frac{((d-1)n)!v}{(u+sd)n + v}\binom{(r+s+1)dn + s}{(d-1)n}\binom{((u+sd)p+1)n + vp - 1}{(u+sd)pn + vp - 1} t^{dn}\right),$$

$$\frac{t}{H_1(t)} = 1 + \sum_{n\geq 1} \frac{((d-1)n)!v}{un + v}\binom{(r+1)dn + s}{(d-1)n}\binom{(up+1)n + vp - 1}{upn + vp - 1} t^{dn}.$$

Example 11. Set $\mathbf{x} = (x_1, x_2, \ldots)$ with $x_n = p\binom{j+p-1}{p-1}^{-1}\left[{p+q+j-1 \atop p+q}\right]_q$, $p \geq 1$, $q \geq 0$, in Theorem 9. The identity [16, Example 13]

$$B_{n,k}(\mathbf{x}) = \binom{kp}{k}\binom{n + (p-1)k}{(p-1)k}^{-1}\left[{n + (p+q-1)k \atop (p+q)k}\right]_{kq} \tag{14}$$

implies for the power series

$$H_1(t) = t\left(1 - \sum_{n\geq 1} v\frac{((r+1)dn - s)!}{((rd+1)n - s)!}\frac{\binom{pA}{A}\left[{n+(p+q)A \atop (p+q)A}\right]_{Aq}}{A\binom{n+A}{A}\binom{n+pA}{(p-1)A}}\frac{t^{dn}}{n!}\right), \quad A = un - v,$$

we have

$$H_1^{(-1)}(t) = t\left(1 + \sum_{n\geq 1} v\frac{((r+s+1)dn + s)!}{(((r+s)d+1)n + s)!}\frac{\binom{pB}{B}\left[{n+(p+q)B \atop (p+q)B}\right]_{Bq}}{B\binom{n+B}{B}\binom{n+pB}{(p-1)B}}\frac{t^{dn}}{n!}\right), \quad B = (u+sd)n + v,$$

$$\frac{t}{H_1(t)} = 1 + \sum_{n\geq 1} v\frac{((r+1)dn + s)!}{((rd+1)n + s)}\frac{\binom{pC}{C}\left[{n+(p+q)C \atop (p+q)C}\right]_{Cq}}{C\binom{n+C}{C}\binom{n+pC}{(p-1)C}}\frac{t^{dn}}{n!}, \quad C = un + v,$$

Example 12. Set $\mathbf{x} = (x_1, x_2, \ldots)$ with $x_n = p\binom{j+p-1}{p-1}^{-1}\left\{ {p+q+j-1 \atop p+q} \right\}_q$, $p \geq 1$, $q \geq 0$, in Theorem 9. The identity [16, Example 13]

$$B_{n,k}(\mathbf{x}) = \binom{kp}{k}\binom{n+(p-1)k}{(p-1)k}^{-1}\left\{ {n+(p+q-1)k \atop (p+q)k} \right\}_{kq} \tag{15}$$

implies for the power series

$$H_1(t) = t\left(1 - \sum_{n\geq 1} v\frac{((r+1)dn-s)!}{((rd+1)n-s)!}\frac{\binom{pA}{A}\left\{ {n+(p+q)A \atop (p+q)A} \right\}_{Aq}}{A\binom{n+A}{A}\binom{n+pA}{(p-1)A}}\frac{t^{dn}}{n!}\right), \quad A = un - v,$$

we have

$$H_1^{(-1)}(t) = t\left(1 + \sum_{n\geq 1} v\frac{((r+s+1)dn+s)!}{(((r+s)d+1)n+s)!}\frac{\binom{pB}{B}\left\{ {n+(p+q)B \atop (p+q)B} \right\}_{Bq}}{B\binom{n+B}{B}\binom{n+pB}{(p-1)B}}\frac{t^{dn}}{n!}\right), \quad B = (u+sd)n + v,$$

$$\frac{t}{H_1(t)} = 1 + \sum_{n\geq 1} v\frac{((r+1)dn+s)!}{((rd+1)n+s)}\frac{\binom{pC}{C}\left\{ {n+(p+q)C \atop (p+q)C} \right\}_{Cq}}{C\binom{n+C}{C}\binom{n+pC}{(p-1)C}}\frac{t^{dn}}{n!}, \quad C = un + v,$$

3 The second family of power series and their inverses

We give in this section the compositional inverse of a power series given from a second family of power series which have coefficients can be expressed in terms of partial Bell polynomials.

Lemma 13. *Let $\boldsymbol{x} = (x_0, x_1, \ldots)$ with $x_0 = 1$ and $\boldsymbol{y} = (y_1, y_2, \ldots)$ be sequences of real numbers with $y_j = jx_{j-1}$. Then, the compositional inverse of*

$$H(t) = \frac{t}{1 + \sum_{n\geq 1} x_n \frac{t^n}{n!}} \tag{16}$$

is given by

$$H^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{n!}{(2n+1)!}B_{2n+1,n+1}(\boldsymbol{y})t^n\right). \tag{17}$$

Proof. Let $(f_n(\varphi))$ be a sequence of polynomials such that $f_n(1) = x_n$ and assume that $x_1 \neq 0$ to ensure that $Df_1(\varphi) \neq 0$. Then we get $f_n(k) = \binom{n+k}{k}^{-1}B_{n+k,k}(\mathbf{y})$. In [13, Theorem 1], we proved that

$$H(t) = \frac{t}{1 + \sum_{n\geq 1} f_n(1)\frac{t^n}{n!}} = \sum_{n\geq 1} nf_{n-1}(-1)\frac{t^n}{n!}, \quad H^{\langle -1\rangle}(t) = \sum_{n\geq 1} f_{n-1}(n)\frac{t^n}{n!}.$$

Then, from the last identity, $f_{n-1}(n)$ can be expressed by partial Bell polynomials as

$$f_{n-1}(n) = \binom{2n-1}{n}^{-1}B_{2n-1,n}(\mathbf{y}).$$

The lemma holds for the case $x_1 = 0$ by continuity. $\square$

Proposition 14. *Let $\boldsymbol{x} = (x_1, x_2, \ldots)$, $\boldsymbol{y} = (y_1, y_2, \ldots)$ be sequences of real numbers with $x_1 = 1, y_j = jx_{j-1}$ and d be a positive integer. Then, we have the pair of compositional inverse power series*

$$H(t) = \frac{t}{1 + \sum_{n\geq 1} x_n \frac{t^{dn}}{n!}}, \tag{18}$$

$$H^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{(dn)!}{((d+1)n+1)!} B_{(d+1)n+1,dn+1}(\boldsymbol{y}) t^{dn}\right). \tag{19}$$

Proof. For $d \geq 2$, choice in Lemma 13 $x_n = 0$ if $d \nmid n$. On using the notations of Lemma 2 we get $y_n = nx_{n-1} = 0$ if $d \nmid n-1$ and

$$1 + \sum_{n\geq 1} x_n \frac{t^{dn}}{n!} = 1 + \sum_{n\geq 1} x_{dn} \frac{t^{dn}}{(dn)!}.$$

Let $\boldsymbol{z} = (z_1, z_2, \ldots)$ with $z_j = j!y_{d(j-1)+1}/(d(j-1)+1)!$.
Then, on using Lemma 2 we obtain:
If $d \nmid n-1$ then $B_{2n+1,n+1}(\boldsymbol{y}) = 0$ and if $d \nmid n-1$ we get

$$\frac{(dn)!}{(2dn+1)!} B_{2dn+1,dn+1}(\boldsymbol{y}) = \frac{(dn)!}{((d+1)n+1)!} B_{(d+1)n+1,dn+1}(\boldsymbol{z}).$$

Therefore, the pair of the power series given in Lemma 13 can be written as

$$H(t) = \frac{t}{1 + \sum_{n\geq 1} x_{dn} t^{dn}/(dn)!}, \quad H^{\langle -1\rangle}(t) = 1 + \sum_{n\geq 1} \frac{(dn)!}{((d+1)n+1)!} B_{(d+1)n+1,dn+1}(\boldsymbol{z}) t^{dn}.$$

To finish this proof, replace $n!x_{dn}/(dn)!$ by x_n. □

Example 15. For $\mathbf{x} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ in Proposition 14 we get

$$H(t) = \frac{t^{d+1}}{e^{t^d} - 1}, \quad H^{\langle -1\rangle}(t) = t\sum_{n\geq 0} \frac{1}{dn+1} \binom{(d+1)n+1}{dn+1}^{-1} \left\{ {(d+1)n+1 \atop dn+1} \right\} \frac{t^{dn}}{n!},$$

for $\mathbf{x} = (1!, 2!, 3!, \ldots)$ in Proposition 14 we get

$$H(t) = t(1 - t^d), \quad H^{\langle -1\rangle}(t) = t\sum_{n\geq 0} \frac{1}{dn+1} \binom{(d+1)n}{dn} t^{dn}$$

and for $\mathbf{x} = (\frac{1!}{2}, \frac{2!}{3}, \frac{3!}{4}, \ldots)$ in Proposition 14 we get

$$H(t) = -\frac{t^{d+1}}{\ln(1 - t^d)}, \quad H^{\langle -1\rangle}(t) = t\sum_{n\geq 0} \frac{1}{dn+1} \binom{(d+1)n+1}{dn+1}^{-1} \left[{(d+1)n+1 \atop dn+1} \right] \frac{t^{dn}}{n!}.$$

Theorem 16. *Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a sequence of real numbers with $x_1 = 1$, r, s, d be integers such that $r > \max(s, -(d+1)s)$, $d \geq 1$ and let*

$$V_n(r,s) = \frac{s}{rn+s} \frac{B_{(r+1)n+s,\ rn+s}(\boldsymbol{x})}{\binom{(r+1)n+s}{rn+s}}, \quad n \geq 1, \quad V_0(r,s) = 1. \tag{20}$$

Then, we have the pair of compositional inverses

$$H_2(t) = t\left(1 + \sum_{n\geq 1} V_n(r,-s)\frac{t^{dn}}{n!}\right) \tag{21}$$

$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} V_n(r+sd,s)\frac{t^{dn}}{n!}\right), \tag{22}$$

Proof. Let $(f_n(\varphi))$ be a sequence of binomial type such that $f_n(1) = \frac{x_{n+1}}{n+1}$. Necessarily $f_n(\varphi)$ satisfies (7) . Then because

$$\frac{s}{rn+s}\binom{(r+1)n+s}{rn+s}^{-1} B_{(r+1)n+s,rn+s}(\boldsymbol{x}) = f_n(s;r)$$

and

$$H(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{rn+s} \frac{B_{(r+1)n+s,\ rn+s}(\boldsymbol{x})}{\binom{(r+1)n+s}{rn+s}} \frac{t^{dn}}{n!}\right)^{-1} = t\left(1 + \sum_{n\geq 1} f_n(s,r)\frac{t^{dn}}{n!}\right)^{-1}$$

we can state, on using Proposition 14 and Proposition 1 given [12], that

$$H^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{(r+sd)n+s} f_n((r+sd)n+s)\frac{t^{dn}}{n!}\right)$$

and by the identity (7) we obtain

$$H^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{(r+sd)n+s} \frac{B_{(r+sd+1)n+s,\ (r+sd)n+s}(\boldsymbol{x})}{\binom{(r+sd+1)n+s}{(r+sd)n+s}} \frac{t^{dn}}{n!}\right).$$

It suffices to remark that from Proposition 1 we have

$$\left(1 + s\sum_{n\geq 1} \frac{B_{(r+1)n+s,\ rn+s}(\boldsymbol{x})}{(rn+s)\binom{(r+1)n+s}{rn+s}} \frac{t^{dn}}{n!}\right)^{-1} = 1 - s\sum_{n\geq 1} \frac{B_{(r+1)n-s,\ rn-s}(\boldsymbol{x})}{(rn-s)\binom{(r+1)n-s}{rn-s}} \frac{t^{dn}}{n!}.$$

□

Example 17. With $\mathbf{x} = (1!, 2!, \ldots, (q+1)!, 0, \ldots)$ in Theorem 16, the identity (12) gives

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{s}{rn-s}\binom{rn-s}{n}_q t^{dn}\right),$$

$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{(r+sd)n+s}\binom{(r+sd)n+s}{n}_q t^{dn}\right).$$

With $\mathbf{x} = (1!, 2!, \ldots)$ in Theorem 16 we obtain

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{s}{rn-s}\binom{(r+1)n-s-1}{rn-s-1} t^{dn}\right),$$
$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{(r+sd)n+s}\binom{(r+sd+1)n+s-1}{(r+sd)n+s-1} \frac{t^{dn}}{n!}\right).$$

With $\mathbf{x} = (1, 1, \ldots)$ in Theorem 16 we obtain

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{s}{rn-s}\binom{(r+1)n-s}{rn-s}^{-1} \left\{ {(r+1)n-s \atop rn-s} \right\} \frac{t^{dn}}{n!}\right),$$
$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{(r+sd)n+s}\binom{(r+sd+1)n+s}{(r+sd)n+s}^{-1} \left\{ {(r+sd+1)n+s \atop (r+sd)n+s} \right\} \frac{t^{dn}}{n!}\right).$$

With $\mathbf{x} = (0!, 1!, \ldots)$ in Theorem 16 we obtain

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{s}{rn-s}\binom{(r+1)n-s}{rn-s}^{-1} \left[{(r+1)n-s \atop rn-s} \right] \frac{t^{dn}}{n!}\right),$$
$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s}{(r+sd)n+s}\binom{(r+sd+1)n+s}{(r+sd)n+s}^{-1} \left[{(r+sd+1)n+s \atop (r+sd)n+s} \right] \frac{t^{dn}}{n!}\right).$$

Other examples can be derived by using the identities (13), (14) and (15).

Proposition 18. *Let r, s, d be integers with $r > \max(s, -(d+1)s)$, $d \geq 1$ and $(f_n(\varphi))$ be a sequence of binomial type. Then, we have the pair of compositional inverses*

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{\varphi}{\alpha n - \varphi} f_n(\alpha n - \varphi) \frac{t^{dn}}{n!}\right),$$
$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{\varphi}{(\alpha + d\varphi)n + \varphi} f_n((\alpha + d\varphi)n + \varphi) \frac{t^{dn}}{n!}\right).$$

Proof. Set $x_n = nf_{n-1}(\varphi; \alpha)$ in Theorem 16 to get

$$H_2(t) = t\left(1 - \sum_{n\geq 1} \frac{s\varphi}{(\alpha + r\varphi)n - s\varphi} f_n((\alpha + r\varphi)n - s\varphi) \frac{t^{dn}}{n!}\right),$$
$$H_2^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} \frac{s\varphi}{(\alpha + r\varphi + sd\varphi)n + s\varphi} f_n((\alpha + r\varphi + sd\varphi)n + s\varphi) \frac{t^{dn}}{n!}\right)$$

and change $s\varphi$ by φ a and $\alpha - r\varphi$ and $s\varphi$ by φ. $\square$

Example 19. With $f_n(\varphi)=\varphi^n$ in Proposition 18 we get

$$H_2(t)=t\left(1-\varphi\sum_{n\geq 1}(\alpha n-\varphi)^{n-1}\frac{t^{dn}}{n!}\right),$$

$$H_2^{\langle -1\rangle}(t)=t\left(1+x\sum_{n\geq 1}((\alpha+d\varphi)n+x)^{n-1}\frac{t^{dn}}{n!}\right)$$

With $f_n(\varphi)=n!\binom{\varphi}{n}$ in Proposition 18 we get

$$H_2(t)=t\left(1-\sum_{n\geq 1}\frac{\varphi}{\alpha n-\varphi}\binom{\alpha n-\varphi}{n}t^{dn}\right),$$

$$H_2^{\langle -1\rangle}(t)=t\left(1+\sum_{n\geq 1}\frac{\varphi}{(\alpha+d\varphi)n+x}\binom{(\alpha+d\varphi)n+\varphi}{n}t^{dn}\right).$$

4 Consequences and complementary results

We give in this section some properties and complementary remarks on the compositional inverse by taking particular cases of Theorems (3) and (16).

Corollary 20. *Let $\boldsymbol{x}=(x_1,x_2,\ldots)$ be a sequence of real numbers and r,s,d,f be integers with $d\geq 1$ and $f\geq 2$. Then, for $r>\max(s,-2s)$ the inverse power series given by (9) and (10) hold for*

$$U_n(r,s;d)=s\frac{((r+1)dn+s)!}{rdn+s}\binom{(rd+1)n+s}{n}^{-1}\frac{B_{(rd+1)n+s-(f-1)[n/f],rdn+s}(\boldsymbol{x})}{((rd+1)n+s-(f-1)[n/f])!},$$

and for $r>\max(s,-(d+1)s)$ the inverse power series given by (21) and (22) hold for

$$V_n(r,s)=(rn+s-1)!n!s\frac{B_{(r+1)n+s-(f-1)[n/f],\ rn+s}(\boldsymbol{x})}{((r+1)n+s-(f-1)[n/f])!},$$

where $[\varphi]$ is the largest integer $\leq\varphi$.

Proof. Set $y_1=1,\ y_2=\cdots=y_f=0$ and $n=mf+\delta$ $(0\leq\delta\leq f-1)$ in Theorem 3 to get

$$\begin{aligned}
\frac{B_{n+k,k}(y_j)}{\binom{n+k}{k}}&=\frac{B_{mf+\delta+k,k}(y_j)}{\binom{mf+\delta+k}{k}}\\
&=\sum_{i=0}^{mf+\delta}\frac{k!}{(k-i)!}B_{mf+\delta,i}\left(\frac{y_{j+1}}{j+1}\right)\\
&=\frac{(mf+\delta)!}{(m+\delta)!}\sum_{i=0}^{m+\delta}\frac{k!}{(k-i)!}B_{m+\delta,i}\left(\frac{j!y_{j+f}}{(j+f)!}\right)\\
&=\frac{(mf+\delta)!}{(m+\delta)!}\binom{m+\delta+k}{k}^{-1}B_{m+\delta+k,k}\left(\frac{j!y_{j+f-1}}{(j+f-1)!}\right)\\
&=\frac{n!k!}{(n+k-(f-1)[n/f])!}B_{n+k-(f-1)[n/f],k}\left(\frac{j!y_{j+f-1}}{(j+f-1)!}\right).
\end{aligned}$$

Then, for $x_n = n!y_{n+f-1}/(n+f-1)!$ we obtain

$$U_n(r,s;d) = \frac{((d-1)n)!s}{rdn+s}\binom{(rd+1)n+s}{n}^{-1}\binom{(r+1)dn+s}{(d-1)n}B_{(rd+1)n+s,rdn+s}(y_j)$$
$$= s\frac{((r+1)dn+s)!}{rdn+s}\binom{(rd+1)n+s}{n}^{-1}\frac{B_{(rd+1)n+s-(f-1)[n/f],rdn+s}(x_j)}{((rd+1)n+s-(f-1)[n/f])!},$$
$$V_n(r,s) = \frac{s}{rn+s}\frac{B_{(r+1)n+s,\ rn+s}(y_j)}{\binom{(r+1)n+s}{rn+s}} = (rn+s-1)!n!s\frac{B_{(r+1)n+s-(f-1)[n/f],\ rn+s}(x_j)}{((r+1)n+s-(f-1)[n/f])!}.$$

□

Proposition 21. *Let r,s be integers, a,x be real numbers and $(f_n(\varphi))$ be a binomial-type polynomials. Then, for $r > \max(s,-2s)$ the inverse power series given by (9) and (10) hold for*

$$U_n(r,s;d) = \frac{((d-1)n)!s}{rdn+s}\binom{(r+1)dn+s}{(d-1)n}D_{z=0}^{rdn+s}(e^z f_n((rdn+s)\varphi+z;\alpha)),$$

and for $r > \max(s,-(d+1)s)$ the inverse power series given by (21) and (22) hold for

$$V_n(r,s) = \frac{s}{rn+s}D_{z=0}^{rn+s}(e^z f_n((rn+s)\varphi+z;\alpha)).$$

Proof. For $y_n = nD_{z=0}(e^z f_{n-1}(\varphi+z;\alpha))$ in Theorems 3 and 16 and use the identity given in [15, Lemma 1] by $B_{n,k}(jD_{z=0}(e^z f_{j-1}(\varphi+z;\alpha))) = \binom{n}{k}D_{z=0}^k(e^z f_{n-k}(k\varphi+z;\alpha))$. □

Theorems 3 and 16 remain true when one use a finite product of power series as follows:

Proposition 22. *Let $\boldsymbol{x} = (x_1,x_2,\ldots)$ be a sequence of real numbers with $x_1 = 1$, $s_1,\ldots,s_m$, r,s,m be integers and $H_{r,s}$ be power series defined by*

$$H_{r,s}(t) = t\left(1 - \sum_{n\geq 1}\frac{s}{rn-s}\frac{B_{(r+1)n-s,rn-s}(\boldsymbol{x})}{\binom{(r+1)n-s}{n}}\frac{t^n}{n!}\right),\quad r > \max(s,0) := s^+.$$

Then, we have

$$t\prod_{i=1}^m\frac{H_{r,s_i}(t)}{t} = H_{r,s}(t)\quad \textit{with}\quad r > \max(s^+,s_1^+,\ldots,s_m^+),\quad s = \sum_{i=1}^m s_i.$$

Proof. Let $(f_n(\varphi))$ be a sequence of polynomials of binomial type such that $f_n(1) = x_{n+1}/(n+1)$ and let $(f(t;r))^\varphi$ be the exponential generating function of the sequence of binomial type $(f_n(t;r))$. Assume that $x_2 \neq 0$. By Proposition 1 given in [12] we get

$$\prod_{i=1}^m\frac{H_{r,s_i}(t)}{t} = \prod_{i=1}^m\left(\sum_{n\geq 0}f_n(-s_i;r)\frac{t^n}{n!}\right) = \prod_{i=1}^m(f(t;r))^{-s_i} = \sum_{n\geq 0}f_n(-s;r)\frac{t^n}{n!} = \sum_{n\geq 0}f_n(-s;r)\frac{t^n}{n!},$$

and this is exactly

$$1 - \sum_{n\geq 1}\frac{s}{rn-s}\frac{B_{(r+1)n-s,rn-s}(\boldsymbol{x})}{\binom{(r+1)n-s}{n}}\frac{t^n}{n!} = \prod_{i=1}^m\frac{H_{r,s_i}(t)}{t}.$$

The proposition remains true for the case $x_2 = 0$ by continuity. □

Corollary 23. *Let* $\boldsymbol{y} = (y_1, y_2, \ldots)$ *be a sequence of real numbers with* $y_1 = 1$, r, s, d *be integers with* $d \geq 1$ *and let* $U_n(r, s; d)$, $V_n(r, s; d)$ *be given by Theorem 3 and Theorem 16. Then, for* $r > \max(s, -2s)$ *we have*

$$\sum_{k=1}^{n} B_{n,k}(U_j(r, -s; d))(dn + k)_{k-1} = U_n(r + s, s; d),$$

$$\sum_{k=1}^{n} B_{n,k}(U_j(r + s, s; d))(dn + k)_{k-1} = U_n(r, -s; d),$$

and for $r > \max(s, -(1 + d)s)$ *we have*

$$\sum_{k=1}^{n} B_{n,k}(V_j(r, -s))(dn + k)_{k-1} = V_n(r + sd, s),$$

$$\sum_{k=1}^{n} B_{n,k}(V_j(r + sd, s))(dn + k)_{k-1} = V_n(r, -s).$$

Proof. From Comtet [7, pp. 151], for $h(t) = t(1 + \sum_{n\geq 1} a_n \frac{t^{dn}}{n!})$, we have

$$h^{\langle -1\rangle}(t) = t\left(1 + \sum_{n\geq 1} b_n \frac{t^{dn}}{n!}\right) \quad \text{with} \quad b_n = \sum_{k=1}^{n} (-1)^k (dn + k)_{k-1} B_{n,k}(a_1, a_2, \ldots).$$

Then, it suffices to combine with Theorem 3 by taking $a_n = U_n(r, -s; d)$ and $b_n = U_n(r + s, s; d)$ and combine with Theorem 16 by taking $a_n = V_n(r, -s)$ and $b_n = V_n(r + sd, s)$. □

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