Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 623456, 21 pages
doi:10.1155/2011/623456

Research Article

On Generalized Bell Polynomials

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Received 18 June 2011; Accepted 25 July 2011

Academic Editor: Garyfalos Papaschinopoulos

It is shown that the sequence of the generalized Bell polynomials $S_n(x)$ is convex under some restrictions of the parameters involved. A kind of recurrence relation for $S_n(x)$ is established, and some numbers related to the generalized Bell numbers and their properties are investigated.

1. Introduction

Hsu and Shiue [1] defined a kind of generalized Stirling number pair with three free parameters which is introduced via a pair of linear transformations between generalized factorials, viz,

$$\begin{aligned}(t \mid \alpha)_n &= \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t-\gamma \mid \beta)_k,\\ (t \mid \beta)_n &= \sum_{k=0}^{n} S(n,k;\beta,\alpha,-\gamma)(t+\gamma \mid \alpha)_k,\end{aligned} \tag{1.1}$$

where $n \in N$ (set of nonnegative integers), α, β, and γ may be real or complex numbers with $(\alpha,\beta,\gamma) \neq (0,0,0)$, and $(t \mid \alpha)_n$ denotes the generalized factorial of the form

$$(t \mid \alpha)_n = \prod_{j=0}^{n-1}(t-j\alpha), \quad n \geq 1, \ (t \mid \alpha)_0 = 1. \tag{1.2}$$

In particular, $(t \mid 1)_n = (t)_n$ with $(t)_0 = 1$. Various well-known generalizations were obtained by special choices of the parameters α,β, and γ (cf. [1]), and the generalization of some properties of the classical Stirling numbers such as the recurrence relations

$$S(n+1,k;\alpha,\beta,\gamma) = S(n,k-1;\alpha,\beta,\gamma) + (k\beta - n\alpha + \gamma)S(n,k;\alpha,\beta,\gamma), \tag{1.3}$$

the exponential generating function

$$(1+\alpha t)^{\gamma/\alpha}\left[\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right]^k = k!\sum_{n\geq 0} S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!}, \quad (1.4)$$

the explicit formula

$$S(n,k;\alpha,\beta,\gamma) = \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+\gamma \mid \alpha)_n, \quad (1.5)$$

the congruence relation, and a kind of asymptotic expansion was established. As a follow-up study of these numbers, more properties were obtained in [2]. Furthermore, some combinatorial interpretations of $S(n,k;\alpha,\beta,\gamma)$ were given in [3] in terms of occupancy distribution and drawing of balls from an urn.

Hsu and Shiue [1] also defined a kind of generalized exponential polynomials $S_n(x) \equiv S_n(x;\alpha,\beta,\gamma)$ in terms of generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$ with $\alpha,\beta,$ and γ real or complex numbers as follows:

$$S_n(x) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)x^k. \quad (1.6)$$

We may call these polynomials *generalized Bell polynomials*. Note that when $x=1$, we get

$$W_n = S_n(1) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma), \quad (1.7)$$

the *generalized Bell numbers*. A kind of generating function of the sequence $\{S_n(x)\}$ for the generalized exponential polynomials has been established by Hsu and Shiue, viz,

$$\sum_{n\geq 0} S_n(x)\frac{t^n}{n!} = (1+\alpha t)^{\gamma/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right], \quad (1.8)$$

where $\alpha,\beta\neq 0$. In particular, (1.8) gives the generating function for the generalized Bell numbers:

$$\sum_{n\geq 0} W_n\frac{t^n}{n!} = (1+\alpha t)^{\gamma/\alpha}\exp\left[\frac{\left((1+\alpha t)^{\beta/\alpha}-1\right)}{\beta}\right]. \quad (1.9)$$

Note that, when $\alpha \to 0$, $(1+\alpha t)^{\gamma/\alpha} \to \exp(\gamma t)$. Hence,

$$(1+\alpha t)^{\gamma/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right] \longrightarrow e^{\gamma t}\exp\left[\left(e^{\beta t}-1\right)\frac{x}{\beta}\right]. \quad (1.10)$$

If we define the polynomial $G_{n,\beta,r}(x)$ as

$$G_{n,\beta,r}(x) = \lim_{\alpha \to 0} S_n(x; \alpha, \beta, r), \tag{1.11}$$

then its exponential generating function is given by

$$\sum_{n\geq 0} G_{n,\beta,r}(x)\frac{t^n}{n!} = \exp\left[rt + \left(e^{\beta t} - 1\right)\frac{x}{\beta}\right]. \tag{1.12}$$

We may call $G_{n,\beta,r}(x)$ the (r,β)-*Bell polynomial*. Hence, with $x = 1$, this yields the exponential generating function for the (r,β)-Bell numbers. Now, if we use $S(n,k;\beta,\gamma)$ to denote the following limit:

$$S(n,k;\beta,\gamma) = \lim_{\alpha \to 0} S(n,k;\alpha,\beta,\gamma), \tag{1.13}$$

then, by (1.5),

$$S(n,k;\beta,\gamma) = \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + \gamma)^n, \tag{1.14}$$

$$G_{n,\beta,r}(x) = \sum_{k=0}^{n} S(n,k;\beta,\gamma)x^k. \tag{1.15}$$

Also obtained by Hsu and Shiue is an explicit formula for $S_n(x)$ of the form

$$S_n(x) = \left(\frac{1}{e}\right)^{x/\beta}\sum_{k=0}^{\infty}\frac{(x/\beta)^k}{k!}(k\beta + \gamma \mid \alpha)_n. \tag{1.16}$$

Consequently, with $x = 1$, we have

$$W_n = \left(\frac{1}{e}\right)^{1/\beta}\sum_{k=0}^{\infty}\frac{(k\beta + \gamma \mid \alpha)_n}{\beta^k k!}. \tag{1.17}$$

Note that, by taking $\alpha = 0$, (1.16) gives

$$G_{n,\beta,r}(x) = \left(\frac{1}{e}\right)^{x/\beta}\sum_{k=0}^{\infty}\frac{(x/\beta)^k}{k!}(k\beta + \gamma)^n, \tag{1.18}$$

the explicit formula for (r,β)-Bell polynomial. When $x = 1$, this gives

$$G_{n,\beta,r} = \left(\frac{1}{e}\right)^{1/\beta}\sum_{k=0}^{\infty}\frac{(1/\beta)^k}{k!}(k\beta + \gamma)^n, \tag{1.19}$$

a kind of the Dobinski formula for (r, β)-Bell numbers. This reduces further to the Dobinski formula for r-Bell numbers [4] when $\beta = 1$. Moreover, with $\gamma = 0$, we get

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k}{k!}, \tag{1.20}$$

which is the Dobinski formula for the ordinary Bell numbers [5].

In this paper, a recurrence relation and convexity of the generalized Bell numbers will be established and some numbers related to W_n will be investigated. Some theorems on (r, β)-Bell polynomials will be established including the asymptotic approximation of the (r, β)-Bell numbers.

2. More Properties of $S_n(x)$

Recurrence relation is one of the useful tools in constructing tables of values. The recurrence relation for the ordinary Bell numbers [6] is given by

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}, \tag{2.1}$$

with initial condition $B_0 = 1$. Carlitz's Bell numbers [7] also satisfy the recurrence relation:

$$A_{n+1}(\lambda) = -\lambda n A_n(\lambda) + \sum_{k=0}^{n} k! \binom{n}{k} \binom{\mu}{k} \lambda^k A_{n-k}(\lambda), \quad \mu = \frac{1}{\lambda}, \tag{2.2}$$

with $A_0(\lambda) = 1$. Note that for $\lambda = 1$, $A_n(1) = B_n$ and (2.2) will reduce to (2.1). Moreover, Mező [4] obtained certain recurrence relations for the r-Bell polynomials, respectively, as

$$B_{n,r}(x) = r B_{n-1,r}(x) + x \sum_{k=0}^{n-1} \binom{n-1}{k} B_{k,r}(x). \tag{2.3}$$

The following theorem will generalize all of these recurrence relations.

Theorem 2.1. *The generalized exponential polynomials satisfy the following recurrence relation:*

$$S_{n+1}(x) = (\gamma - \alpha n) S_n(x) + \sum_{k=0}^{n} x \binom{n}{k} (\beta \mid \alpha)_k S_{n-k}(x) \tag{2.4}$$

with $S_0(x) = 1$. Moreover, the generalized Bell numbers $W_n = S_n(1)$ satisfy

$$W_{n+1} = (\gamma - \alpha n) W_n + \sum_{k=0}^{n} \binom{n}{k} (\beta \mid \alpha)_k W_{n-k}. \tag{2.5}$$

Proof. Differentiating both sides of (1.8) with respect to t will give

$$\sum_{n\geq 0} S_n(x)\frac{t^{n-1}}{(n-1)!} = (1+\alpha t)^{\gamma/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right]\left(\frac{(1+\alpha t)^{\beta/\alpha}x+\gamma}{1+\alpha t}\right). \tag{2.6}$$

Applying binomial theorem and Cauchy's rule for product of two power series will yield

$$\begin{gathered}(1+\alpha t)\sum_{n\geq 0} S_n(x)\frac{t^{n-1}}{(n-1)!} = \left(\sum_{n\geq 0} S_n(x)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0}\binom{\frac{\beta}{\alpha}}{n}x\alpha^n t^n+\gamma\right), \\ \sum_{n\geq 0} S_n(x)\frac{t^{n-1}}{(n-1)!}+\sum_{n\geq 0} n\alpha S_n(x)\frac{t^n}{n!} = \sum_{n\geq 0}\left(\sum_{k=0}^{n} xk!\binom{n}{k}\binom{\frac{\beta}{\alpha}}{k}\alpha^k S_{n-k}(x)\right)\frac{t^n}{n!}.\end{gathered} \tag{2.7}$$

Comparing the coefficients of $t^n/n!$, we obtain

$$S_{n+1}(x)+\alpha n S_n(x) = \gamma S_n(x)+\sum_{k=0}^{n} xk!\binom{n}{k}\binom{\frac{\beta}{\alpha}}{k}\alpha^k S_{n-k}(x), \tag{2.8}$$

which is precisely equivalent to (1.10). □

By taking $\alpha = 0$, Theorem 2.1 yields the recurrence relations for the (r,β)-Bell polynomials. More precisely,

$$G_{n+1,\beta,r}(x) = rG_{n,\beta,r}(x)+\sum_{k=0}^{n} x\binom{n}{k}\beta^k G_{n-k,\beta,r}(x). \tag{2.9}$$

These further give (2.3) when $\beta = 1$. Surely, (2.2) can be deduced from (2.5) by letting $(\alpha,\beta,\gamma) = (\lambda,1,0)$. Furthermore, for $(\alpha,\beta,\gamma) = (0,0,1)$, (2.4) gives

$$\overline{B}_{n+1}(x) = 2\overline{B}_n(x), \tag{2.10}$$

where $\overline{B}_n(x) = \sum_{k=0}^{n}\binom{n}{k}x^k$. If we let $\overline{B}_n = \overline{B}_n(1)$, we get

$$\overline{B}_{n+1} = 2\overline{B}_n, \tag{2.11}$$

which implies

$$\sum_{k=0}^{n}\binom{n+1}{k} = 2^{n+1}-1, \tag{2.12}$$

the number of distinct partitions of an $(n+2)$-set into 2 nonempty subsets, or simply $S(n+2,2)$, the classical Stirling number of the second kind.

Mathematicians have been aware for quite a while that the global behaviour of combinatorial sequences can be used in asymptotic estimates. One of these interesting behaviours is convexity [5]. A real sequence $v_k, k = 0,1,2,\ldots$ is called *convex* on an interval $[a,b]$ (containing at least 3 consecutive integers) when

$$v_k \leq \frac{1}{2}(v_{k-1} + v_{k+1}), \quad k \in [a+1, b-1]. \tag{2.13}$$

For instance, the sequence of binomial coefficients $\binom{n}{k}$ satisfies the convexity property since

$$\binom{n+2}{k} - 2\binom{n+1}{k} + \binom{n}{k} = \binom{n}{k-2} > 0, \quad \text{for } k \geq 2. \tag{2.14}$$

This implies that

$$\overline{B}_{n+1} \leq \frac{1}{2}\left(\overline{B}_n + \overline{B}_{n+2}\right), \tag{2.15}$$

that is, $\overline{B}_n$ is convex.

The next theorem asserts that the sequence of generalized exponential polynomials as well as the generalized Bell numbers is convex under some restrictions.

Theorem 2.2. *The sequence of generalized exponential polynomials $S_n(x)$ with $x > 0$, $\alpha \leq 0$, and $\beta, \gamma \geq 0$ possesses the convexity property, viz,*

$$S_{n+1}(x) \leq \frac{1}{2}(S_n(x) + S_{n+2}(x)), \quad n = 1,2,\ldots. \tag{2.16}$$

Proof. Since $\alpha \leq 0$ and $(k\beta + \gamma - n\alpha) \geq 0$, we have

$$\begin{gathered} 0 \leq \left[1 - (k\beta + \gamma - n\alpha)\right]^2 - \alpha(k\beta + \gamma - n\alpha), \\ 0 \leq 1 - 2(k\beta + \gamma - n\alpha) + (k\beta + \gamma - n\alpha)^2 - \alpha(k\beta + \gamma - n\alpha), \\ 2(k\beta + \gamma - n\alpha) \leq 1 + (k\beta + \gamma - n\alpha)(k\beta + \gamma - n\alpha - \alpha). \end{gathered} \tag{2.17}$$

Multiplying both sides by $(k\beta + \gamma \mid \alpha)_n$, we get

$$2(k\beta + \gamma \mid \alpha)_{n+1} \leq (k\beta + \gamma \mid \alpha)_n + (k\beta + \gamma \mid \alpha)_{n+2}. \tag{2.18}$$

Thus, making use of (1.16), we obtain (2.16). □

Note that, for $(\alpha, \beta, \gamma, x) = (0, \beta, r, 1)$, (2.16) asserts the convexity of (r,β)-Bell polynomials which further imply the convexity of r-Bell polynomials when $\beta = 1$. Moreover, letting $(\alpha, \beta, \gamma, x) = (0,1,0,1)$, (2.16) yields (2.15) and implies the convexity of $\overline{B}_n$.

3. A Variation of Generalized Bell Numbers

Let us denote $\overline{A}(n,k;\alpha,\beta,\gamma) = k!\beta^k S(n,k;\alpha,\beta,\gamma)$ and define

$$B_n(\alpha,\beta,\gamma) = \sum_{k=1}^{n} \overline{A}(n,k;\alpha,\beta,\gamma). \tag{3.1}$$

The numbers $\overline{A}(n,k;\alpha,\beta,\gamma)$ were given combinatorial interpretation in [2], for nonnegative integers $\alpha, \beta,$ and γ, as the number of ways to distribute n distinct balls, one ball at a time, into $k+1$ distinct cells, first k of which has β distinct compartments and the last cell with γ distinct compartments such that

(i) the compartments in each cell are given cyclic ordered numbering,

(ii) the capacity of each compartment is limited to one ball,

(iii) each successive α available compartments in a cell can only have the leading compartment getting the ball,

(iv) the first k cells are nonempty.

Illustration of (iii)

Suppose the first ball lands in compartment 3 of cell 2. The compartment numbered 4, 5, 6,…, $\alpha, \alpha+1, \alpha+2$ will be closed. And suppose the second ball lands in compartment $\beta-2$ also of cell 2. Then compartments numbered $\beta-1, \beta, 1, 2, \alpha+3, \alpha+4, \alpha+5, \ldots, 2\alpha-3$ of cell 2 will be closed.

If $k+1$ cells will be changed to any number of cells with the last cell containing γ distinct compartments and the rest of the cells each has β distinct compartments such that only the last cell could be empty, then this gives the combinatorial interpretation of $B_n(\alpha,\beta,\gamma)$.

The following theorem contains a kind of exponential generating function for $B_n(\alpha,\beta,\gamma)$.

Theorem 3.1. *The numbers $B_n(\alpha,\beta,\gamma)$ have the following exponential generating function:*

$$\sum_{n\geq 0} B_n(\alpha,\beta,\gamma)\frac{t^n}{n!} = \frac{(1+\alpha t)^{\gamma/\alpha}}{2-(1+\alpha t)^{\beta/\alpha}}. \tag{3.2}$$

Proof. Using the exponential generating function in (1.4), we get

$$\begin{aligned}\sum_{n\geq 0} B_n(\alpha,\beta,\gamma)\frac{t^n}{n!} &= \sum_{n\geq 0}\sum_{k\geq 0}\beta^k k! S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!} \\ &= (1+\alpha t)^{\gamma/\alpha}\sum_{k\geq 0}\left[(1+\alpha t)^{\beta/\alpha}-1\right]^k \\ &= (1+\alpha t)^{\gamma/\alpha}\frac{1}{1-\left[(1+\alpha t)^{\beta/\alpha}-1\right]}.\end{aligned} \tag{3.3}$$

This is exactly the desired generating function. □

Differentiating both sides of (1.9) with respect to t, we yield

$$\overline{A}(n,k;\alpha,\beta,\gamma) = \frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\left((1+\alpha t)^{\beta/\alpha}-1\right)^k\right]_{t=0}. \tag{3.4}$$

Since $\overline{A}(n,k;\alpha,\beta,\gamma)$ vanishes when $k=0$ and $k>n$, we have

$$\begin{aligned}
B_n(\alpha,\beta,\gamma) &= \sum_{k=0}^{\infty}\frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\left((1+\alpha t)^{\beta/\alpha}-1\right)^k\right]_{t=0} \\
&= \frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\left(2-(1+\alpha t)^{\beta/\alpha}\right)^{-1}\right]_{t=0} \\
&= \frac{1}{2}\frac{d^n}{dt^n}\left[(1+\alpha t)^{\gamma/\alpha}\sum_{\nu=0}^{\infty}\left(\frac{1}{2}(1+\alpha t)^{\beta/\alpha}\right)^{\nu}\right]_{t=0} \\
&= \frac{1}{2}\sum_{\nu=0}^{\infty}\frac{d^n}{dt^n}\left[(1+\alpha t)^{(\gamma+\beta\nu)/\alpha}\right]_{t=0}\frac{1}{2^{\nu}}.
\end{aligned} \tag{3.5}$$

This result is embodied in the following theorem.

Theorem 3.2. *The number $B_n(\alpha,\beta,\gamma)$ is equal to*

$$B_n(\alpha,\beta,\gamma) = \frac{1}{2}\sum_{\nu=0}^{\infty}(\gamma+\beta\nu\mid\alpha)_n 2^{-\nu}, \quad n \geq 1. \tag{3.6}$$

The next theorem provides a recurrence relation for the number $B_n(\alpha,\beta,\gamma)$ which can be used as a quick tool in computing its first values.

Theorem 3.3. *The following recurrence relation holds:*

$$B_n(\alpha,\beta,\gamma) = (\gamma\mid\alpha)_n + (\beta\mid\alpha)_n + \sum_{j=1}^{n-1}\binom{n}{j}(\beta\mid\alpha)_j B_{n-j}(\alpha,\beta,\gamma), \tag{3.7}$$

where $n \geq 1$.

Proof. Making use of (3.6), we have

$$\binom{\frac{\beta}{\alpha}}{j}\frac{B_{n-j}(\alpha,\beta,\gamma)}{\alpha^{n-j}(n-j)!} = \frac{1}{2}\sum_{\nu=0}^{\infty}\binom{\frac{\beta}{\alpha}}{j}\binom{\frac{\beta\nu+\gamma}{\alpha}}{n-j}2^{-\nu}. \tag{3.8}$$

Summing up both sides from $j = 0$ to $n - 1$ and using Vandermonde's formula, we get

$$\begin{aligned}\sum_{j=0}^{n-1}\binom{n}{j}(\beta \mid \alpha)_j \frac{B_{n-j}(\alpha,\beta,\gamma)}{\alpha^n n!} &= \frac{1}{2}\sum_{\nu=0}^{\infty}\left(\sum_{j=0}^{n-1}\binom{\frac{\beta}{\alpha}}{j}\binom{\frac{\beta\nu+\gamma}{\alpha}}{n-j}\right)2^{-\nu} \\ &= \frac{1}{2}\sum_{\nu=0}^{\infty}\binom{\frac{\beta+\nu\beta+\gamma}{\alpha}}{n}2^{-\nu} - \frac{1}{2}\sum_{\nu=0}^{\infty}\binom{\frac{\beta}{\alpha}}{n}2^{-\nu}.\end{aligned} \tag{3.9}$$

Hence, we have

$$\sum_{j=0}^{n-1}\binom{n}{j}(\beta \mid \alpha)_j B_{n-j}(\alpha,\beta,\gamma) = \frac{1}{2}\sum_{\nu=0}^{\infty}((\nu+1)\beta+\gamma \mid \alpha)_n 2^{-\nu} - (\beta \mid \alpha)_n \frac{1}{2}\sum_{\nu=0}^{\infty}2^{-\nu}. \tag{3.10}$$

Now, by (3.6),

$$\begin{aligned}\frac{1}{2}\sum_{\nu=0}^{\infty}(\beta(\nu+1)+\gamma \mid \alpha)_n 2^{-\nu} &= \sum_{\nu=0}^{\infty}(\beta(\nu+1)+\gamma \mid \alpha)_n 2^{-(\nu+1)} \\ &= \sum_{\nu=0}^{\infty}(\beta\nu+\gamma \mid \alpha)_n 2^{-\nu} - (\gamma \mid \alpha)_n \\ &= 2B_n(\alpha,\beta,\gamma) - (\gamma \mid \alpha)_n\end{aligned} \tag{3.11}$$

and $(1/2)\sum_{\nu=0}^{\infty} 2^{-\nu} = 1$. Thus,

$$\sum_{j=1}^{n-1}\binom{n}{j}(\beta \mid \alpha)_j B_{n-j}(\alpha,\beta,\gamma) = B_n(\alpha,\beta,\gamma) - (\gamma \mid \alpha)_n - (\beta \mid \alpha)_n \tag{3.12}$$

which is precisely equivalent to (3.7). □

Note that when $n = 1$, (3.7) gives

$$B_1(\alpha,\beta,\gamma) = \gamma + \beta, \tag{3.13}$$

while (3.6) gives

$$B_1(\alpha,\beta,\gamma) = \gamma + \beta\left(\sum_{\nu=1}^{\infty}\frac{\nu}{2^{\nu+1}}\right). \tag{3.14}$$

This implies that

$$\sum_{\nu=1}^{\infty}\frac{\nu}{2^{\nu+1}} = 1. \tag{3.15}$$

The following theorem gives a kind of congruence relation for $B_n(\alpha,\beta,\gamma)$ with the restriction that $\alpha \to 0$. We use $\widehat{G}_{n,\beta,r}$ to denote the following limit:

$$\widehat{G}_{n,\beta,r} = \lim_{\alpha\to 0} B_n(\alpha,\beta,\gamma). \tag{3.16}$$

Theorem 3.4. *Let r and β be integers. Then for any odd prime p and $n \geq 1$, one has the following congruence relation:*

$$\widehat{G}_{n+p-1,\beta,r} - \widehat{G}_{n,\beta,r} \equiv 0 \pmod{2p}. \tag{3.17}$$

Proof. Note that the explicit formula in (1.14) can be expressed in terms of a kth difference operator. That is,

$$\left[\Delta^k(\beta t + r)^n\right]_{t=0} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j + r)^n, \tag{3.18}$$

where Δ^k denotes the kth difference operator. Hence,

$$\widehat{G}_{n+p-1,\beta,r} = \sum_{k=0}^{\infty}\left[\Delta^k(\beta t + r)^{n+p-1}\right]_{t=0}. \tag{3.19}$$

Thus,

$$\widehat{G}_{n+p-1,\beta,r} - \widehat{G}_{n,\beta,r} = \sum_{k=0}^{\infty}\Delta^k\left\{(\beta t + r)^{n-1}\left[(\beta t + r)^p - (\beta t + r)\right]\right\}_{t=0}. \tag{3.20}$$

Since, by Fermat's little theorem, $(\beta t + r)^p - (\beta t + r)$ is divisible by p,

$$(\beta t + r)^{n-1}\left[(\beta t + r)^p - (\beta t + r)\right] = px, \tag{3.21}$$

for some integer x. Also, since $(\beta t + r)^n$ and $(\beta t + r)^{p-1} - 1$ are of different parity, $(\beta t + r)^n[(\beta t + r)^{p-1} - 1]$ is divisible by 2. Hence,

$$(\beta t + r)^n\left[(\beta t + r)^{p-1} - 1\right] = 2py, \tag{3.22}$$

for some integer y. Thus, we have

$$(\beta t + r)^n\left[(\beta t + r)^{p-1} - 1\right] \equiv 0 \pmod{2p}. \tag{3.23}$$

This completes the proof of the theorem. □

4. Some Theorems on (r,β)-Bell Polynomials

The (r,β)-Bell polynomials $G_{n,\beta,r}(x)$ have already possessed numerous properties. Some of them are obtained as special case of the properties of $S_n(x)$. However, there are properties of the ordinary Bell numbers or r-Bell numbers which are difficult to establish in $S_n(x)$ but can be done in $G_{n,\beta,r}(x)$. For instance, using the rational generating function for $S(n,k;\beta,r)$ in [2] which is given by

$$\sum_{n\geq k} S(n,k;\beta,r)t^n = \frac{t^k}{\prod_{j=0}^{k}[1-(\beta j+r)t]}, \tag{4.1}$$

we can have

$$\begin{aligned}\sum_{n\geq 0} S(n,k;\beta,r)t^n &= \frac{1}{\beta^{k+1}t}\frac{1}{\prod_{i=0}^{k}((1-rt)/(\beta t)-i)}\\ &= \frac{1}{\beta^{k+1}t}\frac{1}{((1-rt)/(\beta t))\prod_{i=1}^{k}((1-rt)/(\beta t)-i)}\\ &= \frac{-1}{\beta^{k}(rt-1)}\frac{(-1)^k}{\prod_{i=1}^{k}((rt-1)/(\beta t)+i)}.\end{aligned} \tag{4.2}$$

It can easily be shown that

$$\prod_{i=1}^{k}\left(\frac{rt-1}{\beta t}-i\right) = \left(\frac{(\beta+r)t-1}{\beta t}\right)_k. \tag{4.3}$$

Thus,

$$\begin{aligned}\sum_{k\geq 0}\left(\sum_{n\geq 0} S(n,k;\beta,r)t^n\right)x^k &= \sum_{k\geq 0}\left(\frac{-1}{\beta^k(rt-1)}\frac{(-1)^k}{(((\beta+r)t-1)/\beta t)_k}\right)x^k,\\ \sum_{n\geq 0}\left(\sum_{k=0}^{n} S(n,k;\beta,r)x^k\right)t^n &= \frac{-1}{rt-1}\sum_{k\geq 0}\frac{(1)_k}{(((\beta+r)t-1)/\beta t)_k}\frac{(-x/\beta)^k}{k!}.\end{aligned} \tag{4.4}$$

This can be expressed further as

$$\sum_{n\geq 0} G_{n,\beta,r}(x)t^n = \frac{-1}{rt-1}\cdot {}_1F_1\left(\left.\begin{matrix}1\\ \dfrac{(\beta+r)t-1}{\beta t}\end{matrix}\right| \frac{-x}{\beta}\right), \tag{4.5}$$

where ${}_1F_1$ is the hypergeometric function which is defined by

$${}_pF_q\left(\left.\begin{matrix}a_1, & a_2, & \ldots, & a_p\\ b_1, & b_2, & \ldots, & b_q\end{matrix}\right| t\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{t^k}{k!}, \tag{4.6}$$

where $(a_i)_j = a_i(a_i+1)(a_i+2)\cdots(a_i+j-1)$. Applying Kummer's formula [8],

$$e^x {}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| -x\right) = {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix} \middle| x\right), \tag{4.7}$$

we obtain the following generating function.

Theorem 4.1. *The (r,β)-Bell polynomials satisfy the following generating function:*

$$\sum_{n\geq 0} G_{n,\beta,r}(x)t^n = \frac{-1}{rt-1}\cdot\frac{1}{e^{x/\beta}}\cdot {}_1F_1\left(\begin{matrix} \dfrac{rt-1}{\beta t} \\ \dfrac{\beta t + rt - 1}{\beta t} \end{matrix} \middle| \frac{x}{\beta}\right). \tag{4.8}$$

It will be interesting if one can also obtain a generating function of this form for $S_n(x)$.

Now, using the integral identity in [9],

$$\operatorname{Im}\int_0^{\pi} e^{je^{i\theta}}\sin(n\theta)d\theta = \frac{\pi}{2}\frac{j^n}{n!}, \tag{4.9}$$

and the explicit formula in (1.14), we get

$$\begin{aligned}
\frac{\pi}{2n!}S(n,k;\beta,r) &= \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\operatorname{Im}\int_0^{\pi} e^{(\beta j+r)e^{i\theta}}\sin(n\theta)d\theta \\
&= \frac{1}{\beta^k k!}\operatorname{Im}\int_0^{\pi}\left[\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(e^{\beta e^{i\theta}}\right)^j\right]e^{re^{i\theta}}\sin(n\theta)d\theta \\
&= \operatorname{Im}\int_0^{\pi}\frac{\left[\left(e^{\beta e^{i\theta}}-1\right)/\beta\right]^k}{k!}e^{re^{i\theta}}\sin(n\theta)d\theta.
\end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty}S(n,k;\beta,r)x^k &= \frac{2n!}{\pi}\operatorname{Im}\int_0^{\pi}\left\{\sum_{k=0}^{\infty}\frac{\left[\left(e^{\beta e^{i\theta}}-1\right)/\beta\right]^k}{k!}x^k\right\}e^{re^{i\theta}}\sin(n\theta)d\theta \\
&= \frac{2n!}{\pi}\operatorname{Im}\int_0^{\pi}e^{x(e^{\beta e^{i\theta}}-1)/\beta}e^{re^{i\theta}}\sin(n\theta)d\theta.
\end{aligned} \tag{4.11}$$

Thus,

$$G_{n,\beta,r}(x) = \frac{2n!}{\pi e^{x/\beta}}\operatorname{Im}\int_0^{\pi}e^{x\beta^{-1}e^{\beta e^{i\theta}}}e^{re^{i\theta}}\sin(n\theta)d\theta, \tag{4.12}$$

where $\beta\neq 0$. By simple algebraic manipulation, this can further be expressed as follows.

Theorem 4.2. *The (r,β)-Bell polynomials have the following integral representation:*

$$G_{n,\beta,r}(x) = \frac{2n!}{\pi e^{x/\beta}} \int_0^{\pi} e^{J_1(\theta)} \sin(J_2(\theta)) \sin(n\theta) d\theta, \tag{4.13}$$

where

$$\begin{aligned} J_1(\theta) &= r\cos\theta + \frac{xe^{\beta\cos\theta}\cos(\beta\sin\theta)}{\beta}, \\ J_2(\theta) &= r\sin\theta + \frac{xe^{\beta\cos\theta}\sin(\beta\sin\theta)}{\beta}. \end{aligned} \tag{4.14}$$

It will also be compelling to establish such integral representation for $S_n(x)$.

The Bell polynomials $B_n(\lambda)$ are known to be connected to the Poisson distribution. More precisely, $B_n(\lambda)$ can be expressed in terms of the moment of the Poisson random variable Z with parameter $\lambda > 0$ as

$$B_n(\lambda) = E_\lambda[Z^n]. \tag{4.15}$$

The exponential generating function for the (r,β)-Bell polynomials in (1.12) can be written as follows:

$$\begin{aligned} e^{(r/\beta)\beta t} e^{(x/\beta)(e^{\beta t}-1)} &= e^{(r/\beta)\beta t} E_{x/\beta}\left[e^{(\beta t)Z}\right] \\ &= \sum_{n\geq 0} \left\{ \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k E_{x/\beta}\left[Z^k\right] \right\} \frac{t^n}{n!}. \end{aligned} \tag{4.16}$$

Hence, we can also express the (r,β)-Bell polynomials in terms of the following moment:

$$G_{n,\beta,r}(x) = E_{x/\beta}\left[(\beta Z + r)^n\right]. \tag{4.17}$$

Now,

$$\begin{aligned} G_{n,\beta,r}(x) &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k E_{x/\beta}\left[Z^k\right] \\ &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k B_k\left(\frac{x}{\beta}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \beta^k \sum_{j=0}^{k} S(k,j) \left(\frac{x}{\beta}\right)^j. \end{aligned} \tag{4.18}$$

Thus, we have the following theorem.

Theorem 4.3. *The (r, β)-Bell polynomials equal*

$$G_{n,\beta,r}(x) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} \beta^{k-j} S(k,j) x^j. \quad (4.19)$$

An extension of the Bell polynomials $B_n(y, \lambda)$, defined by Privault [10] as

$$\sum_{n=0}^{\infty} B_n(y, \lambda) \frac{t^n}{n!} = e^{yt - \lambda(e^t - t - 1)}, \quad (4.20)$$

can be expressed in terms of the (r, β)-Bell polynomials as

$$B_n(y, \lambda) = G_{n,1,\lambda+y}(-\lambda). \quad (4.21)$$

Using Theorem 4.3, we obtain

$$B_n(y, -\lambda) = G_{n,1,-\lambda+y}(\lambda) = \sum_{k=0}^{n} \binom{n}{k} (y - \lambda)^{n-k} \sum_{j=0}^{k} S(k,j) \lambda^j. \quad (4.22)$$

This is exactly the identity obtained by Privault in [10].

5. An Asymptotic Approximation for $G_{n,\beta,r}$

Using the exponential generating function for $G_{n,r,\beta}$ in (1.12) with $x = 1$ and Cauchy's theorem for integrals, we obtain the integral representation

$$G_{n,r,\beta} = \frac{n!}{2\pi i} \int_{\gamma} \frac{\exp\left[rz + \left(e^{\beta z - 1}/\beta\right)\right]}{z^{n+1}} dz, \quad (5.1)$$

where γ is the circle $z = Re^{i\theta}, -\pi \le \theta \le \pi$. Contour integration yields

$$G_{n,r,\beta} = \frac{n!}{2\pi i R^n} \int_{-\pi}^{\pi} \exp\left(\beta^{-1} e^{\beta R e^{i\theta}} + rRe^{i\theta} - in\theta - \beta^{-1}\right) d\theta, \quad (5.2)$$

which can be written into the compact form

$$G_{n,r,\beta} = A \int_{-\pi}^{\pi} \exp(F(\theta)) d\theta, \quad (5.3)$$

where

$$A = \frac{n!\exp\left(rR + \beta^{-1}e^{\beta R} - \beta^{-1}\right)}{2\pi R^n},$$
$$F(\theta) = \beta^{-1}e^{\beta Re^{i\theta}} + rRe^{i\theta} - in\theta - \left(rR + \beta^{-1}e^{\beta R}\right). \tag{5.4}$$

Define $\epsilon = e^{-3R/8}$ and let

$$J_1 = \int_{-\pi}^{\epsilon} \exp(F(\theta))d\theta, \qquad J_2 = \int_{\epsilon}^{\pi} \exp(F(\theta))d\theta. \tag{5.5}$$

Thus (5.3) can be written as

$$G_{n,r,\beta} = AJ_1 + A\int_{\epsilon}^{\epsilon} \exp(F(\theta))d\theta + AJ_2. \tag{5.6}$$

Lemma 5.1. *There exists a constant $k > 0$ such that*

$$|J_2| < e^{-k\beta^{-1}e^{\beta R}}(\pi - \epsilon). \tag{5.7}$$

Proof. It can be shown that

$$\left|\exp(F(\theta))\right| = e^{-[(rR+\beta^{-1}e^{\beta R})+\beta^{-1}\cos(\beta R\sin\theta)e^{\beta R\cos\theta}]}. \tag{5.8}$$

Since $\cos\theta < 1$ for $0 < \epsilon < \theta \leq \pi$, we have

$$\left|\exp(F(\theta))\right| = e^{-\beta^{-1}e^{\beta R}}\left[1 - \cos\left(\beta R\sin\theta\right)\right]. \tag{5.9}$$

Since $[1 - \cos(\beta R\sin\theta)] > 0$ for $\cos\theta < 1$ for $0 < \epsilon < \theta \leq \pi$, there exists a constant $k > 0$ such that $[1 - \cos(\beta R\sin\theta)] < k$. Hence

$$|J_2| < e^{-k\beta^{-1}e^{\beta R}}(\pi - \epsilon). \tag{5.10}$$

□

It will be seen later that $R \to \infty$ as $n \to \infty$. With the result in Lemma 5.1 we see that J_1 and J_2 will tend to zero as $n \to \infty$. Hence

$$G_{n,r,\beta} \sim A\int_{-\epsilon}^{\epsilon} \exp(F(\theta))d\theta. \tag{5.11}$$

Observe that $F(\theta)$ is analytic at $\theta = 0$.Thus $F(\theta)$ has a Maclaurin series expansion about $\theta = 0$. This Maclaurin expansion can be written in the form

$$F(\theta) = \left(Re^{\beta R} + rR - n\right)i\theta + \frac{1}{2}\left(\beta R^2 + Re^{\beta R} + rR\right)i^2\theta + \sum_{k=3}^{\infty}\left[\beta^{-1}\rho^k\left(e^{\beta R}\right) + rR\right](i\theta)^k, \tag{5.12}$$

where we define ρ to be the operator $\rho = R(d\theta/dR)$. Choose R such that $Re^{\beta R} + rR - n = 0$; that is, R satisfies the equation $xe^{\beta R} + rx - n = 0$. This R is shown to exist in the following lemma.

Lemma 5.2. *There exists a unique positive real solution to the equation* $xe^{\beta R} + rx - n = 0$*.*

Proof. We can rewrite the given equation in the form

$$\frac{x}{n - rx} = e^{-\beta x}. \tag{5.13}$$

The desired solution is the x-coordinate of the intersection of the functions $h(x) = x/(n - rx)$ and $g(x) = e^{-\beta x}$. □

It can be seen from the preceding lemma that $R \to \infty$ as $n \to \infty$. With this choice of R, we have

$$F(\theta) = -\frac{1}{2}\left(\beta R^2 + Re^{\beta R} + rR\right)\theta + \sum_{k=3}^{\infty}\left[\beta^{-1}\rho^k\left(e^{\beta R}\right) + rR\right](i\theta)^k. \tag{5.14}$$

We now introduce the following notations:

$$\begin{gathered} \phi = \left[(1/2)\left(\beta R^2 e^{\beta R} + Re^{\beta R} + rR\right)^{1/2}\right]\theta, \\ a_k = \frac{[\beta^{-1}e^{-\beta R}\rho^{k+2}(e^{\beta R}) + rRe^{-\beta R}]\,(i\phi)^{k+2}}{(k+1)!\,[1/2(\beta R^2 + R + rRe^{-\beta R})]^{k+2/2}}, \\ z = e^{-\beta R/2}, \\ f(z) = \sum_{k=1}^{\infty} a_k z^k. \end{gathered} \tag{5.15}$$

Then $F(\theta) = -\phi^2 + f(z)$ and

$$G_{n,r,\beta} \sim C\int_{-h}^{h} \exp\left[-\phi^2 + f(z)\right]dz, \tag{5.16}$$

where $h = (1/2)(\beta R^2 e^{\beta R} + Re^{\beta R} + rR)^{1/2}e^{-3R/8}$ and $C = A/[(1/2)(\beta R^2 e^{\beta R} + rR)]^{1/2}$.

We have defined z as a function of R. However, for the moment we consider z to be an independent variable and expand $e^{f(z)}$ into a convergent Maclaurin series expansion of the form

$$e^{f(z)} = \sum_{k=0}^{\infty} b_k z^k, \tag{5.17}$$

where $b_0 = e^{f(0)} = 1$, $b_1 = e^{f(0)} f'(0) = a_1$, and $b_2 = a_2 + (a_1^2/2)$.

Lemma 5.3. *There is a constant R_o such that for all $R > R_o$,*

$$|a_k| < |2\phi|^{k+2}. \tag{5.18}$$

Proof. We see that

$$|a_k| = \frac{R^{k+2}\left[1 + o\left(R^{k+2}\right)\right](2)^{(k+2)/2}}{(k+2)!\left(\beta R^2\right)^{(k+2)/2}\left[1 + o\left(R^2\right)\right]} |\phi|^{k+2} \tag{5.19}$$

which tends to

$$\frac{2^{k+2/2}}{(k+2)!} < 2^{k+2}|\phi|^{k+2} \tag{5.20}$$

as $R \to \infty$. From this, it follows that there is a constant R_o satisfying (5.18). □

Now, it will follow from Lemma 5.3 that the radius of convergence of (5.17) becomes large when θ is near zero. Thus, $z = e^{-\beta R/2}$ is within the domain of convergence.

With $z = e^{-\beta R/2}$,

$$G_{n,r,\beta} \sim C \sum_{k=0}^{s-1} \left(\int_{-h}^{h} e^{-\phi^2} b_k \, d\phi \right) z^k + Q_s, \tag{5.21}$$

where

$$Q_s = \int_{-h}^{h} \left(\sum_{k=s}^{\infty} e^{-\phi^2} b_k z^k \right) d\phi. \tag{5.22}$$

Note that $R \to \infty$ as $n \to \infty$. Furthermore with

$$\begin{aligned} h &= \frac{1}{2}\left(\beta R^2 e^{\beta R} + R e^{\beta R} + rR\right)^{1/2} e^{-3R/8} \\ &= \frac{1}{2}\left(\beta R^2 + R + rRe^{-\beta R}\right)^{1/2} e^{(R(4\beta - 3))/8}, \end{aligned} \tag{5.23}$$

$h \to \infty$ as $R \to \infty$. From these facts and the known asymptotic expansion of the function of the form

$$\int_{-h}^{h} e^{-\phi^2} (\text{polynomial in } |\phi|) \, d\phi, \tag{5.24}$$

the replacement of h by ∞ in (5.16) is easily justified (see [11]). Hence

$$G_{n,r,\beta} \sim C\sum_{k=0}^{s-1}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k \, d\phi\right) z^k + Q_s. \tag{5.25}$$

It remains to show that $Q_s = o(|z|^s)$ as $R \to \infty$, that is, $z \to 0$. From a lemma in [12], $|b_k| \le |2\phi|^{k+2}(1+|2\phi|^2)^{k-1}$. Thus,

$$\left|\sum_{k=s}^{\infty} b_k z^k\right| \le \left[|2\phi|^{s+2}\left(1+|2\phi|^2\right)^{s-1}|z|^s\right]\left[1+\mu+\mu^2+\cdots\right], \tag{5.26}$$

where $\mu = |2\phi|(1+|2\phi|^2)|z|$.

Now, for $\mu < 1$, we have

$$\left|\sum_{k=s}^{\infty} b_k z^k\right| \le \frac{|2\phi|^{s+2}\left(1+|2\phi|^2\right)^{s-1}|z|^s}{1-|z||2\phi|\left(1+|2\phi|^2\right)}. \tag{5.27}$$

Let M and $P_s(|\phi|)|z|^s$ denote the denominator and the numerator, respectively, in (5.27). Since $|\phi| \le h$ and $z = e^{-\beta R/2}$, we have

$$\left|\phi^3\right||z| \le \frac{1}{8}\left(\beta R^2 + R + rRe^{-\beta R}\right)^{3/2} e^{-3R/8} \longrightarrow 0 \text{ as } R \longrightarrow \infty. \tag{5.28}$$

Hence for sufficiently large R, $M \ge 1/2$. Moreover,

$$\int_{-\infty}^{\infty} e^{-\phi^2} P_s(|\phi|)\, d\phi \tag{5.29}$$

exists and tends to zero as $R \to \infty$. Therefore,

$$\frac{|Q_s|}{|z|^s} \le \int_{-\infty}^{\infty} \frac{e^{-\phi^2} P_s(|\phi|)}{M}\, d\phi. \tag{5.30}$$

Thus, $|Q_s| = o(|z|^s)$. Consequently,

$$G_{n,r,\beta} \sim C\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_k \, d\phi\right) e^{(-k\beta R)/2}. \tag{5.31}$$

Since $\int_{-\infty}^{\infty} e^{-x^2} x^n = 0$ for odd n, and b_{2k+1}, as a polynomial in ϕ, contain only odd powers of ϕ, it follows that

$$G_{n,r,\beta} \sim C\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\phi^2} b_{2k} d\phi\right) e^{-k\beta R}. \tag{5.32}$$

Calculation yields

$$a_1 = \frac{\beta R^3 + 3R^2 + \beta^{-1}R + rRe^{-\beta R}}{3!\left[(1/2)\left(\beta R^2 + R + rRe^{-\beta R}\right)\right]^{3/2}}(i\phi)^3,$$
$$a_2 = \frac{\beta R^4 + 6\beta R^3 + 7R^2 + \beta^{-1}R + rRe^{-\beta R}}{4!\left[(1/2)\left(\beta R^2 + R + rRe^{-\beta R}\right)\right]^{2}}(i\phi)^4. \tag{5.33}$$

Taking the first two terms of the asymptotic expansion of (5.32), we have

$$G_{n,r,\beta} \sim C\int_{-\infty}^{\infty} e^{-\phi^2} b_o\, d\phi + Cz^2\int_{-\infty}^{\infty} e^{-\phi^2} b_2 d\phi. \tag{5.34}$$

Since $b_2 = a_2 + a_1^2/2$ and $b_o = 1$,

$$G_{n,r,\beta} \sim C\int_{-\infty}^{\infty} e^{-\phi^2} d\phi + Cz^2\int_{-\infty}^{\infty} a_2 e^{-\phi^2} d\phi + C\frac{z^2}{2}\int_{-\infty}^{\infty} e^{-\phi^2}\left(a_1^2\right)d\phi. \tag{5.35}$$

Let I_1, I_2, and I_3 denote, respectively, the integrals in (5.35). Then evaluating the last two integrals by parts and since $\int_{-\infty}^{\infty} e^{-\phi^2} d\phi = \sqrt{\pi}$, we obtain

$$I_1 = C\sqrt{\pi},$$
$$I_2 = \frac{Ce^{-R}\sqrt{\pi}\left(\beta R^3 + 6\beta R^2 + \beta^{-1} + re^{-\beta R}\right)}{8R\left(\beta R + 1 + re^{-\beta R}\right)^2},$$
$$I_3 = \frac{-5Ce^{-R}\sqrt{\pi}\left(\beta R^2 + 3\beta^{-1}R^2 + re^{-\beta R}\right)^2}{24R\left(\beta R + 1 + re^{-\beta R}\right)^3}. \tag{5.36}$$

Substituting the results in (5.35) and simplifying, we obtain

$$G_{n,r,\beta} \sim C\sqrt{\pi}\left(1 + \frac{D+E}{F}\right), \tag{5.37}$$

where

$$D = \left(3\beta^2R^3 + 8\beta R^3 + 3\beta R + 3 - 10\beta^{-1} - 2re^{-\beta R}\right)re^{-\beta R},$$
$$E = \left(3\beta^3 - 5\beta^2\right)R^4 + \left(21\beta^2 - 30\beta\right)R^3 + (39\beta - 55)R^2 + \left(24 - 30\beta^{-1}\right)R + \left(3\beta^{-1} - 5\beta^{-2}\right),$$
$$F = 24Re^{\beta R}\left(\beta R + 1 + re^{-\beta R}\right)^3. \tag{5.38}$$

Since $Re^{\beta R} = (n - rR)\beta^{-1}$ and $R^n = n^n(\beta e^{\beta R} + r)^{-n}$,

$$C = \frac{n! \exp\left(rR + \beta^{-1}e^{\beta R} - \beta\right)}{\pi\left[n^n\left(\beta e^{\beta R} + r\right)^{-n}\right]\left[2(n - rR)\beta^{-1}\right]^{1/2}\left(\beta R + 1 + re^{-\beta R}\right)^{1/2}}. \tag{5.39}$$

Using Stirling's approximation for $n!$, viz,

$$n! \sim (2\pi)e^{-n}n^{n+(1/2)}\left(1 + \frac{1}{12n}\right), \tag{5.40}$$

we obtain

$$C \sim \frac{n^{1/2}(1 + (1/12n))\exp\left(rR + \beta^{-1}e^{\beta R} - \beta\right)\left(\beta^{\beta R} + r\right)^n}{\pi^{1/2}\left[(n - rR)\beta^{-1}\right]^{1/2}\left(\beta R + 1 + re^{-\beta R}\right)^{1/2}e^n}. \tag{5.41}$$

Using (5.37), we obtain

$$G_{n,r,\beta} \sim \frac{n^{1/2}(1 + (1/12n))\exp\left(rR + \beta^{-1}e^{\beta R} - \beta - n\right)\left(\beta^{\beta R} + r\right)^n}{\left[(n - rR)\beta^{-1}\right]^{1/2}\left(\beta R + 1 + re^{-\beta R}\right)^{1/2}}\left(1 + \frac{D + E}{F}\right). \tag{5.42}$$

Acknowledgments

The authors wish to thank the referee for reading and evaluating the manuscript. They would also like to thank the Office of the President and the Office of the Vice Chancellor for Research and Extension of Mindanao State University-Main Campus for the support extended to this research.

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