BARNES TYPE MULTIPLE q-ZETA FUNCTIONS AND q-EULER POLYNOMIALS

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Abstract. The purpose of this paper is to present a systemic study of some families of multiple q-Euler numbers and polynomials and we construct multiple q-zeta function which interpolates multiple q-Euler numbers at negative integer. This is a partial answer of the open question in a previous publication(see J. Physics A: Math. Gen. 34(2001)7633-7638).

§1. Introduction/ Preliminaries

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1+q}, \text{ (see [4, 5, 6, 7]).}$$

The q-factorial is defined as $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. For a fixed $d \in \mathbb{N}$ with $(p,d)=1$, $d \equiv 1$ $(mod\ 2)$, we set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

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where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. The q-binomial formulae are known as

$$(b;q)_n = (1-b)(1-bq)\cdots(1-bq^{n-1}) = \sum_{i=0}^{n} \binom{n}{i}_q q^{\binom{i}{2}}(-1)^i b^i,$$

and

$$\frac{1}{(b;q)_n} = \frac{1}{(1-b)(1-bq)\cdots(1-bq^{n-1})} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q b^i,$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}$, (see [4, 8, 9]).

Recently, many authors have studied the q-extension in the various area(see [4, 5, 6]). In this paper, we try to consider the theory of q-integrals in the p-adic number field associated with Euler numbers and polynomials closely related to fermionic distribution. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$(1) \qquad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [7, 8, 9]).}$$

Thus, we note that

$$(2) \qquad \lim_{q\to 1} I_q(f) = I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x).$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we have

$$(3) \qquad I_1(f_n) = (-1)^n I_1(f) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l).$$

Using formula (3), we can readily derive the Euler polynomials, $E_n(x)$, namely,

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ (see [16-20]).}$$

In the special case $x = 0$, the sequence $E_n(0) = E_n$ are called the n-th Euler numbers. In one of an impressive series of papers(see[1, 2, 3, 21, 23]), Barnes developed the so-called multiple zeta and multiple gamma functions. Barnes' multiple zeta function $\zeta_N(s, w|a_1, \cdots, a_N)$ depend on the parameters $a_1, \cdots, a_N$ that will be assumed to be positive. It is defined by the following series:

(4)

$$\zeta_N(s, w|a_1, \cdots, a_N) = \sum_{m_1,\cdots,m_N=0}^{\infty} (w+m_1a_1+\cdots+m_Na_N)^{-s} \text{ for } \Re(s) > N, \Re(w) > 0.$$

From (4), we can easily see that

$$\zeta_{M+1}(s,w+a_{M+1}|a_1,\cdots,a_{N+1})-\zeta_{M+1}(s,w|a_1,\cdots,a_{N+1})=-\zeta_M(s,w|a_1,\cdots,a_N),$$

and $\zeta_0(s,w)=w^{-s}$(see [11]). Barnes showed that ζ_N has a meromorphic continuation in s (with simple poles only at $s=1,2,\cdots,N$ and defined his multiple gamma function $\Gamma_N(w)$ in terms of the s-derivative at $s=0$, which may be recalled here as follows: $\psi_n(w|a_1,\cdots,a_N)=\partial_s\zeta_N(s,w|a_1,\cdots,a_N)|_{s=0}$(see[11]). Barnes' multiple Bernoulli polynomials $B_n(x,r|a_1,\cdots,a_r)$ are defined by

$$(5)\qquad \frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}e^{xt}=\sum_{n=0}^{\infty}B_n(x,r|a_1,\cdots,a_r)\frac{t^n}{n!},\ (|t|<\max_{1\leq i\leq r}\frac{2\pi}{|a_i|}),\ \text{(see [11]).}$$

By (4) and (5), we see that

$$\zeta_N(-m,w|a_1,\cdots,a_N)=\frac{(-1)^Nm!}{(N+m)!}B_{N+m}(w,N|a_1,\cdots,a_N),$$

where $w>0$ and m is a positive integer. By using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we consider the Barnes' type multiple q-Euler polynomials and numbers in this paper. The main purpose of this paper is to present a systemic study of some families of Barnes' type multiple q-Euler polynomials and numbers. Finally, we construct multiple q-zeta function which interpolates multiple q-Euler numbers at negative integer. This is a partial answer of the open question in [6, p.7637]

§2. Barnes type multiple q-Euler numbers and polynomials

Let $x,w_1,w_2,\cdots,w_r$ be complex numbers with positive real parts. In $\mathbb{C}$, the Barnes type multiple Euler numbers and polynomials are defined by
(6)

$$\frac{2^r}{\prod_{j=1}^r(e^{w_jt}+1)}e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x|w_1,\cdots,w_r)\frac{t^n}{n!},\ \text{for } |t|<max\{\frac{\pi}{|w_i|}|i=1,\cdots,r\},$$

and $E_n^{(r)}(w_1,\cdots,w_r)=E_n^{(r)}(0|w_1,\cdots,w_r)$(see [11, 12, 14]). In this section, we assume that $q\in\mathbb{C}_p$ with $|1-q|_p<1$. We first consider the q-extension of Euler polynomials as follows:

$$(7)\qquad \sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=\int_{\mathbb{Z}_p}e^{[x+y]_qt}d\mu_1(y)=2\sum_{m=0}^{\infty}(-1)^me^{[m+x]_qt},\ \text{(see [7, 8, 17])}\ .$$

Thus, we have $E_{n,q}(x)=\frac{2}{(1-q)^n}\sum_{l=0}^n\frac{\binom{n}{l}(-1)^lq^{lx}}{1+q^l}$(see [7]). In the special case $x=0$, $E_{n,q}=E_{n,q}(0)$ are called the q-Euler numbers. The q-Euler polynomials of order

$r \in \mathbb{N}$ are also defined by

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{[x+x_1+\cdots+x_r]_q t} d\mu_1(x_1)\cdots d\mu_1(x_r)$$
(8)
$$= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}(-1)^m e^{[m+x]_q t}, \text{ (see [7, 8]).}$$

In the special case $x=0$, the sequence $E_{n,q}^{(r)}(0) = E_{n,q}^{(r)}$ are refereed as the q-extension of the Euler numbers of order r. Let $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. Then we have
(9)

$$E_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} [x+x_1+\cdots+x_r]_q^n d\mu_1(x_1)\cdots d\mu_1(x_r)$$
$$= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx} \sum_{a_1,\cdots,a_r=0}^{f-1} \sum_{m_1,\cdots,m_r=0}^{\infty} q^{l(\sum_{i=1}^{r}(a_i+fm_i))}(-1)^{\sum_{i=1}^{r}(a_i+fm_i)}$$
$$= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n.$$

By (8) and (9), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$E_{n,q}^{(r)}(x) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n$$
$$= 2^r \sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[m+x]_q^n.$$

Let $F_q^{(r)}(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!}$. Then we have

$$F_q^{(r)}(t,x) = 2^r \sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m e^{[m+x]_q t}$$
(10)
$$= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} e^{[m_1+\cdots+m_r+x]_q t}.$$

Let χ be the Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. Then the generalized q-Euler polynomials attached to χ are defined by

(11) $$\sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}(-1)^m\chi(m)e^{[m+x]_q t}.$$

Thus, we have
(12)

$$E_{n,\chi,q}(x)=\sum_{a=0}^{f-1}\chi(a)(-1)^a\int_{\mathbb{Z}_p}[x+a+fy]_q^n d\mu_1(y)=[f]_q^n\sum_{a=0}^{f-1}\chi(a)(-1)^a E_{n,q^f}(\frac{x+a}{f}).$$

In the special case $x=0$, the sequence $E_{n,\chi,q}(0)=E_{n,\chi,q}$ are called the n-th generalized q-Euler numbers attached to χ. From (2) and (3), we can easily derive the following equation.

$$E_{m,\chi,q}(nf)-(-1)^n E_{m,\chi,q}=2\sum_{l=0}^{nf-1}(-1)^{n-1-l}\chi(l)[l]_q^m.$$

Let us consider higher-order generalized q-Euler polynomials attached to χ as follows:

$$(13)\qquad \int_X\cdots\int_X\left(\prod_{i=1}^r\chi(x_i)\right)e^{[x_1+\cdots+x_r+x]_q t}d\mu_1(x_1)\cdots d\mu_1(x_r)=\sum_{n=0}^\infty E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!},$$

where $E_{n,\chi,q}^{(r)}(x)$ are called the n-th generalized q-Euler polynomials of order r attached to χ. By (13), we see that
(14)

$$E_{n,\chi,q}^{(r)}(x)=\frac{2^r}{(1-q)^n}\sum_{l=0}^n\binom{n}{l}q^{lx}(-1)^l\sum_{a_1,\cdots,a_r=0}^{f-1}\left(\prod_{j=1}^r\chi(a_j)\right)\frac{(-q^l)^{\sum_{i=1}^r a_i}}{(1+q^{lf})^r}$$

$$=2^r\sum_{m=0}^\infty\binom{m+r-1}{m}(-1)^m\sum_{a_1,\cdots,a_r=0}^{f-1}\left(\prod_{j=1}^r\chi(a_j)\right)(-1)^{\sum_{i=1}^r a_i}[\sum_{j=1}^r a_j+x+mf]_q^n,$$

and

$$(15)\qquad \sum_{n=0}^\infty E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}=2^r\sum_{m_1,\cdots,m_r=0}^\infty(-1)^{\sum_{j=1}^r m_j}\left(\prod_{i=1}^r\chi(m_i)\right)e^{[x+\sum_{j=1}^r m_j]_q t}.$$

In the special case $x=0$, the sequence $E_{n,\chi,q}^{(r)}(0)=E_{n,\chi,q}^{(r)}$ are called the n-th generalized q-Euler numbers of order r attached to χ.

By (14) and (15), we obtain the following theorem.

Theorem 2. *Let χ be the Dirichlet's character with conductor $f\in\mathbb{N}$ with $f\equiv 1$*

(*mod* 2). *For* $n \in \mathbb{Z}_+$, $r \in \mathbb{N}$, *we have*

$$E_{n,\chi,q}^{(r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \sum_{a_1,\cdots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) \frac{(-q^l)^{\sum_{i=1}^{r} a_i}}{(1+q^{lf})^r}$$
$$= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \sum_{a_1,\cdots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j) \right) (-1)^{\sum_{i=1}^{r} a_i} [\sum_{j=1}^{r} a_j + x + mf]_q^n$$
$$= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \left(\prod_{i=1}^{r} \chi(m_i) \right) [x+m_1+\cdots+m_r]_q^n.$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, we introduce the extended higher-order q-Euler polynomials as follows:
(16)
$$E_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x_j} [x+x_1+\cdots+x_r]_q^n d\mu_1(x_1)\cdots d\mu_1(x_r), \text{ (see [8])} .$$

From (16), we note that
(17)
$$E_{n,q}^{(h,r)}(x) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} q^{(h-1)m_1+\cdots+(h-r)m_r} (-1)^{m_1+\cdots+m_r} [x+m_1+\cdots+m_r]_q^n.$$

It is known in [8] that
(18)
$$E_{n,q}^{(h,r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-q^x)^l}{(-q^{h-r+l};q)_r} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-q^{h-r})^m [x+m]_q^n.$$

Let $F_q^{(h,r)}(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(h,r)}(x) \frac{t^n}{n!}$. Then we have

(19)
$$F_q^{(h,r)}(t,x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q q^{(h-r)m} (-1)^m e^{[m+x]_q t}$$
$$= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j)m_j} (-1)^{\sum_{j=1}^{r} m_j} e^{[x+m_1+\cdots+m_r]_q t}.$$

Therefore, we obtain the following theorem.

Theorem 3. *For* $h, \in \mathbb{Z}$, $r \in \mathbb{N}$, *and* $x \in \mathbb{Q}^+$, *we have*

$$E_{n,q}^{(h,r)}(x) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} q^{(h-1)m_1+\cdots+(h-r)m_r} (-1)^{m_1+\cdots+m_r} [m_1+\cdots+m_r+x]_q^n.$$

For $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$, it is easy to show that the following distribution relation for $E_{n,q}^{(h,r)}(x)$.

$$E_{n,q}^{(h,r)}(x) = [f]_q^n \sum_{a_1,\cdots,a_r=0}^{f-1} (-1)^{a_1+\cdots+a_r} q^{\sum_{j=1}^r (h-j)a_j} E_{n,q^f}(\frac{x+a_1+\cdots+a_r}{f}).$$

Let us consider Barnes' type multiple q-Euler polynomials. For $w_1, \cdots, w_r \in \mathbb{Z}_p$, we define the Barnes' type q-multiple Euler polynomials as follow:

$$(20) \qquad E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [\sum_{j=1}^r w_j x_j + x]_q^n d\mu_1(x_1)\cdots d\mu_1(x_r).$$

From (20), we can easily derive the following equation.

$$(21) \qquad E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l}(-q^x)^l}{(1+q^{lw_1})\cdots(1+q^{lw_r})}, \text{ (see [8])}.$$

Thus, we have
(22)

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l}(-q^x)^l \sum_{a_1,\cdots,a_r=0}^{f-1} \frac{(-1)^{\sum_{i=1}^r a_i} q^{l\sum_{j=1}^r w_j a_j}}{(1+q^{lfw_1})\cdots(1+q^{lfw_r})},$$

where $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. By (22), we see that

$$(23) \quad E_{n,q}^{(r)}(x|w_1,\cdots,w_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} [x+w_1m_1+\cdots+w_rm_r]_q^n.$$

In the special case $x = 0$, the sequence $E_{n,q}^{(r)}(w_1,\cdots,w_r) = E_{n,q}^{(r)}(0|w_1,\cdots,w_r)$ are called the n-th Barnes' type multiple q-Euler numbers. Let $F_q^{(r)}(t,x|w_1,\cdots,w_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1,\cdots,w_r)\frac{t^n}{n!}$. Then we have

$$(24) \qquad F_q^{(r)}(t,x|w_1,\cdots,w_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} e^{[x+w_1m_1+\cdots+w_rm_r]_q t}.$$

Therefore we obtain the following theorem.

Theorem 4. *For $w_1,\cdots,w_r \in \mathbb{Z}_p$, $r \in \mathbb{N}$, and $x \in \mathbb{Q}^+$, we have*

$$\begin{aligned} E_{n,q}^{(r)}(x|w_1,\cdots,w_r) &= 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}[x+m_1w_1+\cdots+m_rw_r]_q^n \\ &= \frac{2^r}{(1-q)^n}\sum_{l=0}^n \frac{\binom{n}{l}(-q^x)^l}{(1+q^{lw_1})\cdots(1+q^{lw_r})}. \end{aligned}$$

For $w_1, \cdots, w_r \in \mathbb{Z}_p$, $a_1, \cdots, a_r \in \mathbb{Z}$, we consider another q-extension of Barnes' type multiple q-Euler polynomials as follows:
(25)

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+\sum_{j=1}^{r}w_jx_j]_q^n q^{\sum_{i=1}^{r}a_ix_i}\left(\prod_{i=1}^{r}d\mu_1(x_i)\right).$$

Thus, we have

$$(26)\quad E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)=\frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^lq^{lx}}{(1+q^{lw_1+a_1})\cdots(1+q^{lw_r+a_r})}.$$

From (25) and (26), we can derive the following equation.
(27)

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)=2^r\sum_{m_1,\cdots,m_r=0}(-1)^{\sum_{j=1}^{r}m_j}q^{\sum_{i=1}^{r}a_im_i}[x+\sum_{j=1}^{r}w_jx_j]_q^n.$$

Let $F_q^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!}$. Then, we have

$$(28)\quad \begin{aligned}&F_q^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)\\&=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}e^{[x+w_1m_1+\cdots+w_rm_r]_qt}.\end{aligned}$$

Theorem 5. *For $r \in \mathbb{N}$, $w_1, \cdots, w_r \in \mathbb{Z}_p$, and $a_1, \cdots, a_r \in \mathbb{Z}$, we have*

$$E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots a_r)=2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{\sum_{j=1}^{r}m_j}q^{\sum_{i=1}^{r}a_im_i}[x+\sum_{j=1}^{r}w_jm_j]_q^n.$$

Let χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1$ $(mod\ 2)$. Now we consider the generalized Barnes' type q-multiple Euler polynomials attached to χ as follows:

$$\begin{aligned}&E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\\&=\int_X\cdots\int_X[x+w_1x_1+\cdots+w_rx_r]_q^n\left(\prod_{j=1}^{r}\chi(x_j)\right)q^{a_1x_1+\cdots+a_rx_r}d\mu_1(x_1)\cdots d\mu_1(x_r).\end{aligned}$$

Thus, we have
(29)

$$\begin{aligned}&E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\\&=\frac{2^r}{(1-q)^n}\sum_{b_1,\cdots,b_r=0}^{f-1}\left(\prod_{i=1}^{r}\chi(b_i)\right)(-1)^{\sum_{j=1}^{r}b_j}q^{\sum_{i=1}^{r}(lw_i+a_i)b_i}\frac{\sum_{l=0}^{n}\binom{n}{l}(-1)^lq^{lx}}{\prod_{j=1}^{r}(1+q^{(lw_j+a_j)f})}.\end{aligned}$$

From, (29), we note that

$$E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty}\left(\prod_{j=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}[x+\sum_{j=1}^{r}w_jm_j]_q^n.$$

Therefore we obtain the following theorem.

Theorem 6. *For $r\in\mathbb{N}$, $w_1,\cdots,w_r\in\mathbb{Z}_p$, and $a_1,\cdots,a_r\in\mathbb{Z}$, we have*

$$E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty}\left(\prod_{j=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}[x+\sum_{j=1}^{r}w_jm_j]_q^n.$$

Let $F_{q,\chi}^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)=\sum_{n=0}^{\infty}E_{n,\chi,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!}$. By Theorem 6, we see that

$$F_{q,\chi}^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty}\left(\prod_{j=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}e^{[x+\sum_{j=1}^{r}w_jm_j]_qt}. \tag{30}$$

§3. Barnes type multiple q-zeta functions

In this section, we assume that $q\in\mathbb{C}$ with $|q|<1$ and the parameters $w_1,\cdots,w_r$ are positive. From (28), we consider the Barnes' type multiple q-Euler polynomials in $\mathbb{C}$ as follows:

$$\begin{aligned}F_q^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r) &= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}q^{a_1m_1+\cdots+a_rm_r}e^{[x+w_1m_1+\cdots+w_rm_r]_qt}\\ &= \sum_{n=0}^{\infty}E_{n,q}^{(r)}(x|w_1,\cdots,w_r;a_1,\cdots,a_r)\frac{t^n}{n!},\ \text{for } |t|<\max_{1\le i\le r}\{\frac{\pi}{|w_i|}\}\ .\end{aligned} \tag{31}$$

For $s,x\in\mathbb{C}$ with $\Re(x)>0$, $a_1,\cdots,a_r\in\mathbb{C}$, we can derive the following Eq.(32) from the Mellin transformation of $F_q^{(r)}(t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)$.

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}F_q^{(r)}(-t,x|w_1,\cdots,w_r;a_1,\cdots,a_r)dt\\ &= 2^r\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_r}q^{m_1a_1+\cdots+m_ra_r}}{[x+w_1m_1+\cdots+w_rm_r]_q^s}.\end{aligned} \tag{32}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, $a_1, \cdots, a_r \in \mathbb{C}$, we define Barnes' type multiple q-zeta function as follows:

$$\zeta_{q,r}(s, x|w_1, \cdots, w_r; a_1, \cdots, a_r) = 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{m_1 a_1+\cdots+m_r a_r}}{[x + w_1 m_1 + \cdots + w_r m_r]_q^s}. \tag{33}$$

Note that $\zeta_{q,r}(s, x|w_1, \cdots, w_r)$ is meromorphic function in whole complex s-plane. By using the Mellin transformation and the Cauchy residue theorem, we obtain the following theorem which is a part of answer of open question in [6, p.7637] .

Theorem 7. *For $x \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have*

$$\zeta_{q,r}(-n, x|w_1, \cdots, w_r; a_1, \cdots, a_r) = E_{n,q}^{(r)}(x|w_1, \cdots, w_r; a_1, \cdots, a_r).$$

Let χ be a Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \ (mod\ 2)$. From (30), we can define the generalized Barnes' type multiple q-Euler polynomials attached to χ in $\mathbb{C}$ as follows:

$$\begin{aligned} & F_{q,\chi}^{(r)}(t, x|w_1, \cdots, w_r; a_1, \cdots, a_r) \\ &= 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \left(\prod_{j=1}^{r} \chi(m_i) \right) (-1)^{m_1+\cdots+m_r} q^{a_1 m_1+\cdots+a_r m_r} e^{[x+\sum_{j=1}^{r} w_j m_j]_q t} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \cdots, w_r; a_1, \cdots, a_r) \frac{t^n}{n!}. \end{aligned} \tag{34}$$

From (34) and Mellin transformation of $F_{q,\chi}^{(r)}(t, x|w_1, \cdots, w_r; a_1, \cdots, a_r)$, we can easily derive the following equation (35) .

$$\begin{aligned} & \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_{q,\chi}^{(r)}(-t, x|w_1, \cdots, w_r; a_1, \cdots, a_r) dt \\ &= 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \frac{\left(\prod_{j=1}^{r} \chi(m_i) \right) (-1)^{m_1+\cdots+m_r} q^{m_1 a_1+\cdots+m_r a_r}}{[x + w_1 m_1 + \cdots + w_r m_r]_q^s}. \end{aligned} \tag{35}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, we also define Barnes' type multiple q-l-function as follows:

$$\begin{aligned} & l_{q,\chi}^{(r)}(s, x|w_1, \cdots, w_r; a_1, \cdots, a_r) \\ &= 2^r \sum_{m_1, \cdots, m_r=0}^{\infty} \frac{\left(\prod_{j=1}^{r} \chi(m_j) \right) (-1)^{m_1+\cdots+m_r} q^{m_1 a_1+\cdots+m_r a_r}}{[x + w_1 m_1 + \cdots + w_r m_r]_q^s}. \end{aligned} \tag{36}$$

Note that $l_{q,\chi}^{(r)}(s, x|w_1, \cdots, w_r)$ is meromorphic function in whole complex s-plane. By using (34), (35), (36), and the Cauchy residue theorem, we obtain the following theorem.

Theorem 8. *For $x, s \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have*

$$l_{q,\chi}^{(r)}(-n, x|w_1, \cdots, w_r; a_1, \cdots, a_r) = E_{n,\chi,q}^{(r)}(x|w_1, \cdots, w_r; a_1, \cdots, a_r).$$

We note that Theorem 8 is r-multiplication of Dirichlet's type q-l-series. Theorem 8 seems to be interesting and worthwhile for doing study in the area of multiple p-adic l-function or mathematical physics related to Knot theory and ζ-function (see [4-20]).

References

[1] I. N. Cangul,V. Kurt, H. Ozden, Y. Simsek, *On the higher-order w-q-Genocchi numbers*, Adv. Stud. Contemp. Math. **19** (2009), 39–57.

[2] L. Comtet, *Advanced combinatories, Reidel*, Dordrecht, 1974.

[3] E.Deeba, D.Rodriguez, *Stirling's series and Bernoulli numbers*, Amer. Math. Monthly **98** (1991), 423-426.

[4] N. K. Govil, V. Gupta,, *Convergence of q-Meyer-Konig-Zeller-Durrmeyer operators*, Adv. Stud. Contemp. Math. **19** (2009), 97-108.

[5] L.-C. Jang, *A study on the distribution of twisted q-Genocchi polynomials*, Adv. Stud. Contemp. Math. **18** (2009), 181–189.

[6] T. Kim, D.-W. Park, S.-H. Rim, *On multivariate p-adic q-integrals*, J. Phys. A: Math. Gen. **34** (2001), 7633-7638.

[7] T. Kim, *The modified q-Euler numbers and polynomials*, Adv. Stud. Contemp. Math. **16** (2008), 161-170.

[8] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-stirling numbers by the fermionic p-adic integrals on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **16** (2009), 484-491.

[9] T. Kim, *q−Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), 288–299.

[10] T. Kim, *A Note on p-Adic q-integral on $\mathbb{Z}_p$ Associated with q-Euler Numbers*, Adv. Stud. Contemp. Math. **15** (2007), 133–138.

[11] T. Kim, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Phys. **10** (2003), 261-267.

[12] T. Kim, *q-Extension of the Euler formula and trigonometric functions*, Russ. J. Math. Phys. **14** (2007), 275–278.

[13] T. Kim, *Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function*, Russ. J. Math. Phys. **12** (2005), 186–196.

[14] T. Kim, *Non-Archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials*, Russ. J. Math. Phys. **10** (2003), 91–98.

[15] T. Kim, *Note on the Euler q-zeta functions*, J. Number Theory **129** (2009), 1798–1804.

[16] Y. H. Kim, K. W. Hwang, *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math. **18** (2009), 127–133.

[17] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. **18** (2009), 41-48.

[18] H. Ozden, Y. Simsek, S.-H. Rim, I.N. Cangul, *A note on p-adic q-Euler measure*, Adv. Stud. Contemp. Math. **14** (2007), 233–239.

[19] K. Shiratani, S. Yamamoto, *On a p-adic interpolation function for the Euler numbers and its derivatives*, Mem. Fac. Sci., Kyushu University Ser. A **39** (1985), 113-125.

[20] Y. Simsek, *Theorems on twisted L-function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. **11** (2005), 205–218.

[21] H.J.H. Tuenter, *A Symmetry of power sum polynomials and Bernoulli numbers*, Amer. Math. Monthly **108** (2001), 258-261.

[22] Z. Zhang, Y. Zhang, *Summation formulas of q-series by modified Abel's lemma*, Adv. Stud. Contemp. Math. **17** (2008), 119–129.

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