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Note

A Characterization of the bell numbers

Martin Aigner

II. Math. Institut, Freie Universität Berlin, Arnimallee 3, D-14195 Berlin, Germany

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Abstract

Let B_n be the Bell numbers, and $\tilde{A}_n\,(n\geqslant 0)$, $\tilde{B}_n\,(n\geqslant 1)$ be the matrices defined by $\tilde{A}_n(i,j) = B_{i+j}\,(0\leqslant i,j\leqslant n)$, $\tilde{B}_n(i,j) = B_{i+j+1}\,(0\leqslant i,j\leqslant n)$. It is shown that (B_n) is the unique sequence of real numbers such that $\det\tilde{A}_n = \det\tilde{B}_n = n!!$ for all n, where $n!! = \prod_{k=0}^{n}(k!)$.

The Bell number B_n counts the number of partitions of an n-set, with the first values $B_0=1$, $B_1=1$, $B_2=2$, $B_3=5$, $B_4=15$, $B_5=52$. The purpose of this note is to provide a characterization of the sequence $(B_0,B_1,B_2,\ldots)$ by means of the determinants of the Hankel matrices

$$\tilde{A}_n = \begin{pmatrix} B_0 & B_1 & \ldots & B_n \\ B_1 & B_2 & \ldots & B_{n+1} \\ & & \ldots & \\ B_n & B_{n+1} & \ldots & B_{2n} \end{pmatrix}, \qquad \tilde{B}_n = \begin{pmatrix} B_1 & B_2 & \ldots & B_{n+1} \\ B_2 & B_3 & \ldots & B_{n+2} \\ & & \ldots & \\ B_{n+1} & B_{n+2} & \ldots & B_{2n+1} \end{pmatrix}.$$

It is clear that any sequence of real numbers is uniquely determined by the determinant sequence $\det\tilde{A}_0$, $\det\tilde{B}_0$, $\det\tilde{A}_1$, $\det\tilde{B}_1,\ldots$ as long as these are different from 0. For example, the Catalan numbers are the unique sequence such that $\det\tilde{A}_n = \det\tilde{B}_n = 1$ for all n (see e.g. [4]). See also the related problem 36 [3, p. 50].

Theorem. *The Bell numbers B_n are the unique sequence of real numbers such that*

$$\det\tilde{A}_n = \det\tilde{B}_n = n!!$$

where $n!! = \prod_{i=0}^{n}(k!)$.

E-mail address: aigner@math.fu.berlin.de (M. Aigner)

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Table 1

	0	1	2	3	4	5
0	1					
1	1	1				
2	2	3	1			
3	5	10	6	1		
4	15	37	31	10	1	
5	52	151	160	75	15	1

This characterization can be obtained by the continued fraction approach in [1], but maybe the present short proof is also of interest.

Before proceeding to the proof let us recall some facts on exponential generating functions. If $C(x)=\sum_{n\geqslant 0}(C_n/n!)x^n$, then $C'(x)=\sum_{n\geqslant 0}(C_{n+1}/n!)x^n$ where $C'(x)$ is the derivative of $C(x)$. Further, it is a classical result that the exponential generating function of the Bell numbers is $B(x)=\sum_{n\geqslant 0}(B_n/n!)x^n=\mathrm{e}^{\mathrm{e}^x-1}$.

On the way to proving the theorem we use the fact that the Bell numbers can be obtained by a recursive procedure. This goes as follows:

Let $A=(a_{n,k})$ be the infinite lower triangular matrix defined recursively by

$$\begin{aligned} &a_{0,0}=1, \quad a_{0,k}=0 \ (k>0),\\ &a_{n,k}=a_{n-1,k-1}+(k+1)a_{n-1,k}+(k+1)a_{n-1,k+1} \quad (n\geqslant 1). \end{aligned} \tag{1}$$

The first rows of A are given in the Table 1 (omitting the zeroes).

We see that the numbers in the 0-column are the Bell numbers $B_0,\dots,B_5$, and we proceed to show that this holds in general. For the proof we make use of the Riordan group method introduced in [2].

Lemma 1. *Let $A_k(x)$ be the exponential generating function of the kth column of A, then*

$$A_k(x)=\mathrm{e}^{\mathrm{e}^x-1}\frac{(\mathrm{e}^x-1)^k}{k!} \quad (k\geqslant 0).$$

In particular, $A_0(x)=B(x)$.

Proof. The recursion (1) translates into

$$\begin{aligned} &A_k'(x)=A_{k-1}(x)+(k+1)A_k(x)+(k+1)A_{k+1}(x)\\ &A_k(0)=[k=0]. \end{aligned} \tag{2}$$

It is now easily seen that the functions $A_k(x)=\mathrm{e}^{\mathrm{e}^x-1}(\mathrm{e}^x-1)^k/k!$ satisfy precisely the functional equation (2). $\square$

Lemma 2. *Let r_n be the nth row of $A=(a_{n,k})$. Define $r_n \circ r_\ell := \sum_{k\geqslant 0} a_{n,k}a_{\ell,k}k!$, then $r_n \circ r_\ell = a_{n+\ell,0}$ $(=B_{n+\ell})$ for all n and ℓ.*

Proof. For $n=0$ we have $r_0 \circ r_\ell = a_{\ell,0}$ for all ℓ. Suppose that $r_m \circ r_\ell = a_{m+\ell,0}$ holds for $m\leqslant n-1$ and all ℓ. Then by (1) and interchanging the summation

$$\begin{aligned} r_n \circ r_\ell &= \sum_k a_{n,k}a_{\ell,k}k! = \sum_k (a_{n-1,k-1} + (k+1)a_{n-1,k} + (k+1)a_{n-1,k+1})a_{\ell,k}k! \\ &= \sum_k (a_{\ell,k+1}(k+1)! + a_{\ell,k}(k+1)! + a_{\ell,k-1}k!)a_{n-1,k} \\ &= \sum_k (a_{\ell,k-1} + (k+1)a_{\ell,k} + (k+1)a_{\ell,k+1})a_{n-1,k}k! \\ &= \sum_k a_{\ell+1,k}a_{n-1,k}k! = r_{n-1}\circ r_{\ell+1} = a_{n+\ell,0}. \qquad \square \end{aligned}$$

Proof of the theorem. The proof proceeds by providing an LDU decomposition of $\tilde{A}_n$. Let A_n be the submatrix of A consisting of the rows and columns numbered 0 to n. Since A_n is lower triangular with diagonal 1, we have $\det A_n = 1$. Now multiply the kth column of A_n by $k!$, for $0\leqslant k\leqslant n$, and call the new matrix $\overline{A}_n$; hence $\det \overline{A}_n = n!!$. Since $r_k \circ r_\ell = B_{k+\ell}$ by Lemma 1, we infer for the first Hankel matrix $\tilde{A}_n = \overline{A}_n A_n^{\mathrm{T}}$, and thus $\det \tilde{A}_n = n!!$.

If we replace recursion (1) by

$$\begin{aligned} &b_{0,0}=1, \quad b_{0,k}=0\ (k>0), \\ &b_{n,k} = b_{n-1,k-1} + (k+2)b_{n-1,k} + (k+1)b_{n-1,k+1} \quad (n\geqslant 1) \end{aligned} \tag{1'}$$

then we derive by the same method

$$B_k(x) = \mathrm{e}^{\mathrm{e}^x-1+x}(\mathrm{e}^x-1)^k/k!. \tag{2'}$$

In particular, $B_0(x) = \mathrm{e}^{\mathrm{e}^x-1}\mathrm{e}^x = B'(x)$. Hence the entries in the 0-column of the new matrix B are $(B_1, B_2, B_3, \ldots)$. As in Lemma 2, we find again $r_k \circ r_\ell = b_{k+\ell,0} = B_{k+\ell+1}$. If B_n, $\overline{B}_n$ denote the submatrices of B corresponding to $A_n, \overline{A}_n$ as before, then $\tilde{B}_n = \overline{B}_n B_n^{\mathrm{T}}$ and thus $\det \tilde{B}_n = n!!$. $\square$

Remark 1. By the same method one can prove

$$\det \begin{pmatrix} B_2 & B_3 & \ldots & B_{n+2} \\ B_3 & B_4 & \ldots & B_{n+3} \\ & & \ldots & \\ B_{n+2} & B_{n+3} & \ldots & B_{2n+2} \end{pmatrix} = c_{n+1}(n!!),$$

where $c_n = \sum_{k=0}^n n^{\underline{k}}$ is the total number of permutations of n things.

Remark 2. Let $S_n = \sum_{k=0}^n a_{n,k}$ be the sum of the nth row of A, with the first values $S_0=1$, $S_1=2$, $S_2=6$, $S_3=22$, $S_4=94$, $S_5=454$. The exponential generating function of the sequence (S_n) is by Lemma 1

$$S(x) = \mathrm{e}^{\mathrm{e}^x - 1} \sum_{k \geqslant 0} \frac{(\mathrm{e}^x - 1)^k}{k!} = (\mathrm{e}^{\mathrm{e}^x - 1})^2$$

and we find $S_n = \sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k}$, the convoluted Bell number. Using the same method as before one can show that

$$\det \begin{pmatrix} S_0 & S_1 & \dots & S_n \\ S_1 & S_2 & \dots & S_{n+1} \\ & & \dots & \\ S_n & S_{n+1} & \dots & S_{2n} \end{pmatrix} = 2^{\binom{n+1}{2}} (n!!).$$

References

[1] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 32 (1980) 125–161.
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[4] R. Stanley, Enumerative Combinatorics, vol. II, in press.