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Harmonic number identities via hypergeometric series and Bell polynomials

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From the Kummer $_2F_1$-summation theorem, the Dixon–Kummer $_4F_3$-summation theorem and the Dougall–Dixon $_5F_4$-summation theorem, we establish, by means of the Bell polynomials, three general formulas related to the generalized harmonic numbers and the Riemann zeta function. Based on these three general formulas, we further find series of harmonic number identities. Some of these identities involve both finite summation and infinite series, so that we can determinate the explicit expressions of numerous infinite series. In particular, we show that several interesting analogues of the Euler sums can be evaluated.

Keywords: harmonic numbers; hypergeometric series; Bell polynomials; Riemann zeta function; combinatorial identities; Euler sums

1. Introduction

The *generalized harmonic numbers* are defined by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \dots.$$

When $r = 1$, they reduce to the classical harmonic numbers, denoted as $H_n = H_n^{(1)}$.

There are several ways of dealing with finite summations or infinite series involving (generalized) harmonic numbers, such as the Newton–Andrews method, that is, applying the differential operator to classical hypergeometric identities [1,9,11,12,32], the partial fraction decomposition method [7,8,10] and the computer algebra method [17,18,21–23]. Sprugnoli [30] presented some harmonic number identities by means of the theory of Riordan arrays, and Sofo [27–29] obtained many from the integral representations of some special binomial sums. Borwein and Borwein [2], Coffey [14], De Doelder [16] and Flajolet and Salvy [19] established numerous infinite series (Euler sums or analogues of Euler sums) related to harmonic numbers and the Riemann zeta function by evaluating some integrals, while Choi and Srivastava [5], Chu [6],

Chu and Zheng [13], Shen [25] and Zheng [33] derived a large number of infinite series of the same types by systematically analysing the hypergeometric series. Additionally, based on the Bell polynomials and the Riordan array method, we established series of harmonic number identities, some of which also involve other special combinatorial sequences, such as the Stirling numbers of both kinds, the Lah numbers, the Bernoulli numbers and polynomials and the Cauchy numbers of both kinds (see [31]).

Recently, also by means of the Bell polynomials, Chen and Chu [3,4] studied the Gauss $_2F_1$-summation theorem and the Dixon $_3F_2$-summation theorem. They established general formulas involving (generalized) harmonic numbers and the Riemann zeta function, and illustrated as examples many interesting harmonic number identities. It should be noticed that unlike the works given in the past decade or so, some of Chen and Chu's harmonic number identities have both finite summation and infinite series.

The purpose of this paper is to explore Chu and Chen's method further, and we are interested in three well-known results in the theory of classical hypergeometric series, that is, the Kummer $_2F_1$-summation theorem, the Dixon–Kummer $_4F_3$-summation theorem and the Dougall–Dixon $_5F_4$-summation theorem. By introducing an extra indeterminate x, we reformulate these three summation theorems and establish the corresponding general harmonic number identities. Then, by specifying some parameters, we obtain numerous finite and infinite series identities involving (generalized) harmonic numbers and the Riemann zeta function. The identities in this paper contain the summation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\mathbb{H}_k,\quad \sum_{k=0}^{n}\binom{n}{k}^3\mathbb{H}_k\quad\text{or}\quad \sum_{k=0}^{n}\binom{n}{k}^4\mathbb{H}_k,$$

where $\mathbb{H}_k$ is a combination of some (generalized) harmonic numbers. In particular, it seems that harmonic number identities of the form $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\mathbb{H}_k$ have not been studied before.

The *Euler sums* are infinite series whose general term is a product of (generalized) harmonic numbers of index k and a power of k^{-1}, such as $\sum_{k=1}^{\infty}H_k^{(p)}/k^q$. The readers may find that the identities in this paper give the expressions of many infinite series, such as

$$\sum_{k=0}^{\infty}\frac{(-1)^kH_k}{(k+1)_{n+1}^2},\quad \sum_{k=0}^{\infty}\frac{(-1)^kH_{n+1+k}}{(k+1)_{n+1}^2}\quad\text{and}\quad \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_{n+1}^2}\{2H_k^2-H_k^{(2)}\},$$

which can be viewed as analogues of the Euler sums.

This paper is organized as follows. Some basic definitions and results are introduced at the end of this section. Sections 2 and 3 consider the Kummer $_2F_1$-summation theorem. Similarly, Sections 4 and 5 study the Dixon–Kummer $_4F_3$-summation theorem, and Sections 6 and 7 study the Dougall–Dixon $_5F_4$-summation theorem. Now, let us introduce some definitions and results briefly; for more details, the readers are referred to [3,4].

For the sequence $t:=(t_1,t_2,\ldots)$, the Bell polynomials (or the cycle indicators of symmetric groups) $\Omega_l(t):=\Omega_l(t_1,t_2,\ldots,t_l)$ are defined by the generating function

$$\sum_{l=0}^{\infty}\Omega_l(t)\frac{x^l}{l!}=\exp\left\{\sum_{i=1}^{\infty}t_i\frac{x^i}{i}\right\},\tag{1.1}$$

and their exact expression is

$$\Omega_l(t)=\sum\frac{l!}{k_1!\cdots k_l!}\left(\frac{t_1}{1}\right)^{k_1}\left(\frac{t_2}{2}\right)^{k_2}\cdots\left(\frac{t_l}{l}\right)^{k_l},$$

where the summation takes place over all non-negative integers $k_1,k_2,\ldots,k_l$ such that $k_1+2k_2+\cdots+lk_l=l$ (see [15, Section 3.3] and [24, Section 4.2]). From the expression, the first few Bell

polynomials can be obtained:

$$\Omega_0(t) = 1, \quad \Omega_1(t) = t_1,$$
$$\Omega_2(t) = t_1^2 + t_2, \quad \Omega_3(t) = t_1^3 + 3t_1 t_2 + 2t_3,$$
$$\Omega_4(t) = t_1^4 + 6t_1^2 t_2 + 8t_1 t_3 + 3t_2^2 + 6t_4.$$

The following lemma given in [3,4] characterizes the relations between the binomial coefficients, the Bell polynomials and the harmonic numbers.

LEMMA 1.1 *Let λ be an indeterminate, then we have*

$$[x^l]\binom{n + \lambda x}{n} = \frac{1}{l!}\Omega_l(u) \quad with \quad u_i := (-1)^{i-1}\lambda^i H_n^{(i)},$$
$$[x^l]\binom{n - \lambda x}{n}^{-1} = \frac{1}{l!}\Omega_l(v) \quad with \quad v_i := \lambda^i H_n^{(i)}.$$

On the other hand, with the notation from Slater [26], the *hypergeometric series* reads as

$$_{1+p}F_q\left[\begin{array}{cccc} a_0, & a_1, & \ldots, & a_p \\ & b_1, & \ldots, & b_q \end{array}\bigg| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_p)_k}{k!(b_1)_k \cdots (b_q)_k} z^k,$$

where the *shifted factorial* (rising factorial) is defined, for a complex number x, by $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$ when $n = 1, 2, \ldots$. Most of the hypergeometric summation theorems evaluate infinite series as a fraction of Γ-functions in the following form:

$$\Gamma\left[\begin{array}{cccc} a, & b, & \ldots, & c \\ A, & B, & \ldots, & C \end{array}\right] = \frac{\Gamma(a)\Gamma(b)\cdots\Gamma(c)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

According to the definitions, there hold two relations between the Γ-functions and the shifted factorials:

$$\frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n \quad and \quad \frac{\Gamma(x)}{\Gamma(x-n)} = (-1)^n(1-x)_n. \tag{1.2}$$

Furthermore, the Γ-function has the following power series expansions (e.g. see [3,4,6]):

$$\Gamma(1-x) = \exp\left\{\sum_{i=1}^{\infty} \sigma_i \frac{x^i}{i}\right\} \quad and \quad \Gamma\left(\frac{1}{2} - x\right) = \sqrt{\pi}\exp\left\{\sum_{i=1}^{\infty} \tau_i \frac{x^i}{i}\right\}, \tag{1.3}$$

where σ_i and τ_i are defined, respectively, by

$$\sigma_1 = \gamma \quad and \quad \sigma_i = \zeta(i) \text{ for } i \geq 2,$$
$$\tau_1 = \gamma + 2\log 2 \quad and \quad \tau_i = (2^i - 1)\zeta(i) \text{ for } i \geq 2,$$

with γ being the *Euler–Mascheroni constant* given by $\gamma = \lim_{n\to\infty}(H_n - \log n)$ and $\zeta(x)$ the *Riemann zeta function* defined by $\zeta(x) = \sum_{k=1}^{\infty} 1/k^x$.

2. Reformulation of the Kummer summation theorem

The *Kummer summation theorem* [26, p. 51] is

$$
{}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b \end{matrix}\middle| -1\right] = \Gamma\left[\begin{matrix} 1+\frac{a}{2}, & 1+a-b \\ 1+a, & 1+\frac{a}{2}-b \end{matrix}\right].
$$

By introducing an indeterminate x and the parameters λ, θ and ν with the condition $\nu = \lambda - \theta$, we may reformulate the Kummer summation theorem as follows:

$$
{}_2F_1\left[\begin{matrix} -\lambda x - n, & -\theta x - n \\ & 1-\nu x \end{matrix}\middle| -1\right] = \Gamma\left[\begin{matrix} 1-\nu x, & 1-\dfrac{\lambda x + n}{2} \\ 1-\lambda x - n, & 1-\dfrac{\lambda x - n}{2}+\theta x \end{matrix}\right]. \tag{2.1}
$$

According to the relations

$$
(1-x)_k = k!\binom{k-x}{k} \quad \text{and} \quad (-y-n)_k = (-1)^k k!\binom{n+y}{k},
$$

the left-hand side of Equation (2.1) can be expressed as

$$
\sum_{k=0}^{\infty} \frac{(-1)^k (-\lambda x - n)_k (-\theta x - n)_k}{k!(1-\nu x)_k} = \sum_{k=0}^{\infty} \frac{(-1)^k \binom{n+\lambda x}{k}\binom{n+\theta x}{k}}{\binom{k-\nu x}{k}}.
$$

In view of the relations given in (1.2), the right-hand side of Equation (2.1) can also be rewritten. For $n = 2m$, we have

$$
\Gamma\left[\begin{matrix} 1-\nu x, & 1-\dfrac{\lambda x}{2}-m \\ 1-\lambda x - 2m, & 1-\dfrac{\lambda x}{2}+\theta x + m \end{matrix}\right] = \frac{(-1)^m \binom{2m}{m}\binom{\lambda x+2m}{2m}}{\binom{\frac{\lambda x}{2}+m}{m}\binom{m-\frac{\lambda x}{2}+\theta x}{m}} \Gamma\left[\begin{matrix} 1-\nu x, & 1-\dfrac{\lambda x}{2} \\ 1-\lambda x, & 1-\dfrac{\lambda x}{2}+\theta x \end{matrix}\right].
$$

For $n = 2m+1$, using

$$
\frac{1}{((1-\lambda x)/2 + \theta x)_{m+1}} = \frac{2^{2m+1} m!\binom{\theta x - \frac{\lambda x}{2}+m}{m}}{(2m+1)!\binom{2\theta x-\lambda x+2m+1}{2m+1}},
$$

we have

$$
\Gamma\left[\begin{matrix} 1-\nu x, & \dfrac{1-\lambda x}{2}-m \\ 1-\lambda x - 2m - 1, & \dfrac{1-\lambda x}{2}+\theta x + m + 1 \end{matrix}\right]
$$
$$
= \frac{(-1)^{m+1} 2^{4m+1} m!^2 \lambda x \binom{m+\frac{\lambda x}{2}}{m}\binom{m+\theta x-\frac{\lambda x}{2}}{m}}{(2m+1)!\binom{2m+1+2\theta x-\lambda x}{2m+1}} \Gamma\left[\begin{matrix} 1-\nu x, & \dfrac{1-\lambda x}{2} \\ 1-\lambda x, & \dfrac{1-\lambda x}{2}+\theta x \end{matrix}\right].
$$

Dividing both sides of Equation (2.1) by $\binom{n+\lambda x}{n}\binom{n+\theta x}{n}$ and taking into account that

$$
\frac{\binom{n+\lambda x}{k}}{\binom{n+\lambda x}{n}} = \frac{\binom{n}{k}}{\binom{n-k+\lambda x}{n-k}} \quad \text{and} \quad \frac{\binom{n+\lambda x}{n+k+1}}{\binom{n+\lambda x}{n}} = \frac{k!\binom{\lambda x - 1}{k}\lambda x}{(n+1)_{k+1}},
$$

we finally reformulate the Kummer summation theorem in terms of binomial sums:

$$x^2 \lambda \theta \sum_{k=0}^{\infty} \frac{(-1)^{n+1+k} k!^2}{(n+1)_{k+1}^2} W_k(x) + \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 T_k(x)$$

$$= \begin{cases} (-1)^m \dbinom{2m}{m} U(x), & n = 2m, \\[3mm] \dfrac{(-1)^{m+1} 2^{4m+1} m!^2}{(2m+1)!} \lambda x\, V(x), & n = 2m+1, \end{cases} \tag{2.2}$$

where $W_k(x)$, $T_k(x)$, $U(x)$ and $V(x)$ are defined, respectively, by

$$W_k(x) = \binom{k - \lambda x}{k} \binom{k - \theta x}{k} \binom{n+k+1-\nu x}{n+k+1}^{-1},$$

$$T_k(x) = \left\{ \binom{n-k+\lambda x}{n-k} \binom{n-k+\theta x}{n-k} \binom{k-\nu x}{k} \right\}^{-1},$$

$$U(x) = \frac{1}{\binom{m+\frac{\lambda x}{2}}{m} \binom{m+\theta x - \frac{\lambda x}{2}}{m} \binom{2m+\theta x}{2m}} \Gamma \begin{bmatrix} 1 - \nu x, & 1 - \dfrac{\lambda x}{2} \\[2mm] 1 - \lambda x, & 1 - \dfrac{\lambda x}{2} + \theta x \end{bmatrix},$$

$$V(x) = \frac{\binom{m+\frac{\lambda x}{2}}{m} \binom{m+\theta x - \frac{\lambda x}{2}}{m}}{\binom{2m+1+\lambda x}{2m+1} \binom{2m+1+\theta x}{2m+1} \binom{2m+1+2\theta x - \lambda x}{2m+1}} \Gamma \begin{bmatrix} 1 - \nu x, & \dfrac{1 - \lambda x}{2} \\[2mm] 1 - \lambda x, & \dfrac{1 - \lambda x}{2} + \theta x \end{bmatrix}.$$

By appealing to Lemma 1.1 as well as Equation (1.3), we can obtain the following coefficients:

$$[x^l] W_k(x) = \frac{\Omega_l(w_k)}{l!} \quad \text{with} \quad w_{k,i} = \nu^i H_{n+k+1}^{(i)} - (\lambda^i + \theta^i) H_k^{(i)},$$

$$[x^l] T_k(x) = \frac{\Omega_l(t_k)}{l!} \quad \text{with} \quad t_{k,i} = \nu^i H_k^{(i)} + (-1)^i (\lambda^i + \theta^i) H_{n-k}^{(i)},$$

$$[x^l] U(x) = \frac{\Omega_l(u)}{l!},$$

where

$$u_i = \left\{ (\lambda - \theta)^i + \left(\frac{\lambda}{2} \right)^i \right\} \sigma_i - \left\{ \lambda^i + \left(\frac{\lambda}{2} - \theta \right)^i \right\} \sigma_i + (-1)^i \left\{ \left(\frac{\lambda}{2} \right)^i + \left(\theta - \frac{\lambda}{2} \right)^i \right\} H_m^{(i)}$$

$$+ (-1)^i \theta^i H_{2m}^{(i)},$$

and

$$[x^l] V(x) = \frac{\Omega_l(v)}{l!},$$

where

$$v_i = \{ (\lambda - \theta)^i - \lambda^i \} \sigma_i + \left\{ \left(\frac{\lambda}{2} \right)^i - \left(\frac{\lambda}{2} - \theta \right)^i \right\} \tau_i$$

$$- (-1)^i \left\{ \left(\frac{\lambda}{2} \right)^i + \left(\theta - \frac{\lambda}{2} \right)^i \right\} H_m^{(i)} + (-1)^i \{ \lambda^i + \theta^i + (2\theta - \lambda)^i \} H_{2m+1}^{(i)}.$$

Equating the coefficients of x^l on both sides of (2.2) gives the following general harmonic number identity related to the Kummer summation theorem.

THEOREM 2.1 (General harmonic number identity) *Let $\{w_k, t_k, u, v\}$ be the four sequences defined above; then for $m, l \in \mathbb{N}$, we have*

$$
\lambda\theta l(l-1)\sum_{k=0}^{\infty}\frac{(-1)^{n+1+k}k!^2}{(n+1)_{k+1}^2}\Omega_{l-2}(w_k) + \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\Omega_l(t_k)
$$

$$
= \begin{cases} (-1)^m\dbinom{2m}{m}\Omega_l(u), & n = 2m, \\[2ex] \dfrac{(-1)^{m+1}2^{4m+1}m!^2}{(2m+1)!}\lambda l\Omega_{l-1}(v), & n = 2m+1. \end{cases} \tag{2.3}
$$

3. Harmonic number identities related to the Kummer summation theorem

As applications of Theorem 2.1, we illustrate here a series of harmonic number identities.

3.1. $l = 0, 1$

Setting $l = 0$ in Theorem 2.1, we have

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} (-1)^m\dbinom{2m}{m}, & n = 2m, \\[2ex] 0, & n = 2m+1, \end{cases} \tag{3.1}
$$

which is an identity presented in Gould's famous collection (see [20, Equation (3.81)]).

The substitution $l = 1$ in Theorem 2.1 yields

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\left\{\begin{matrix}(\lambda-\theta)H_k \\ -(\lambda+\theta)H_{n-k}\end{matrix}\right\} = \begin{cases} (-1)^m\dbinom{2m}{m}(-\theta)(H_m+H_{2m}), & n = 2m, \\[2ex] \lambda\dfrac{(-1)^{m+1}2^{4m+1}m!^2}{(2m+1)!}, & n = 2m+1. \end{cases}
$$

Under the involution $k \to n - k$, the last identity can be simplified as

$$
\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 H_k = \begin{cases} \dfrac{(-1)^m}{2}\dbinom{2m}{m}(H_m+H_{2m}), & n = 2m, \\[2ex] \dfrac{(-1)^{m+1}2^{4m}m!^2}{(2m+1)!} = \dfrac{(-1)^{m+1}2^{4m}}{2m+1}\dbinom{2m}{m}^{-1}, & n = 2m+1. \end{cases} \tag{3.2}
$$

In Section 3, many identities containing the summation $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\mathbb{H}_k$ will be displayed, where $\mathbb{H}_k$ is a combination of some (generalized) harmonic numbers, and we have not found harmonic number identities of this kind in the literature.

3.2. $l = 2$

The general identity obtained from Theorem 2.1 by the substitution $l = 2$ will not be presented here due to its complexity. Instead, we concentrate on its further specializations.

Example 3.1 ($\theta = 0$ *or* $\lambda = 0$)   When $\theta = 0$, it can be found that λ is a redundant parameter, and the final harmonic number identity is

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^2\{H_k^2 + H_k^{(2)} - H_k H_{2m-k}\} = \frac{(-1)^m}{4}\binom{2m}{m}H_m^{(2)}. \tag{3.3}$$

Note that to obtain this identity, the replacement $k \to n - k$ is used again, and in fact, we will make use of this replacement frequently in the paper.

When $\lambda = 0$, the parameter θ turns to be redundant, and we have

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^2\{H_k^2 + H_k^{(2)} + H_k H_{2m-k}\} = \frac{(-1)^m}{2}\binom{2m}{m}\{(H_m + H_{2m})^2 + H_m^{(2)} + H_{2m}^{(2)}\}. \tag{3.4}$$

Combining Equations (3.3) and (3.4), we obtain

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^2\{H_k^2 + H_k^{(2)}\} = \frac{(-1)^m}{4}\binom{2m}{m}\left\{(H_m + H_{2m})^2 + \frac{3}{2}H_m^{(2)} + H_{2m}^{(2)}\right\}, \tag{3.5}$$

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^2 H_k H_{2m-k} = \frac{(-1)^m}{4}\binom{2m}{m}\left\{(H_m + H_{2m})^2 + \frac{1}{2}H_m^{(2)} + H_{2m}^{(2)}\right\}. \tag{3.6}$$

Example 3.2 ($\lambda = 1$ *and* $\theta = \sqrt{-1}$)   By equating the imaginary parts, we arrive at explicit identities involving infinite series and finite summation as follows:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_{2m+1}^2} = \frac{1}{(2m)!^2}\sum_{k=0}^{2m}(-1)^{k+1}\binom{2m}{k}^2 H_k^{(2)} + \frac{(-1)^m}{2(2m)!^2}\binom{2m}{m}\{\zeta(2) + H_m^{(2)}\}, \tag{3.7}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_{2m+2}^2} = \frac{1}{(2m+1)!^2}\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}^2\{2H_k^2 + H_k^{(2)}\}$$
$$+ \frac{(-1)^{m+1}2^{4m+1}m!^2}{(2m+1)!^3}\{2\log 2 + H_m - 3H_{2m+1}\}. \tag{3.8}$$

These two identities also give us explicit expressions for infinite series of the forms

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_{n+1}^2} \quad \text{and} \quad \sum_{k=0}^{\infty}\frac{(-1)^k}{(n+1+k)^2\binom{n+k}{k}^2}.$$

For example, when $n = 2, 3, 4, 5$, the corresponding evaluations are

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_3^2} = \frac{21 - 2\pi^2}{48}, \quad \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_4^2} = \frac{-11 + 16\log 2}{54},$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_5^2} = \frac{-235 + 24\pi^2}{27648}, \quad \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)_6^2} = \frac{3553 - 5120\log 2}{2160000}.$$

Of course, these four series can be obtained immediately with a system of computer algebra, such as Mathematica 7. But our aim is to show that from the hypergeometric method, the explicit expression of a more general series $\sum_{k=0}^{\infty}(-1)^k/(k+1)_{n+1}^2$ can be derived, while these four series are just the simplest specializations. In this paper, the expressions of other general infinite series will be presented. In particular, some specializations cannot be obtained easily with Mathematica 7.

3.3. $l = 3$

Example 3.3 ($\theta = 0$ or $\lambda = 0$)   For simplicity, define two abbreviated notations:

$$\Phi_k^{\langle l \rangle} = \Omega_l(H_k, H_k^{(2)}, H_k^{(3)}, \ldots),$$

$$\Psi_k^{\langle l \rangle} = \Omega_l(H_k + H_{2k}, H_k^{(2)} + H_{2k}^{(2)}, H_k^{(3)} + H_{2k}^{(3)}, \ldots);$$

then by setting $\theta = 0$ and $\lambda = 0$, respectively, we can derive the next two identities:

$$\sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^2 \{\Phi_k^{\langle 3 \rangle} - 3(H_k^2 + H_k^{(2)})H_{2m+1-k}\}$$

$$= \frac{(-1)^{m+1} 2^{4m-1} m!^2}{(2m+1)!} \cdot 3\{4H_{2m+1}^{(2)} - H_m^{(2)}\}, \tag{3.9}$$

$$\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^2 \{\Phi_k^{\langle 3 \rangle} + 3(H_k^2 + H_k^{(2)})H_{2m-k}\} = \frac{(-1)^m}{2} \binom{2m}{m} \Psi_m^{\langle 3 \rangle}. \tag{3.10}$$

Example 3.4 ($\lambda = 1$ and $\theta = 1$)   When $n = 2m$, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_{2m+1}^2} = \frac{1}{3(2m)!^2} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^2 \{2H_k^3 + 3H_k H_k^{(2)} + H_k^{(3)}\}$$

$$+ \frac{(-1)^{m+1}}{12(2m)!^2} \binom{2m}{m} \left\{(H_m + H_{2m})^3 + 3(H_m + H_{2m})\right.$$

$$\left. \times \left(-\zeta(2) + \frac{1}{2}H_m^{(2)} + H_{2m}^{(2)}\right) + 2\left(\frac{3}{4}\zeta(3) + \frac{1}{4}H_m^{(3)} + H_{2m}^{(3)}\right)\right\}. \tag{3.11}$$

Combining Equation (3.11) with Equation (3.10), we obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_{2m+1}^2} = \frac{1}{2(2m)!^2} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^2 (H_k^2 + H_k^{(2)})(H_k - H_{2m-k})$$

$$+ \frac{(-1)^{m+1}}{8(2m)!^2} \binom{2m}{m} \{\zeta(3) - H_m^{(3)} - (H_m + H_{2m})(2\zeta(2) + H_m^{(2)})\}. \tag{3.12}$$

When $n = 2m + 1$, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_{2m+2}^2} = \frac{1}{3(2m+1)!^2} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^2 \{2H_k^3 + 3H_k H_k^{(2)} + H_k^{(3)}\}$$

$$+ \frac{(-1)^m 2^{4m-1} m!^2}{(2m+1)!^3} \left\{(2\log 2 + H_m - 3H_{2m+1})^2 - \zeta(2)\right.$$

$$\left. - \frac{1}{2}H_m^{(2)} + 3H_{2m+1}^{(2)}\right\}. \tag{3.13}$$

These three identities give us explicit expressions of the infinite series

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_{n+1}^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(n+1+k)^2 \binom{n+k}{k}^2}.$$

For example, the following evaluations hold

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)^2} = -\frac{1}{8}\zeta(3),$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_2^2} = 4 - \frac{\pi^2}{12} - 6\log 2 + 2\log^2 2,$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_3^2} = \frac{42 - 5\pi^2 + 6\zeta(3)}{96},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_k}{(k+1)_4^2} = \frac{-1757 + 24\pi^2 + 2592\log 2 - 576\log^2 2}{3888}.$$

Example 3.5 ($\lambda = 1$ and $\theta = -1$)   For $n = 2m$, we obtain the identity

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_{2m+1+k}}{(k+1)_{2m+1}^2} = \frac{1}{3(2m)!^2} \sum_{k=0}^{2m} (-1)^{k+1} \binom{2m}{k}^2 \{2H_k^3 + 6H_k H_k^{(2)} + 4H_k^{(3)} + 3H_k^{(2)} H_{2m-k}\}$$
$$+ \frac{(-1)^m}{12(2m)!^2} \binom{2m}{m} \left\{ (H_m + H_{2m})^3 + 3(H_m + H_{2m}) \right.$$
$$\times \left. \left(\zeta(2) + \frac{5}{2}H_m^{(2)} + H_{2m}^{(2)} \right) + \frac{1}{2}(15\zeta(3) + 13H_m^{(3)} + 4H_{2m}^{(3)}) \right\}, \tag{3.14}$$

which, in conjunction with (3.10), immediately yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_{2m+1+k}}{(k+1)_{2m+1}^2} = \frac{1}{(2m)!^2} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^2 (2H_k^2 + H_k^{(2)}) H_{2m-k}$$
$$+ \frac{(-1)^{m+1}}{8(2m)!^2} \binom{2m}{m} \{2(H_m + H_{2m})^3 + (H_m + H_{2m})(3H_m^{(2)} + 6H_{2m}^{(2)}$$
$$- 2\zeta(2)) - 5\zeta(3) + H_m^{(3)} + 4H_{2m}^{(3)}\}. \tag{3.15}$$

On the other hand, for $n = 2m + 1$, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_{2m+2+k}}{(k+1)_{2m+2}^2} = \frac{1}{3(2m+1)!^2} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^2 \{2H_k^3 + 6H_k H_k^{(2)} + 4H_k^{(3)}$$
$$- 3H_k^{(2)} H_{2m+1-k}\} + \frac{(-1)^m 2^{4m-1} m!^2}{(2m+1)!^3} \left\{ (2\log 2 + H_m - 3H_{2m+1})^2 \right.$$
$$- 3\zeta(2) - \frac{5}{2}H_m^{(2)} + 11H_{2m+1}^{(2)} \Big\}, \tag{3.16}$$

which, together with (3.9), gives

$$
\sum_{k=0}^{\infty} \frac{(-1)^k H_{2m+2+k}}{(k+1)_{2m+2}^2} = \frac{1}{(2m+1)!^2} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^2 (2H_k^2 + H_k^{(2)}) H_{2m+1-k}
$$
$$
+ \frac{(-1)^m 2^{4m-1} m!^2}{(2m+1)!^3} \left\{ (2\log 2 + H_m - 3H_{2m+1})^2 - 3\zeta(2) \right.
$$
$$
\left. - \frac{1}{2} H_m^{(2)} + 3H_{2m+1}^{(2)} \right\}. \tag{3.17}
$$

These identities lead us to the explicit expressions for the infinite series

$$
\sum_{k=0}^{\infty} \frac{(-1)^k H_{n+1+k}}{(k+1)_{n+1}^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k H_{n+1+k}}{(n+1+k)^2 \binom{n+k}{k}^2}.
$$

For example, it can be derived that

$$
\sum_{k=0}^{\infty} \frac{(-1)^k H_{k+1}}{(k+1)^2} = \frac{5}{8} \zeta(3),
$$

$$
\sum_{k=0}^{\infty} \frac{(-1)^k H_{k+2}}{(k+1)_2^2} = 6 - \frac{\pi^2}{4} - 6\log 2 + 2\log^2 2,
$$

$$
\sum_{k=0}^{\infty} \frac{(-1)^k H_{k+3}}{(k+1)_3^2} = -\frac{5}{96} \{-18 + \pi^2 + 6\zeta(3)\},
$$

$$
\sum_{k=0}^{\infty} \frac{(-1)^k H_{k+4}}{(k+1)_4^2} = \frac{-739 + 24\pi^2 + 864\log 2 - 192\log^2 2}{1296}.
$$

3.4. $l = 4$

Example 3.6 ($\lambda = 1$ *and* $\theta = 1$)   Recall the notations $\Phi_k^{\langle l \rangle}$ and $\Psi_k^{\langle l \rangle}$ defined in Example 3.3. When $n = 2m$, we have

$$
\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)_{2m+1}^2} \{2H_k^2 - H_k^{(2)}\} = \frac{1}{6(2m)!^2} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^2 \{\Phi_k^{\langle 4 \rangle} + 3H_k^4 + 6H_k^2 H_k^{(2)} - 3H_k^{(4)}\}
$$
$$
+ \frac{(-1)^{m+1}}{24(2m)!^2} \binom{2m}{m} \left\{ \Psi_m^{\langle 4 \rangle} - 6(H_m + H_{2m})^2 \left(\zeta(2) + \frac{1}{2} H_m^{(2)} \right) \right.
$$
$$
+ 6(H_m + H_{2m})(\zeta(3) - H_m^{(3)}) + 3 \left(\zeta(2) + \frac{1}{2} H_m^{(2)} \right)
$$
$$
\left. \times \left(\zeta(2) - \frac{3}{2} H_m^{(2)} - 2H_{2m}^{(2)} \right) - 6 \left(\zeta(4) + \frac{7}{8} H_m^{(4)} \right) \right\},
$$

and when $n = 2m + 1$, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)_{2m+2}^2}\{2H_k^2 - H_k^{(2)}\} = \frac{1}{6(2m+1)!^2} \sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}^2$$

$$\times \{\Phi_k^{\langle 4\rangle} + 3H_k^4 + 6H_k^2 H_k^{(2)} - 3H_k^{(4)}\}$$

$$+ \frac{(-1)^{m+1}2^{4m}m!^2}{3(2m+1)!^3}\left\{(2\log 2 + H_m - 3H_{2m+1})^3\right.$$

$$+ 3(2\log 2 + H_m - 3H_{2m+1})\left(-\zeta(2) - \frac{1}{2}H_m^{(2)} + 3H_{2m+1}^{(2)}\right)$$

$$\left.+ 2\left(\frac{3}{4}\zeta(3) + \frac{1}{4}H_m^{(3)} - 3H_{2m+1}^{(3)}\right)\right\}.$$

From these identities, we can evaluate the infinite series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)_{n+1}^2}\{2H_k^2 - H_k^{(2)}\} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+1+k)^2\binom{n+k}{k}^2}\{2H_k^2 - H_k^{(2)}\}.$$

For example, the next four series hold

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2}\{2H_k^2 - H_k^{(2)}\} = -\frac{\pi^4}{1440},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)_2^2}\{2H_k^2 - H_k^{(2)}\} = \frac{1}{6}\{90 - 144\log 2 + 72\log^2 2 - 16\log^3 2 + \pi^2(-3 + \log 4)$$

$$- 3\zeta(3)\},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)_3^2}\{2H_k^2 - H_k^{(2)}\} = \frac{3555 - 480\pi^2 + \pi^4 + 900\zeta(3)}{2880},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)_4^2}\{2H_k^2 - H_k^{(2)}\} = \frac{1}{3888}\{-8929 + 13728\log 2 - 5184\log^2 2 + 768\log^3 2$$

$$- 24\pi^2(-9 + \log 16) + 144\zeta(3)\}.$$

The readers may find that these series cannot be evaluated with Mathematica 7 by the function 'Sum'. Additionally, it can be seen from the $l = 4$ case that the specializations of Theorem 2.1 are becoming more and more complicated, so we choose not to present others.

4. Reformulation of the Dixon–Kummer summation theorem

The *Dixon–Kummer summation theorem* [26, p. 56] is

$$_4F_3\left[\begin{matrix} a, & 1 + \dfrac{a}{2}, & d, & e \\ & \dfrac{a}{2}, & 1 + a - d, & 1 + a - e \end{matrix}\,\middle|\,-1\right] = \Gamma\left[\begin{matrix} 1 + a - d, & 1 + a - e \\ 1 + a, & 1 + a - d - e \end{matrix}\right].$$

Similar to Section 2, let us introduce an indeterminate x and the parameters λ, θ, ε, ν and δ with the conditions $\nu = \lambda - \theta$ and $\delta = \lambda - \varepsilon$; then by some transformations, the Dixon–Kummer

summation theorem can be rewritten as

$$\sum_{k=0}^{\infty}(\lambda x + n - 2k)\frac{\binom{n+\lambda x}{k}\binom{n+\theta x}{k}\binom{n+\varepsilon x}{n}}{\binom{k-\nu x}{k}\binom{k-\delta x}{k}} = \frac{(-1)^{n}\lambda x\binom{n+\lambda x}{n}}{\binom{n-\lambda x+\theta x+\varepsilon x}{n}}\,\Gamma\begin{bmatrix}1-\nu x, & 1-\delta x\\ 1-\lambda x, & 1-\lambda x+\theta x+\varepsilon x\end{bmatrix}.$$

Dividing both sides by $\binom{n+\lambda x}{n}\binom{n+\theta x}{n}\binom{n+\varepsilon x}{n}$ and then making some further transformations, we may reformulate the Dixon–Kummer summation theorem as

$$x^{3}\lambda\theta\varepsilon\sum_{k=0}^{\infty}\frac{(-1)^{k}k!^{3}(\lambda x-2k-n-2)}{(n+1)_{k+1}^{3}}\bar{W}_{k}(x) + \sum_{k=0}^{n}\binom{n}{k}^{3}(\lambda x+n-2k)\bar{T}_{k}(x)$$
$$= (-1)^{n}\lambda x\bar{U}(x), \tag{4.1}$$

where $\bar{W}_{k}(x)$, $\bar{T}_{k}(x)$ and $\bar{U}(x)$ are explicitly given by

$$\bar{W}_{k}(x) = \frac{\binom{k-\lambda x}{k}\binom{k-\theta x}{k}\binom{k-\varepsilon x}{k}}{\binom{n+k+1-\nu x}{n+k+1}\binom{n+k+1-\delta x}{n+k+1}},$$

$$\bar{T}_{k}(x) = \frac{1}{\binom{n-k+\lambda x}{n-k}\binom{n-k+\theta x}{n-k}\binom{n-k+\varepsilon x}{n-k}\binom{k-\nu x}{k}\binom{k-\delta x}{k}},$$

$$\bar{U}(x) = \frac{1}{\binom{n+\theta x}{n}\binom{n+\varepsilon x}{n}\binom{n-\lambda x+\theta x+\varepsilon x}{n}}\,\Gamma\begin{bmatrix}1-\nu x, & 1-\delta x\\ 1-\lambda x, & 1-\lambda x+\theta x+\varepsilon x\end{bmatrix}.$$

According to Equation (1.3) and Lemma 1.1, the following coefficients can be deduced:

$$[x^{l}]\bar{W}_{k}(x) = \frac{\Omega_{l}(\bar{w}_{k})}{l!} \quad\text{with}\quad \bar{w}_{k,i} = (\nu^{i}+\delta^{i})H_{n+k+1}^{(i)} - (\lambda^{i}+\theta^{i}+\varepsilon^{i})H_{k}^{(i)},$$

$$[x^{l}]\bar{T}_{k}(x) = \frac{\Omega_{l}(\bar{t}_{k})}{l!} \quad\text{with}\quad \bar{t}_{k,i} = (\nu^{i}+\delta^{i})H_{k}^{(i)} + (-1)^{i}(\lambda^{i}+\theta^{i}+\varepsilon^{i})H_{n-k}^{(i)}$$

and

$$[x^{l}]\bar{U}(x) = \frac{\Omega_{l}(\bar{u})}{l!},$$

where

$$\bar{u}_{i} = \{(\lambda-\theta-\varepsilon)^{i}+(-1)^{i}(\theta^{i}+\varepsilon^{i})\}H_{n}^{(i)} + \{\nu^{i}+\delta^{i}-\lambda^{i}-(\lambda-\theta-\varepsilon)^{i}\}\sigma_{i}.$$

Thus, equating the coefficients of x^{l} on both sides of (4.1) gives us the general harmonic number identity related to the Dixon–Kummer summation theorem.

THEOREM 4.1 (General harmonic number identity)   *Let* $\{\bar{w}_{k}, \bar{t}_{k}, \bar{u}\}$ *be the three sequences defined above*; *then for* $l \in \mathbb{N}$, *we have*

$$\lambda\theta\varepsilon\sum_{k=0}^{\infty}\frac{(-1)^{k}k!^{3}}{(n+1)_{k+1}^{3}}\{\lambda(l-3)_{4}\Omega_{l-4}(\bar{w}_{k}) - (l-2)_{3}(n+2k+2)\Omega_{l-3}(\bar{w}_{k})\}$$

$$+ \lambda l\sum_{k=0}^{n}\binom{n}{k}^{3}\Omega_{l-1}(\bar{t}_{k}) + \sum_{k=0}^{n}\binom{n}{k}^{3}(n-2k)\Omega_{l}(\bar{t}_{k}) = (-1)^{n}\lambda l\,\Omega_{l-1}(\bar{u}). \tag{4.2}$$

5. Harmonic number identities related to the Dixon–Kummer summation theorem

By specifying the parameters l, λ, θ and ε in Theorem 4.1, many interesting harmonic number identities can be established. In particular, all of these identities involve the summation $\sum_{k=0}^{n} \binom{n}{k}^3 \mathbb{H}_k$, where $\mathbb{H}_k$ is a combination of some generalized harmonic numbers.

5.1. $l = 1, 2$

It can be verified that the $l = 0$ case of Theorem 4.1 is trivial, so let us begin with $l = 1$ and $l = 2$. When $l = 1$, Theorem 4.1 reduces to

$$\lambda \sum_{k=0}^{n} \binom{n}{k}^3 + \sum_{k=0}^{n} \binom{n}{k}^3 (n - 2k)\{(2\lambda - \theta - \varepsilon)H_k - (\lambda + \theta + \varepsilon)H_{n-k}\} = (-1)^n \lambda.$$

Under the replacement $k \rightarrow n - k$, this identity can be simplified as

$$\sum_{k=0}^{n} \binom{n}{k}^3 \{1 + 3(n - 2k)H_k\} = (-1)^n, \tag{5.1}$$

which can be found in [11, Table 1, Entry (16)] and [23, Equation (3)]. Similarly, the substitution $l = 2$ in Theorem 4.1 yields

$$\sum_{k=0}^{n} \binom{n}{k}^3 \left\{ H_k + \left(\frac{n}{2} - k \right) (3H_k^2 + H_k^{(2)}) \right\} = (-1)^n H_n, \tag{5.2}$$

which is a result presented in [12, Example 20].

5.2. $l = 3$

Example 5.1 ($\lambda = 2, \varepsilon = 1 + \sqrt{-3}$ *and* $\theta = 0$)   Now, Theorem 4.1 gives the following identity:

$$\sum_{k=0}^{n} \binom{n}{k}^3 \{3H_k^2 - 6H_k H_{n-k} + H_k^{(2)} - 3(n - 2k)(3H_k^2 + H_k^{(2)})H_{n-k}\} = (-1)^{n+1}\{3H_n^2 + H_n^{(2)}\}. \tag{5.3}$$

Example 5.2   The substitutions $\lambda = 1, \varepsilon = -1$ and $\theta = 0$ give

$$\sum_{k=0}^{n} \binom{n}{k}^3 \{9H_k^2 + 7H_k^{(2)} + 3(n - 2k)\{3H_k^3 + H_k^{(2)}(5H_k - 2H_{n-k}) + 2H_k^{(3)}\}\}$$
$$= (-1)^n \{9H_n^2 + 5H_n^{(2)}\}; \tag{5.4}$$

the substitutions $\lambda = 2, \varepsilon = 1$ and $\theta = 0$ yield

$$\sum_{k=0}^{n} \binom{n}{k}^3 \{18H_k(H_k - H_{n-k}) + 10H_k^{(2)} + 3(n - 2k)\{3H_k^2(H_k - 3H_{n-k})$$
$$+ 5H_k^{(2)}(H_k - H_{n-k}) + 2H_k^{(3)}\}\} = (-1)^n 2H_n^{(2)}. \tag{5.5}$$

Moreover, setting $\lambda = \varepsilon$ and $\theta = 0$ results in

$$\sum_{k=0}^{n} \binom{n}{k}^3 \{H_k(5H_k - 4H_{n-k}) + 3H_k^{(2)} + (n - 2k)\{3H_k^2(H_k - 2H_{n-k})$$
$$+ H_k^{(2)}(5H_k - 4H_{n-k}) + 2H_k^{(3)}\}\} = (-1)^n\{H_n^2 + H_n^{(2)}\}. \tag{5.6}$$

Note that subtracting (5.4) from (5.5) gives us identity (5.3) in Example 5.1. Additionally, combining (5.6) with (5.4) or (5.5), we can also obtain (5.3).

Example 5.3   For $l = 3$ case, set further $\lambda = 4$, $\varepsilon = 1 + \sqrt{-3}$ and $\theta = 1 - \sqrt{-3}$; then we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k(n + 2k + 2)}{(k + 1)_{n+1}^3} = \frac{1}{n!^3} \sum_{k=0}^{n} \binom{n}{k}^3 \left\{ 9H_k(H_k - H_{n-k}) + 3H_k^{(2)} + \frac{n - 2k}{2}\{2H_k^{(3)} \right.$$
$$\left. + 9H_k^2(H_k - 3H_{n-k}) + 9H_k^{(2)}(H_k - H_{n-k})\} \right\} + \frac{(-1)^n}{n!^3}\zeta(2). \tag{5.7}$$

Alternately, set further $\lambda = 4$, $\varepsilon = 1 + 3\sqrt{-1}$ and $\theta = 1 - 3\sqrt{-1}$; then we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k(n + 2k + 2)}{(k + 1)_{n+1}^3} = \frac{1}{5n!^3} \sum_{k=0}^{n} \binom{n}{k}^3 \{(n - 2k)\{-4H_k^{(3)} + 9H_k^2(H_k - 3H_{n-k})\}$$
$$+ 18H_k(H_k - H_{n-k})\} + \frac{(-1)^n}{5n!^3}\{5\zeta(2) + 3H_n^{(2)}\}. \tag{5.8}$$

From (5.5), we establish an expression of $H_n^{(2)}$, which can be substituted into the right-hand side of (5.8) to obtain (5.7) once again. Furthermore, adding (5.5) to (5.8), we obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k(n + 2k + 2)}{(k + 1)_{n+1}^3} = -\frac{1}{n!^3} \sum_{k=0}^{n} \binom{n}{k}^3 \{2H_k^{(2)} + (n - 2k)\{2H_k^{(3)} + 3H_k^{(2)}(H_k - H_{n-k})\}\}$$
$$+ \frac{(-1)^n}{n!^3}\{\zeta(2) + H_n^{(2)}\}. \tag{5.9}$$

From Equations (5.7)–(5.9), the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k(n + 2k + 2)}{(k + 1)_{n+1}^3} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(-1)^k(n + 2k + 2)}{(n + k + 1)^3\binom{n+k}{k}^3}$$

can be evaluated. For example, when $n = 1, 2, 3, 4$, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k + 3)}{(k + 1)_2^3} = 2 - \frac{\pi^2}{6}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k(2k + 4)}{(k + 1)_3^3} = \frac{-9 + \pi^2}{48},$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k + 5)}{(k + 1)_4^3} = \frac{31 - 3\pi^2}{3888}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k(2k + 6)}{(k + 1)_5^3} = \frac{-115 + 12\pi^2}{995328}.$$

The first three series can be obtained immediately with Mathematica 7, while the last one as well as the other specializations of the general series cannot be evaluated by the function 'Sum'.

5.3. $l = 4$

For $l = 4$ case, we illustrate two identities.

Example 5.4 ($\lambda = 2$, $\varepsilon = 1 + \sqrt{-3}$ *and* $\theta = 0$)   The corresponding identity is

$$\sum_{k=0}^{n} \binom{n}{k}^3 \{4(3H_k^2 H_{n-k} - 2H_k^3 + H_k^{(2)} H_{n-k} - 2H_k H_k^{(2)}) - (n - 2k)\{3H_k^4 - 12H_k^3 H_{n-k}$$

$$+ 6H_k^2 H_k^{(2)} - 12H_k H_{n-k} H_k^{(2)} - 2H_k^{(4)} + (H_k^{(2)})^2\}\} = (-1)^n 4H_n\{H_n^2 + H_n^{(2)}\}. \qquad (5.10)$$

Example 5.5 ($\lambda = 4$, $\varepsilon = 1 + \sqrt{-3}$ *and* $\theta = 1 - \sqrt{-3}$)   These substitutions give

$$\sum_{k=0}^{\infty} \frac{(-1)^k \{2 - 3(n + 2k + 2)(H_{n+k+1} - H_k)\}}{(k+1)_{n+1}^3}$$

$$= \frac{1}{n!^3} \sum_{k=0}^{n} \binom{n}{k}^3 \{2H_k^{(3)} - 3(n - 2k)(H_{n-k} H_k^{(3)} - H_k H_k^{(3)} - H_k^{(4)})\}$$

$$+ \frac{(-1)^n}{n!^3} \{H_n^{(3)} - 3\zeta(3)\}. \qquad (5.11)$$

By means of the fact that $H_{n+k+1} - H_k = \sum_{i=1}^{n+1} 1/(k+i)$, we will obtain many series, such as

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k+3)(2k+4)(2k+5)}{(k+1)_3^4} = \frac{1}{128}\{23 - 24\zeta(3)\}.$$

The interested readers may find more series from (5.11) by choosing special values for n.

6.   Reformulation of the Dougall–Dixon summation theorem

The *Dougall–Dixon summation theorem* [26, p. 56] is

$${}_5F_4 \left[\begin{array}{ccccc} a, & 1 + \dfrac{a}{2}, & b, & c, & d \\ & \dfrac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d \end{array} \middle| 1 \right]$$

$$= \Gamma \left[\begin{array}{cccc} 1+a-b, & 1+a-c, & 1+a-d, & 1+a-b-c-d \\ 1+a, & 1+a-b-c, & 1+a-b-d, & 1+a-c-d \end{array} \right].$$

Similarly, by setting $a \to -\lambda x - n, b \to -\theta x - n, c \to -\varepsilon x - n$ and $d \to -\eta x - n$ and writing $\nu = \lambda - \theta$, $\delta = \lambda - \varepsilon$ and $\mu = \lambda - \eta$, we finally reformulate the theorem as

$$x^4 \lambda \theta \varepsilon \eta \sum_{k=0}^{\infty} \frac{k!^4(\lambda x - n - 2k - 2)}{(n+1)_{k+1}^4} \tilde{W}_k(x) + \sum_{k=0}^{n} \binom{n}{k}^4 (\lambda x + n - 2k) \tilde{T}_k(x)$$

$$= (-1)^n \lambda x \binom{2n}{n} \tilde{U}(x), \qquad (6.1)$$

where $\tilde{W}_k(x)$, $\tilde{T}_k(x)$ and $\tilde{U}(x)$ are explicitly given by

$$\tilde{W}_k(x) = \frac{\binom{k-\lambda x}{k}\binom{k-\theta x}{k}\binom{k-\varepsilon x}{k}\binom{k-\eta x}{k}}{\binom{n+k+1-\nu x}{n+k+1}\binom{n+k+1-\delta x}{n+k+1}\binom{n+k+1-\mu x}{n+k+1}},$$

$$\tilde{T}_k(x) = \frac{1}{\binom{n-k+\lambda x}{n-k}\binom{n-k+\theta x}{n-k}\binom{n-k+\varepsilon x}{n-k}\binom{n-k+\eta x}{n-k}\binom{k-\nu x}{k}\binom{k-\delta x}{k}\binom{k-\mu x}{k}},$$

$$\tilde{U}(x) = \frac{\binom{2n-\lambda x+\theta x+\varepsilon x+\eta x}{2n}}{\binom{n+\theta x}{n}\binom{n+\varepsilon x}{n}\binom{n+\eta x}{n}\binom{n-\lambda x+\theta x+\varepsilon x}{n}\binom{n-\lambda x+\theta x+\eta x}{n}\binom{n-\lambda x+\varepsilon x+\eta x}{n}}$$

$$\cdot \Gamma\begin{bmatrix} 1-\nu x, & 1-\delta x, & 1-\mu x, & 1-\lambda x+\theta x+\varepsilon x+\eta x \\ 1-\lambda x, & 1-\lambda x+\theta x+\varepsilon x, & 1-\lambda x+\theta x+\eta x, & 1-\lambda x+\varepsilon x+\eta x \end{bmatrix}.$$

The coefficients of $\tilde{W}_k(x)$, $\tilde{T}_k(x)$ and $\tilde{U}(x)$ are

$$[x^l]\tilde{W}_k(x) = \frac{\Omega_l(\tilde{w}_k)}{l!} \quad \text{with} \quad \tilde{w}_{k,i} = (\nu^i + \delta^i + \mu^i)H_{n+k+1}^{(i)} - (\lambda^i + \theta^i + \varepsilon^i + \eta^i)H_k^{(i)},$$

$$[x^l]\tilde{T}_k(x) = \frac{\Omega_l(\tilde{t}_k)}{l!} \quad \text{with} \quad \tilde{t}_{k,i} = (\nu^i + \delta^i + \mu^i)H_k^{(i)} + (-1)^i(\lambda^i + \theta^i + \varepsilon^i + \eta^i)H_{n-k}^{(i)}$$

and

$$[x^l]\tilde{U}(x) = \frac{\Omega_l(\tilde{u})}{l!},$$

where

$$\tilde{u}_i = \{\nu^i + \delta^i + \mu^i + (\lambda - \theta - \varepsilon - \eta)^i - \lambda^i - (\lambda - \theta - \varepsilon)^i - (\lambda - \theta - \eta)^i - (\lambda - \varepsilon - \eta)^i\}\sigma_i$$

$$+ (-1)^i(\theta^i + \varepsilon^i + \eta^i)H_n^{(i)} + \{(\lambda - \theta - \varepsilon)^i + (\lambda - \theta - \eta)^i + (\lambda - \varepsilon - \eta)^i\}H_n^{(i)}$$

$$- (\lambda - \theta - \varepsilon - \eta)^i H_{2n}^{(i)}.$$

Equating the coefficients of x^l on both sides of (6.1) leads us to the general harmonic number identity related to the Dougall–Dixon summation theorem.

THEOREM 6.1 (General harmonic number identity) *Let $\{\tilde{w}_k, \tilde{t}_k, \tilde{u}\}$ be the three sequences defined above; then for $l \in \mathbb{N}$, we have*

$$\lambda\theta\varepsilon\eta \sum_{k=0}^{\infty} \frac{k!^4}{(n+1)_{k+1}^4}\{\lambda(l-4)_5\Omega_{l-5}(\tilde{w}_k) - (l-3)_4(n+2k+2)\Omega_{l-4}(\tilde{w}_k)\}$$

$$+ \lambda l \sum_{k=0}^{n}\binom{n}{k}^4 \Omega_{l-1}(\tilde{t}_k) + \sum_{k=0}^{n}\binom{n}{k}^4(n-2k)\Omega_l(\tilde{t}_k) = (-1)^n\lambda l\binom{2n}{n}\Omega_{l-1}(\tilde{u}). \qquad (6.2)$$

7. Harmonic number identities related to the Dougall–Dixon summation theorem

By specifying the parameters l, λ, θ, ε and η in Theorem 6.1, we obtain many harmonic number identities, which have the summation $\sum_{k=0}^{n}\binom{n}{k}^4\mathbb{H}_k$, where $\mathbb{H}_k$ is a combination of some generalized harmonic numbers.

7.1. $l = 1, 2$

When $l = 1$, Theorem 6.1 gives

$$\sum_{k=0}^{n} \binom{n}{k}^4 \{1 + 4(n - 2k)H_k\} = (-1)^n \binom{2n}{n}, \tag{7.1}$$

which can be found in [11, Table 1, Entry (17)], [12, Example 24] and [23, Equation (4)]. When $l = 2$, Theorem 6.1 yields

$$\sum_{k=0}^{n} \binom{n}{k}^4 \{2H_k + (n - 2k)(4H_k^2 + H_k^{(2)})\} = (-1)^{n+1} \binom{2n}{n} \{H_{2n} - 3H_n\}, \tag{7.2}$$

which is a result presented in [12, Example 25].

7.2. $l = 3$

Example 7.1 Let $\theta = \varepsilon = \eta = \lambda$; then we have $\nu = \delta = \mu = 0$ and

$$\sum_{k=0}^{n} \binom{n}{k}^4 \left\{ 6H_k^2 + \frac{3}{2}H_k^{(2)} + (n - 2k)(H_k^{(3)} + 6H_k H_k^{(2)} + 8H_k^3) \right\}$$

$$= (-1)^n \frac{3}{4} \binom{2n}{n} \{2(H_{2n} - 3H_n)^2 - 2H_{2n}^{(2)} + 3H_n^{(2)}\}. \tag{7.3}$$

Set $\theta = -\lambda$, $\varepsilon = \lambda$ and $\eta = -\lambda$; then we have $\nu = \mu = 2\lambda$ and $\delta = 0$. The identity is

$$\sum_{k=0}^{n} \binom{n}{k}^4 \left\{ 3H_k^2 + \frac{9}{4}H_k^{(2)} + (n - 2k)\{4H_k^3 + 3H_k^{(2)}(2H_k - H_{n-k}) + 2H_k^{(3)}\} \right\}$$

$$= (-1)^n \frac{3}{8} \binom{2n}{n} \{2(H_{2n} - 3H_n)^2 - 2H_{2n}^{(2)} + 7H_n^{(2)}\}, \tag{7.4}$$

which, in conjunction with (7.3), yields

$$\sum_{k=0}^{n} \binom{n}{k}^4 \{H_k^{(2)} + (n - 2k)\{H_k^{(3)} + 2H_k^{(2)}(H_k - H_{n-k})\}\} = (-1)^n \binom{2n}{n} H_n^{(2)}. \tag{7.5}$$

Note that (7.5) is equivalent to an identity given by Chu and Fu [12, Example 29]. Furthermore, subtracting (7.5) from (7.3) yields

$$\sum_{k=0}^{n} \binom{n}{k}^4 \left\{ 4H_k^2 - \frac{1}{2}H_k^{(2)} + \frac{1}{6}(n - 2k)\{32H_k^3 + 6H_k^{(2)}(H_k + 3H_{n-k}) - 5H_k^{(3)}\} \right\}$$

$$= (-1)^n \binom{2n}{n} \{(H_{2n} - 3H_n)^2 - H_{2n}^{(2)}\}. \tag{7.6}$$

Example 7.2 Setting $l = 3$ and $\eta = 0$ leads us to a result of Chu and Fu [12, Theorem 5]. Thus, by specifying further the parameters λ, θ and ε, we can obtain Examples 26–30 in [12]. More

identities can be obtained from the $l = 3$ and $\eta = 0$ case. For example, let $\theta = \lambda$ and $\varepsilon = 0$; then we have

$$\sum_{k=0}^{n} \binom{n}{k}^4 \left\{ 2H_k(H_k - H_{n-k}) + H_k^{(2)} + \frac{2}{3}(n - 2k)\{2H_k^2(H_k - 3H_{n-k}) \right.$$
$$\left. + 3H_k^{(2)}(H_k - H_{n-k}) + H_k^{(3)}\} \right\} = (-1)^n \frac{1}{2} \binom{2n}{n} H_n^{(2)}. \tag{7.7}$$

Subtracting (7.5) from (7.7) gives

$$\sum_{k=0}^{n} \binom{n}{k}^4 \left\{ 2H_k(H_k - H_{n-k}) + \frac{1}{3}(n - 2k)\{4H_k^2(H_k - 3H_{n-k}) - H_k^{(3)}\} \right\}$$
$$= (-1)^{n+1} \frac{1}{2} \binom{2n}{n} H_n^{(2)}. \tag{7.8}$$

Adding (7.8) to (7.7) yields

$$\sum_{k=0}^{n} \binom{n}{k}^4 \left\{ 4H_k(H_k - H_{n-k}) + H_k^{(2)} + \frac{1}{3}(n - 2k)\{8H_k^2(H_k - 3H_{n-k}) \right.$$
$$\left. + 6H_k^{(2)}(H_k - H_{n-k}) + H_k^{(3)}\} \right\} = 0. \tag{7.9}$$

The last two identities are equivalent to Examples 28 and 30 in [12], respectively.

7.3. $l = 4, 5$

As examples, we give one identity for $l = 4$ case and one for $l = 5$ case, respectively.

Example 7.3 ($l = 4, \theta = \varepsilon = \eta = \lambda$)   Now, Theorem 6.1 gives

$$\sum_{k=0}^{\infty} \frac{n + 2k + 2}{(k + 1)_{n+1}^4} = -\frac{1}{3n!^4} \sum_{k=0}^{n} \binom{n}{k}^4 \{4(8H_k^3 + 6H_k H_k^{(2)} + H_k^{(3)}) + (n - 2k)$$
$$\times \{16H_k(2H_k^3 + H_k^{(3)}) + 6H_k^{(2)}(8H_k^2 + H_k^{(2)}) + 3H_k^{(4)}\}\}$$
$$+ \frac{(-1)^{n+1}2}{3n!^4} \binom{2n}{n} \{2(H_{2n} - 3H_n)^3 - 3(H_{2n} - 3H_n)(2H_{2n}^{(2)} - 3H_n^{(2)})$$
$$+ 4H_{2n}^{(3)} - 3H_n^{(3)} - 3\zeta(3)\},$$

from which we can evaluate the infinite series

$$\sum_{k=0}^{\infty} \frac{n + 2k + 2}{(k + 1)_{n+1}^4} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{n + 2k + 2}{(n + k + 1)^4 \binom{n+k}{k}^4}.$$

For example, the next four series hold

$$\sum_{k=0}^{\infty} \frac{2k + 3}{(k + 1)_2^4} = 5 - 4\zeta(3), \quad \sum_{k=0}^{\infty} \frac{2k + 4}{(k + 1)_3^4} = -\frac{115}{128} + \frac{3}{4}\zeta(3),$$

$$\sum_{k=0}^{\infty} \frac{2k + 5}{(k + 1)_4^4} = \frac{5195}{139968} - \frac{5}{162}\zeta(3), \quad \sum_{k=0}^{\infty} \frac{2k + 6}{(k + 1)_5^4} = -\frac{290785}{573308928} + \frac{35}{82944}\zeta(3).$$

Example 7.4 ($l = 5, \theta = \varepsilon = \eta = \lambda$)   Now, we have

$$\sum_{k=0}^{\infty} \frac{1 + 4(n + 2k + 2)H_k}{(k+1)_{n+1}^4} = -\frac{1}{30n!^4} \sum_{k=0}^{n} \binom{n}{k}^4 \{10\{16H_k(2H_k^3 + H_k^{(3)}) + 6H_k^{(2)}(8H_k^2 + H_k^{(2)})$$

$$+ 3H_k^{(4)}\} + 8(n - 2k)\{32H_k^5 + 80H_k^3 H_k^{(2)} + 40H_k^2 H_k^{(3)}$$

$$+ 30H_k(H_k^{(2)})^2 + 15H_k H_k^{(4)} + 10H_k^{(2)} H_k^{(3)} + 3H_k^{(5)}\}\}$$

$$+ \frac{(-1)^n}{6n!^4} \binom{2n}{n} \{4(H_{2n} - 3H_n)^4 - 12(H_{2n} - 3H_n)^2$$

$$\times (2H_{2n}^{(2)} - 3H_n^{(2)}) + 8(H_{2n} - 3H_n)(4H_{2n}^{(3)} - 3H_n^{(3)}$$

$$- 3\zeta(3)) + 3(2H_{2n}^{(2)} - 3H_n^{(2)})^2 + 3(-8H_{2n}^{(4)} + 3H_n^{(4)} + 6\zeta(4))\},$$

which gives the explicit expressions for the infinite series

$$\sum_{k=0}^{\infty} \frac{1 + 4(n + 2k + 2)H_k}{(k+1)_{n+1}^4} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1 + 4(n + 2k + 2)H_k}{(n + k + 1)^4 \binom{n+k}{k}^4}.$$

Moreover, in view of the values of the series $\sum_{k=0}^{\infty} 1/(k+1)_{n+1}^4$, we have

$$\sum_{k=0}^{\infty} \frac{H_k}{(k+1)^3} = \frac{\pi^4}{360},$$

$$\sum_{k=0}^{\infty} \frac{(2k+3)H_k}{(k+1)_2^4} = 14 - \frac{5\pi^2}{6} - \frac{\pi^4}{45} - 3\zeta(3),$$

$$\sum_{k=0}^{\infty} \frac{(2k+4)H_k}{(k+1)_3^4} = \frac{103}{64} - \frac{35\pi^2}{128} + \frac{29}{32}\zeta(3),$$

$$\sum_{k=0}^{\infty} \frac{(2k+5)H_k}{(k+1)_4^4} = \frac{766349}{1679616} - \frac{5005\pi^2}{139968} - \frac{7\pi^4}{14580} - \frac{61}{1296}\zeta(3).$$

It can be found that the last three series cannot be evaluated by the function 'Sum' of Mathematica 7.

Acknowledgements

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