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ScienceDirect
Journal of Computational and Applied Mathematics 211 (2008) 45 – 56
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Generalized Bochner theorem: Characterization of the Askey–Wilson polynomials

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Received 10 May 2006; received in revised form 1 November 2006

Abstract

Assume that there is a set of monic polynomials $P_n(z)$ satisfying the second-order difference equation

$$A(s)P_n(z(s+1)) + B(s)P_n(z(s)) + C(s)P_n(z(s-1)) = \lambda_n P_n(z(s)), \quad n = 0, 1, 2, \ldots, N,$$

where $z(s)$, $A(s)$, $B(s)$, $C(s)$ are some functions of the discrete argument s and N may be either finite or infinite. The irreducibility condition $A(s-1)C(s) \neq 0$ is assumed for all admissible values of s. In the finite case we assume that there are $N+1$ distinct grid points $z(s)$, $s = 0, 1, \ldots, N$ such that $z(i) \neq z(j)$, $i \neq j$. If $N = \infty$ we assume that the grid $z(s)$ has infinitely many different values for different values of s. In both finite and infinite cases we assume also that the problem is non-degenerate, i.e., $\lambda_n \neq \lambda_m$, $n \neq m$. Then we show that necessarily: (i) the grid $z(s)$ is at most quadratic or q-quadratic in s; (ii) corresponding polynomials $P_n(z)$ are at most the Askey–Wilson polynomials corresponding to the grid $z(s)$. This result can be considered as generalizing of the Bochner theorem (characterizing the ordinary classical polynomials) to generic case of arbitrary difference operator on arbitrary grids.

MSC: 33C45; 42C05

Keywords: Classical orthogonal polynomials in discrete argument; Askey–Wilson polynomials; Bochner theorem; Duality

1. Introduction

General orthogonal polynomials (OP) $P_n(x)$ can be characterized by the three-term recurrence relation [6]

$$P_{n+1}(x) + b_n P_n(x) + u_n P_n(x) = x P_n(x) \tag{1.1}$$

with initial conditions $P_0 = 1$, $P_1 = x - b_0$.

The polynomials $P_n(x)$ are *monic* polynomials, i.e., $P_n(x) = x^n + \mathrm{O}(x^{n-1})$.

It is well known [1] that all polynomial solutions $P_n(x)$ of the second-order differential equation

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) = \lambda_n P_n(x) \tag{1.2}$$

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.11.004

are classical orthogonal polynomials (COP), i.e., Jacobi, Laguerre, Hermite and Bessel polynomials. In (1.2) it appears that $\sigma(x)$ and $\tau(x)$ are polynomials such that $\deg(\sigma(x)) \leqslant 2$, $\deg(\tau(x)) = 1$. This result is known as the Bochner theorem [5].

It is natural to consider generalization of the Bochner theorem replacing the second-order differential operator with the second-order difference operator. Namely, we are seeking polynomial solutions $P_n(z)$ of the problem

$$A(s)P_n(z(s+1)) + B(s)P_n(z(s)) + C(s)P_n(z(s-1)) = \lambda_n P_n(z(s)), \quad n = 0, 1, 2, \ldots, N, \tag{1.3}$$

where $z(s)$, $A(s)$, $B(s)$, $C(s)$ are some functions of the discrete argument s and N may be either finite or infinite. The irreducibility condition $A(s-1)C(s) \neq 0$ is assumed for all admissible values of s. In the finite case we assume that there are $N+1$ distinct grid point $z(s)$, $s = 0, 1, \ldots, N$ such that $z(i) \neq z(j)$, $i \neq j$. If $N = \infty$ we assume that the grid $z(s)$ has infinitely many different values for different values of s. In both finite and infinite cases we assume also that the problem is non-degenerate, i.e., $\lambda_n \neq \lambda_m$, $n \neq m$. We assume also that there are polynomial solutions of all degrees $n = 0, 1, \ldots, N$ (i.e., we assume that the polynomial $P_n(x)$ always has exact degree n for all $n = 0, 1, \ldots, N$).

Askey and Wilson [2] discovered (OP) (the Askey–Wilson polynomials, or briefly, AWP) which satisfy Eq. (1.3) for quadratic $z(s) = as^2 + bs + c$ or q-quadratic grid $z(s) = aq^s + bq^{-s} + c$, where q is some parameter such that $|q| \neq 1$. Finite-dimensional case (i.e., when there exists only N mutually OP $n = 0, 1, \ldots, N-1$) corresponds to the so-called q-Racah polynomials [9].

In [7] it was shown that the only OP satisfying (1.3) for AW-grids are the AWP. Leonard [10] showed that in the finite-dimensional case the only OP satisfying (1.3) are the q-Racah polynomials. For further development of the Leonard result and its new algebraic interpretation see, e.g., [3,17]. In [8] Ismail obtained more strong result: he showed that all polynomial (i.e., not necessarily orthogonal, ab initio) solutions of Eq. (1.3) for the AW-grid are AWP. In the finite-dimensional case Terwilliger obtained the result that the AW-grid is the most general for polynomials satisfying (1.3).

So far, the open problem was: in the infinite-dimensional case characterize all possible grids $z(s)$ for which polynomial solutions of Eq. (1.3) are obtained. In this paper we solve this problem and show that there are no grids more general than AW-grids. Hence, all polynomial solutions for (1.3) *should be* orthogonal AWP. Although for the finite-dimensional case the problem was effectively solved by Terwilliger in [18], we present here the finite-dimensional version of the generalized Bochner theorem as well. The main reason is that our method of proof is essentially different and deals directly with difference equation (1.3) for polynomials, whereas in the Terwilliger paper [18] another (a purely algebraic) approach is presented.

2. Finite-dimensional case

In this section we show that if N is finite then the problem is essentially equivalent to the Leonard theorem [3,10]. Indeed, consider $(N+1) \times (N+1)$ tri-diagonal matrix J which acts on a basis e_k, $k = 0, 1, \ldots, N$ by

$$Je_k = C(k+1)e_{k+1} + B(k)e_k + A(k-1)e_{k-1}. \tag{2.1}$$

It is assumed that $C(N+1) = A(-1) = 0$ which means merely that the matrix J acts in linear space of dimension $N+1$. We will assume the non-degeneracy condition:

$$C(i)A(i-1) \neq 0, \ \ i = 1, 2, \ldots, N. \tag{2.2}$$

Find the eigenvectors $v^{(k)}$, $k = 0, 1, \ldots, N$ of the matrix J, i.e.,

$$Jv^{(k)} = \lambda_k v^{(k)}$$

with some eigenvalues λ_k. We assume that all eigenvalues are distinct: $\lambda_i \neq \lambda_j$ if $i \neq j$. Then all vectors $v^{(k)}$, $k = 0, 1, \ldots, N$ are independent and we have

$$v^{(k)} = \sum_{s=0}^{N} v_{ks} e_s, \tag{2.3}$$

where v_{ks}, $s = 0, 1, \dots, N$ are components of the vector $v^{(k)}$ in the basis e_s. For them we have relation

$$A(s)v_{k,s+1} + B(s)v_{ks} + C(s)v_{k,s-1} = \lambda_k v_{ks}. \tag{2.4}$$

Now we can identify components v_{ks} with $P_k(z_s)$, i.e., we merely put $v_{ks} = P_k(z_s)$ for all values $k, s = 0, 1, \dots, N$. Then difference equation (1.3) coincides with (2.4).

Consider transposed Jacobi matrix J^* defined as

$$J^* e_k = A(k)e_{k+1} + B(k)e_k + C(k)e_{k-1} \tag{2.5}$$

and corresponding eigenvalue vectors $v^{*(k)}$:

$$J^* v^{*(k)} = \lambda_k \ v^{*(k)}, \quad k = 0, 1, 2, \dots, N. \tag{2.6}$$

Vectors $v^{*(k)}$ can be expanded in terms of the same basis e_s:

$$v^{*(k)} = \sum_{s=0}^{N} v^*_{ks} e_s. \tag{2.7}$$

From elementary linear algebra it is known that in non-degenerated case (i.e., if $\lambda_i \neq \lambda_j$ for $i \neq j$) the vectors v^k and $v^{*(j)}$ are biorthogonal:

$$(v^k, v^{*(j)}) \equiv \sum_{s=0}^{N} v_{ks} v^*_{js} = 0 \quad \text{if} \quad k \neq j. \tag{2.8}$$

Introduce now the diagonal matrix M which acts on basis e_s as

$$M e_s = \mu_s e_s, \quad s = 0, 1, 2, \dots, N, \tag{2.9}$$

where

$$\mu_s = \frac{A(0)A(1)\dots A(s-1)}{C(1)C(2)\dots C(s)}, \quad s = 1, 2, \dots, N, \quad \mu_0 = 1. \tag{2.10}$$

Note that all μ_s are well defined due to non-degeneracy condition (2.2).

It is elementary verified that

$$J^* = M^{-1} J M, \tag{2.11}$$

and hence

$$v^{*(k)} = M^{-1} v^{(k)}, \quad k = 0, 1, \dots, N \tag{2.12}$$

(inverse matrix M^{-1} exists due to non-degeneracy condition (2.2)). Relation (2.12) allows one to rewrite biorthogonality condition (2.8) in the form

$$\sum_{s=0}^{N} w_s v_{ks} v_{js} = 0 \quad \text{if} \quad k \neq j, \tag{2.13}$$

where

$$w_s = 1/\mu_s = \prod_{i=1}^{s} \frac{C(i)}{A(i-1)}. \tag{2.14}$$

In terms of polynomials $P_n(x)$ this relation becomes

$$\sum_{s=0}^{N} w_s P_k(z(s)) P_j(z(s)) = 0 \quad \text{if} \quad k \neq j. \tag{2.15}$$

But relation (2.15) means that $P_n(x)$ are polynomials which are orthogonal on a finite distinct set of points $z(s)$, $s = 0, 1, \dots, N$ with discrete weights $w_s \neq 0$. By general elementary theorems concerning OP [6] this means that polynomials $P_n(x)$ should satisfy a three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \quad n = 0, 1, \dots, N. \tag{2.16}$$

The roots $x_s, \ s = 0, 1, \dots, N$ of the polynomial $P_{N+1}(x)$ coincide with spectral points:

$$z(s) = x_s, \quad s = 0, 1, \dots, N.$$

Thus, we proved that (under some non-degeneration conditions) polynomials $P_n(x)$ satisfying relation (1.3) on a grid $z(s)$ for finite N are orthogonal with respect to discrete weight function (2.15) and satisfy three-term recurrence relation (2.16).

Now we are ready to relate our results with Leonard's approach to dual OP [10].

Recall relation between non-degenerated Jacobi matrices and OP (see, e.g., [6]). Let K be an arbitrary Jacobi matrix of dimension $N + 1 \times N + 1$. In some finite-dimensional basis d_n it can be presented as

$$K d_n = \alpha_n d_{n+1} + \beta_n d_n + \gamma_n d_{n-1} \tag{2.17}$$

with some (complex) coefficients with non-degeneracy property

$$\prod_{i=1}^{N} \gamma_i \alpha_{i-1} \neq 0. \tag{2.18}$$

Construct eigenvectors $\pi^{(k)}$ of the matrix K:

$$K \pi^{(k)} = z_k \pi^{(k)}, \quad k = 0, 1, \dots, N. \tag{2.19}$$

We assume that all spectral points $z_k, \ k = 0, 1, \dots, N$ are distinct: $z_k \neq z_j$ for $k \neq j$. Expand eigenvectors $\pi^{(k)}$ in terms of basis d_n:

$$\pi^{(k)} = \sum_{s=0}^{N} \pi_{ks} d_s$$

with some coefficients π_{ks}. For these coefficients we have from (2.19) the recurrence relation

$$\gamma_{s+1} \pi_{k,s+1} + \beta_s \pi_{ks} + \alpha_{s-1} \pi_{k,s-1} = z_k \pi_{ks}, \quad k, s = 0, 1, \dots, N. \tag{2.20}$$

It is assumed that $\alpha_{-1} = \gamma_{N+1} = 0$. Then, for each value k, starting from π_{k0} we can find recursively all further coefficients $\pi_{k1}, \pi_{k,2}, \dots, \pi_{k,N}$. We can always normalize $\pi_{k0} = 1, \ k = 0, 1, \dots, N$. Then it is clear from (2.20) that π_{ks} is a polynomial of degree s in argument z_k.

Introduce polynomials $T_n(x)$ satisfying three-term recurrence relation

$$\gamma_{n+1} T_{n+1}(x) + \beta_n T_n(x) + \alpha_{n-1} T_{n-1}(x) = x T_n(x) \tag{2.21}$$

with initial conditions $\alpha_{-1} P_{-1} = 0, \ P_0(x) = 1$. Then relation (2.21) defines n-degree polynomials $T_n(x) = \kappa_n x^n + O(x^{n-1})$ with the leading coefficient

$$\kappa_n = \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_n}$$

(this leading coefficient is well defined and non-zero do to non-degeneracy condition (2.18)). From general theory of OP it follows that polynomials $T_n(x)$ are orthogonal on a finite set of points x_k [6]

$$\sum_{k=0}^{N} \sigma_k T_n(x_k) T_m(x_k) = 0, \quad n \neq m, \tag{2.22}$$

where x_k are roots of the polynomial $T_{N+1}(x)$.

We can thus associate OP $T_n(x)$ with expansion coefficients of eigenvectors of the arbitrary non-degenerated Jacobi matrix K:

$$T_s(z_k) = \pi_{ks}. \tag{2.23}$$

Return to our polynomials $P_n(x)$ satisfying relation (1.3). We showed that these polynomials are orthogonal and correspond to the Jacobi matrix K whose matrix coefficients can be restored from recurrence relation (2.16): $\gamma_n = 1$, $\beta_n = b_n$, $\alpha_n = u_{n+1}$. On the other hand, we have the Jacobi matrix J defined by (2.1). By just described recipe, we can associate with this Jacobi matrix corresponding OP $Y_n(x)$. These polynomials satisfy three-term recurrence relation

$$A(n)Y_{n+1}(x) + B(n)Y_n(x) + C(n)Y_{n-1}(x) = xY_n(x). \tag{2.24}$$

Now it is seen that polynomials $P_n(x)$ and $Y_n(x)$ are related as

$$P_n(z(s)) = Y_s(\lambda_n). \tag{2.25}$$

We thus have a duality property coinciding with that introduced by Leonard [10]: there are two systems of finite OP and two finite sequences $z(s)$ and λ_n such relation (2.25) holds. Our non-degeneracy conditions: all $z(s)$ and λ_n are distinct and matrices J, T are non-degenerated coincide with similar conditions in the Leonard paper. Hence we can conclude.

Theorem 1. *Under non-degeneracy conditions the finite-dimensional case of relation* (1.3) *generates at most finite Askey–Wilson OP* (*Racah and q-Racah polynomials in other terms*).

3. Infinite-dimensional case. Reducing to a more simple problem

In this section we start to analyze the infinite-dimensional case. We first derive some restrictions upon the coefficients $A(s)$, $B(s)$, $C(s)$.

In what follows we will assume that polynomial solutions $P_n(z)$ of Eq. (1.3) are monic, i.e., $P_n(z) = z^n + O(z^{n-1})$. This is not restriction of our problem, because it is possible to divide all terms in Eq. (1.3) by a (non-zero) leading coefficient of the polynomial $P_n(z)$.

First of all we observe that eigenvalues λ_n can be shifted by an arbitrary constant $\lambda_n \to \lambda_n + \text{const}$. Such shift leads to adding a constant to the coefficient $B(s)$. Using this observation we always can choose λ_n in such a way that

$$\lambda_0 = 0. \tag{3.1}$$

In what follows we will assume that condition (3.1) is fulfilled. We will also assume that the eigenvalue problem (1.3) is non-degenerate, i.e.,

$$\lambda_n \neq \lambda_m, \quad n \neq m. \tag{3.2}$$

The grid $z(s)$ is also assumed to be non-degenerate, i.e.,

$$z(s_1) \neq z(s_2), \quad s_1 \neq s_2. \tag{3.3}$$

Parameter s takes infinite number of integer values: $s = s_0, s_0 + 1, s_0 + 2, \ldots$ where s_0 is either finite or $s_0 = -\infty$. In the first case we deal with semi-infinite grid z_s, whereas in the second case we have the grid which is infinite in both directions.

Taking the case $n = 0$ in (1.3) we see that $A(s) + B(s) + C(s) = 0$. Hence, we can rewrite Eq. (1.3) in the form

$$A(s)\Delta P_n(z(s)) - C(s)\nabla P_n(z(s)) = \lambda_n P_n(z(s)), \tag{3.4}$$

where we use the standard notation [13]

$$\Delta F(s) = F(s+1) - F(s), \quad \nabla F(s) = F(s) - F(s-1)$$

for any function $F(s)$ of the argument s.

Assume that polynomials $P_n(z)$ have the expansion

$$P_n(z) = z^n + \sum_{i=0} \xi_{ni} z^i$$

with some coefficients ξ_{ni}. Then for $n = 1$ we get from (3.4)

$$A(s)\Delta z(s) - C(s)\nabla z(s) = \lambda_1 Q_1(z(s)),$$

where $Q_1(z) = z + \xi_{10} = P_1(z)$. By induction, it can be easily shown that

$$A(s)\Delta z^n(s) - C(s)\nabla z^n(s) = \lambda_n Q_n(z(s)), \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

where $Q_n(z)$ is a monic polynomial of degree n.

Vice versa, assume that property (3.5) holds for some $z(s)$, $A(s)$, $C(s)$ with $Q_n(x)$ being a set of monic polynomials in x of degree n. Then there exists a set of monic polynomials $P_n(x)$ satisfying Eq. (3.4). This statement is almost obvious and follows from the observation that on the given grid $z(s)$ and for *any* monic nth degree polynomial $T_n(x)$ the expression $A(s)\Delta T_n(z(s)) - C(s)\nabla T_n(z(s))$ is again a nth degree polynomial in the argument $z(s)$ with the leading coefficient λ_n. Hence, it is possible to choose a polynomial $P_n(x)$ with property (3.4).

Consider now condition (3.5) for $n \to n + 1$:

$$A(s)\Delta z^{n+1}(s) - C(s)\nabla z^{n+1}(s) = \lambda_{n+1} Q_{n+1}(z(s)), \quad n = 0, 1, \ldots . \tag{3.6}$$

Multiplying (3.5) by $z(s)$ and subtracting (3.6) we get another set of conditions

$$A_1(s) z^n(s+1) + C_1(s) z^n(s-1) = R_{n+1}(z(s)), \quad n = 0, 1, \ldots, \tag{3.7}$$

where $A_1(s) = A(s)\Delta z(s)$, $\quad C_1(s) = C(s)\nabla z(s)$, The polynomials $R_n(z)$ are nth degree polynomials $R_n(z) = \omega_n z^n + O(z^{n-1})$, where $\omega_n = \lambda_n - \lambda_{n-1}$. Note that due to non-degeneracy condition (3.2) we have $\omega_n \neq 0$ and hence every polynomial $R_n(z)$ has exact degree n.

Consider first two conditions (3.7) corresponding to $n=0$ and 1. These two conditions can be considered as equations for two unknowns $A_1(s)$, $C_1(s)$. Solving these equations we have

$$A_1(s) = \frac{R_2(z(s)) - z(s-1)R_1(z(s)}{z(s+1) - z(s-1)},$$
$$C_1(s) = -\frac{R_2(z(s)) - z(s+1)R_1(z(s)}{z(s+1) - z(s-1)}. \tag{3.8}$$

Note that these expressions are well defined for all possible s because, by non-degeneracy condition, $z(s+1) \neq z(s-1)$.

Hence conditions (3.7) can be rewritten as

$$R_2(z(s))Y_n - R_1(z(s))z(s-1)z(s+1)Y_{n-1} = R_{n+1}(z(s)), \ n = 2, 3, \ldots, \tag{3.9}$$

where

$$Y_n = \frac{z^n(s+1) - z^n(s-1)}{z(s+1) - z(s-1)}. \tag{3.10}$$

Introduce the variables

$$u = z(s-1)z(s+1), \quad v = z(s-1) + z(s+1).$$

Clearly Y_n is a symmetric polynomial with respect to $z(s-1)$, $z(s+1)$ and hence it can be expressed in terms of variables u, v only. Indeed, it is easily verified that Y_n satisfy the recurrence relation

$$Y_{n+1} = vY_n - uY_{n-1}, \quad Y_0 = 0, \quad Y_1 = 1. \tag{3.11}$$

This allows us to find an explicit expression for every Y_n in terms of u, v. For example, $Y_2 = v$, $\ Y_3 = v^2 - u$, $\ Y_4 = v^3 - 2uv$, etc.

Return to condition (3.5). We have explicit expressions for coefficients $A(s)$, $C(s)$:

$$A(s) = \frac{R_1 z(s-1) - R_2}{(z(s+1) - z(s-1))(z(s+1) - z(s))},$$
$$C(s) = \frac{R_1 z(s+1) - R_2}{(z(s+1) - z(s-1))(z(s) - z(s-1))}. \tag{3.12}$$

Hence we have

$$A(s)\Delta z(s) - C(s)\nabla z(s) = \sum_{k=0}^{n-1} z^{n-k-1}(z(s+1)z(s-1)R_1 Y_{k-1} - R_2 Y_k) = -\sum_{k=0}^{n-1} R_{k+1} z(s)^{n-k-1}$$

(in the last equality we have used (3.9)). It is seen that this expression is indeed a polynomial of degree n with non-zero leading coefficient λ_n. Thus conditions (3.5) and (3.7) are equivalent and we can use only more simple condition (3.7) for further analysis.

4. Functional equation for the grid $z(s)$

From (3.9) and (3.11) we find the conditions

$$R_{n+2}(z(s)) = v R_{n+1}(z(s)) - u R_n(z(s)), \quad n = 2, 3, \ldots . \tag{4.1}$$

These conditions form a system of linear equations for two unknowns u, v. Consider the first two equations corresponding to $n = 2$ and 3. There are two possibilities:

(i) These equations are not independent. Then we should have $R_{i+1}(x) = \tau(x) R_i(x)$, $i = 1, 2, 3$ where $\tau(x)$ is a linear function. By induction, we then have $R_n(x) = R_1(x)\tau^{n-1}(x)$, $n = 1, 2, \ldots$ for all n, where both $\tau(x)$ and $R_1(x)$ are linear functions in x. Now from (4.1) we have the condition

$$\tau^2(z(s)) - v\tau(z(s)) + u = 0 \tag{4.2}$$

or, equivalently,

$$(\tau(z(s)) - z(s+1))(\tau(z(s)) - z(s-1)) = 0. \tag{4.3}$$

From (4.3) and (3.12) we see that in this case either $A(s) = 0$ or $C(s) = 0$ for every admissible s. But this contradicts our non-degeneracy assumption $A(s-1)C(s) \neq 0$. Thus case (i) should be excluded from consideration.

(ii) These equations are independent. Putting $n = 2, 3$ in (4.1) we obtain a linear system of equations for unknowns u, v from which we find

$$u = \frac{\pi_8(z(s))}{\pi_6(z(s))}, \quad v = \frac{\pi_7(z(s))}{\pi_6(z(s))}, \tag{4.4}$$

where $\pi_i(x)$ are polynomials of degrees $\leqslant i$:

$$\pi_6 = R_3^2 - R_2 R_4, \quad \pi_7 = R_4 R_3 - R_2 R_5, \quad \pi_8 = R_3 R_5 - R_4^2.$$

Thus u, v are some rational functions in the variable $z(s)$. In what follows we will sometimes replace the grid $z(s)$ with independent variable x (this is possible because the grid $z(s)$ takes infinitely many different values).

We first prove an important statement concerning possible solutions of the system of non-linear difference equations of the form

$$z(s-1) + z(s+1) = T_1(z(s)), \quad z(s-1)z(s+1) = T_2(z(s)), \tag{4.5}$$

where $T_{1,2}(x)$ are some rational functions.

Lemma 1. *Assume that system* (4.5) *has a solution* $z(s)$, $s=s_0, s_0+1, \ldots$ *with infinitely many non-coinciding values* $z(s_1) \neq z(s_2)$ *if* $s_1 \neq s_2$. *Then there are two possibilities*:

(i) *Either*

$$T_1(x) = -\frac{\alpha_2 x^2 + \alpha_4 x + \alpha_5}{\alpha_1 x^2 + \alpha_2 x + \alpha_3}, \quad T_2(x) = \frac{\alpha_3 x^2 + \alpha_5 x + \alpha_6}{\alpha_1 x^2 + \alpha_2 x + \alpha_3} \tag{4.6}$$

with some constants α_i, $i=1,\ldots,6$.

(i) *In this case variables* $z(s)$, $z(s+1)$ *satisfy equation*

$$\Phi(z(s), z(s+1)) = 0, \tag{4.7}$$

where $\Phi(x, y)$ *is a non-reducible symmetric biquadratic polynomial*:

$$\Phi(x, y) = \alpha_1 x^2 y^2 + \alpha_2 xy(x+y) + \alpha_3(x^2+y^2) + \alpha_4 xy + \alpha_5(x+y) + \alpha_6 \tag{4.8}$$

or

(ii)

$$T_1(x) = -\frac{\alpha_2 x + \alpha_4}{\alpha_1 x + \alpha_3} - \frac{\alpha_3 x + \alpha_4}{\alpha_1 x + \alpha_2}, \quad T_2(x) = \frac{\alpha_2 x + \alpha_4}{\alpha_1 x + \alpha_3}\frac{\alpha_3 x + \alpha_4}{\alpha_1 x + \alpha_2} \tag{4.9}$$

with some constants α_i, $i=1,\ldots,4$ *such that* $\alpha_2 \neq \alpha_3$. *In this case variables* $z(s)$, $z(s+1)$ *satisfy equation*

$$\alpha_1 z(s)z(s+1) + \alpha_2 z(s) + \alpha_3 z(s+1) + \alpha_4 = 0. \tag{4.10}$$

Remark 1. Case (ii) formally corresponds to a special case of (i) when polynomial $\Phi(x, y)$ can be decomposed as a product of two polynomials of the first degree in both variables x, y.

Proof. Obviously, system (4.5) is equivalent to the statement that both $z(s+1)$ and $z(s-1)$ are roots of the quadratic equation

$$A_2(z(s))z_{s\pm1}^2 + A_1(z(s))z_{s\pm1} + A_0(z(s)) = 0, \tag{4.11}$$

where $A_i(x)$ are non-zero polynomials having no common factors.

Introduce two polynomials in two variables:

$$W_1(x, y) = A_2(x)y^2 + A_1(x)y + A_0(x), \quad W_2(x, y) \equiv W_1(y, x) = A_2(y)x^2 + A_1(x)x + A_0(y).$$

Equation $W_1(x, y) = 0$ and $W_2(x, y) = 0$ define two algebraic curves in complex variables x, y. From (4.11) it is clear that both curves contain infinitely many common distinct points (x_n, y_n), $n = 1, 2, \ldots$. By the Bezout theorem this is possible only if these curves either coincide or have a common component.

The polynomial $W_1(x, y)$ has degree 2 in variable y and hence there are two possibilities:

(i) $W_1(x, y)$ is irreducible, i.e., it cannot be decomposed into irreducible polynomials of a lesser degree in y.
(ii) $W_1(x, y)$ can be presented as a product of two polynomials, each of degree 1 in variable y: $W_1(x, y) = (e_1(x)y + e_2(x))(e_3(x)y + e_4(x))$ with some polynomials $Re_i(x)$, $i = 1, \ldots, 4$.

We consider these two possibilities separately. In case (i) we have that the polynomials $W_1(x, y)$ and $W_2(x, y)$ are both irreducible. Hence, by the Bezout theorem, they should coincide:

$$W_1(x, y) = W_2(x, y) = W_1(y, x). \tag{4.12}$$

But condition (4.12) means that the polynomial $W_1(x, y)$ is *symmetric* in variables x, y. This is possible only if all polynomials $A_i(x)$, $i = 0, 1, 2$ have degree $\leqslant 2$ in variable x. Hence, the most general expression for $W_1(x, y)$ in this case is *symmetric biquadratic polynomial* in x, y:

$$W_1(x, y) = \alpha_1 x^2 y^2 + \alpha_2 xy(x+y) + \alpha_3(x^2+y^2) + \alpha_4 xy + \alpha_5(x+y) + \alpha_6 \tag{4.13}$$

with some constants α_i, $i=1,\dots,6$. We thus have

$$A_2(x)=\alpha_1x^2+\alpha_2x+\alpha_3,\quad A_1(x)=\alpha_2x^2+\alpha_4x+\alpha_5,\ A_0(x)=\alpha_3x^2+\alpha_5x+\alpha_6$$

and

$$T_1(x)=-A_1(x)/A_2(x)=-\frac{\alpha_2x^2+\alpha_4x+\alpha_5}{\alpha_1x^2+\alpha_2x+\alpha_3},\quad T_2(x)=A_0(x)/A_2(x)=\frac{\alpha_3x^2+\alpha_5x+\alpha_6}{\alpha_1x^2+\alpha_2x+\alpha_3}$$

giving expression (4.6).

Consider now the case when polynomials $W_1(x,y)$, $W_2(x,y)$ have a non-trivial common component which does not coincide with both these polynomials. Clearly, this is possible only if $W_1(x,y)$ can be decomposed into two polynomials linear in y:

$$W_1(x,y)=(a_1(x)y+b_1(x))(a_2(x)y+b_2(x)) \tag{4.14}$$

with some polynomials $a_{1,2}(x)$, $b_{1,2}(x)$. By definition $W_2(x,y)=W_1(y,x)$ and hence we have also

$$W_2(x,y)=(a_1(y)x+b_1(y))(a_2(y)x+b_2(y)). \tag{4.15}$$

Without loss of generality we can assume that $a_1(x)y+b_1(x)$ is a common component of two curves $W_1(x,y)=0$ and $W_2(x,y)=0$. Comparing (4.14) and (4.15), we can conclude that there are two possibilities:

(i) either $a_1(x)y+b_1(x)=a_1(y)x+b_1(y)$;
(ii) or $a_1(x)y+b_1(x)=a_2(y)x+b_2(y)$.

In case (i) we have that variables x, y satisfy symmetric polynomial relation

$$\alpha_1xy+\alpha_2(x+y)+\alpha_4=0 \tag{4.16}$$

with some constants $\alpha_1,\alpha_2,\alpha_4$. Substituting $x=z(s)$, $y=z(s+1)$ into (4.16) we find from (4.16) that there are only two non-coinciding points $z(s_0)$ and $z(s_0+1)$. For all further points we find that $z(s_0+2j)=z(s_0)$ and $z(s_0+2j+1)=z(s_0+1)$ for all integer j. But this contradicts our assumption that there are infinitely many distinct points belonging to the curves. Thus case (i) is impossible.

In case (ii) the polynomials $a_{1,2}(x)$, $b_{1,2}(x)$ should be linear in x and we have that $z(s)$, $z(s+1)$ satisfy the relation

$$\alpha_1z(s)z(s+1)+\alpha_2z(s)+\alpha_3z(s+1)+\alpha_4, \tag{4.17}$$

where $\alpha_3\neq\alpha_2$ in order to prevent impossible case (i). This case corresponds to (4.9) and (4.10). Thus, the Lemma is proven. □

It is interesting to find explicit solutions in both cases (i) and (ii) of the Lemma. Case (i) corresponds to a parametrization of symmetric Euler–Baxter biquadratic curve $\Phi(z(s),z(s+1))=0$ with $\Phi(x,y)$ given by (4.8). This problem was already solved by Baxter [4] in his famous solution of the 8-vertex model. Explicitly

$$z(s)=\kappa\,\phi(\beta_1s+\beta_0) \tag{4.18}$$

with some parameters κ,β_1,β_0. Here, $\phi(z)$ is an even elliptic function of the second order (i.e., having exactly two poles in the fundamental parallelogram). Recall that (up to an arbitrary factor) any even elliptic function of the second order can be presented in the form [19]

$$\phi(z)=\frac{\sigma(z-e)\sigma(z+e)}{\sigma(z-d)\sigma(z+d)}. \tag{4.19}$$

Recently it was shown that the elliptic grid $z(s)$ described by (4.18) appears naturally in theory of biorthogonal rational functions with the duality property [15,16]. Note that in a special case $\alpha_1=\alpha_2=0$ the rational functions $T_1(x)$, $T_2(x)$ become linear and quadratic polynomials. In this case solution for $z(s)$ is expressed in terms of elementary functions (see below).

For case (ii) of the Lemma the solution can be easily found in terms of elementary functions of s (we will not describe these solutions in details because they can be obtained from the elliptic solutions by a limiting procedure).

Now return to condition (4.1) and consider first case (i) of the Lemma. We can rewrite (4.1) in the form

$$1 = v(x)R_{n+1}(x)/R_{n+2}(x) - u(x)R_n(x)/R_{n+2}(x), \tag{4.20}$$

where

$$v(x) = -\frac{\alpha_2 x^2 + \alpha_4 x + \alpha_5}{\alpha_1 x^2 + \alpha_2 x + \alpha_3}, \quad u(x) = \frac{\alpha_3 x^2 + \alpha_5 x + \alpha_6}{\alpha_1 x^2 + \alpha_2 x + \alpha_3}.$$

Assume first that $\alpha_1 \neq 0$. Then for $x \to \infty$ it is seen that rhs of (4.20) tends to 0 which contradicts to lhs of (4.20). Thus, necessarily $\alpha_1 = 0$. Assume now that $\alpha_1 = 0$ and $\alpha_2 \neq 0$. Then again for $x \to \infty$ we obtain from (4.20) the condition (recall that $R_n(x) = \omega_n x^n + \mathrm{O}(x^{n-1})$, where $\omega_n = \lambda_n - \lambda_{n-1}$)

$$1 = -\omega_{n+1}/\omega_{n+2}$$

whence $\lambda_n = \lambda_{n+2}$ for all $n = 2, 3, \ldots$. But this contradicts our condition of non-degeneracy of the spectrum λ_n. We thus have necessarily $\alpha_1 = \alpha_2 = 0$. But in this case $v(x), u(x)$ become polynomials of the first and second degrees:

$$v(x) = -\xi x - \eta, \quad u(x) = x^2 + \eta x + \zeta, \tag{4.21}$$

where $\xi = \alpha_4/\alpha_3,\ \eta = \alpha_5/\alpha_3,\ \zeta = \alpha_6/\alpha_3$ and equations for the grid become

$$z(s-1) + z(s+1) = -\xi z(s) - \eta, \quad z(s-1)z(s+1) = z^2(s) + \eta z(s) + \zeta \tag{4.22}$$

with arbitrary complex parameters ξ, η, ζ. Equivalently, variables $z(s), z(s+1)$ belong to a non-degenerating conic (i.e., ellipsis, hyperbola or parabola):

$$z^2(s+1) + z^2(s) + \eta(z(s+1) + z(s)) + \xi z(s)z(s+1) + \zeta = 0 \tag{4.23}$$

which is symmetric with respect to $z(s), z(s+1)$ (this means that the plot of this conic in Cartesian co-ordinates $x = z(s)$, $y = z(s+1)$ is symmetric with respect to the line $y = x$). Eqs. (4.22) and (4.23) were studied in [11–13]. In these works it was shown that all non-degenerate solutions of these equations can be presented in the form

$$z(s) = C_1 q^s + C_2 q^{-s} + C_0 \tag{4.24}$$

or

$$z(s) = C_2 s^2 + C_1 s + C_0 \tag{4.25}$$

or

$$z(s) = (-1)^s\ (C_2 s^2 + C_1 s + C_0) \tag{4.26}$$

with some constants C_0, C_1, C_2. The first case (4.24) occurs if $\xi = q + q^{-1}$, where $q \neq \pm 1$ (i.e., $\xi \neq \pm 2$). The second case (4.25) occurs if $\xi = -2$ and the third case (4.26) occurs if $\xi = 2$. All these cases exhaust possible types of the Askey–Wilson grids [12].

Note that when $C_1 C_2 = 0$ in (4.24) we obtain so-called exponential grids, say $z(s) = C_1 q^s + C_0$. Similarly, when $C_2 = 0$ in (4.25) or (4.26) we obtain the linear grid: $z(s) = C_1 s + C_0$ or $z(s) = (-1)^s (C_1 s + C_0)$. However, in these case the conic (4.23) becomes degenerated—it divided into two lines. This corresponds to case (ii) of the Lemma (see below).

Now substituting $v(x), u(x)$ into (4.1) we obtain that for arbitrary given polynomials $R_1(x)$, $R_2(x)$ one can construct uniquely the polynomial $R_n(x) = \omega_n x^n + \mathrm{O}(x^{n-1})$ with leading coefficient satisfying the recurrence relation (which easily follows from (4.1) for given $v(x)u(x)$):

$$\omega_{n+2} + \xi\omega_{n+1} + \omega_n = 0, \quad n = 2, 3, \ldots. \tag{4.27}$$

General solution for ω_n can be easily found from (4.27): if $\xi = q + q^1 \neq \pm 2$ we have

$$\omega_n = G_1 q^n + G_2 q^{-n} \tag{4.28}$$

with arbitrary G_1, G_2. If $\xi = -2$ then

$$\omega_n = G_1 n + G_0 \tag{4.29}$$

and if $\xi = 2$ then

$$\omega_n = (-1)^n (G_1 n + G_0). \tag{4.30}$$

From $\omega_n = \lambda_n - \lambda_{n-1}$ we can easily reconstruct the spectrum λ_n which has the same functional dependence on n as the Askey–Wilson grid $z(s)$ has on s.

Finally, we should consider case (ii) of the lemma. In this case similar considerations lead to conclusion that $\alpha_1 = 0$. Then condition (4.10) becomes

$$\alpha_2 z(s) + \alpha_3 z(s+1) + \alpha_4 = 0 \tag{4.31}$$

which describes exponential or linear grids $z(s)$. Thus, case (ii) can be considered as a degeneration of case (i). It should be noted that linear and exponential grids are described by *non-symmetric* recurrence relations (4.31) (with respect to $z(s), z(s+1)$). The reason is that in this case conic (4.23) is degenerated to a two straight lines each of which is non-symmetric.

We see that in both cases (i) and (ii) of the lemma solutions $R_n(x)$ of the recurrence relation (4.1) are indeed polynomials of exact degree n. Hence, by previous considerations, we obtain a unique set of polynomials $P_n(x)$ which are solutions of Eq. (3.4). From explicit form (3.12) of the coefficients $A(s), C(s)$ we can conclude that they coincide with those defining the AWP [11–13].

We thus proved that under some non-degeneracy conditions, the only admissible grid is the Askey–Wilson grid and corresponding polynomials $P_n(x)$ coincide with the AWP.

5. Concluding remarks

The authors of [13] exploited relations (3.4) as a starting point in their approach to construction of the AWP. In a slightly different manner, Magnus in [11,12] derived relations (4.23) from the following requirement: find all the grids $z(s)$ and $y(s)$, such that for *any* polynomial $P_n(x)$ of degree n we have the property

$$\frac{P_n(z(s+1)) - P_n(z(s))}{z(s+1) - z(s)} = T_{n-1}(y(s)), \tag{5.1}$$

where $T_{n-1}(x)$ is a polynomial of degree $n-1$ and relation (5.1) should be valid for all $n = 1, 2, \ldots$ and for infinitely many distinct values s of the grids $z(s)$ and $y(s)$. Relation (5.1) can be also presented in the form

$$\mathscr{D}_s P_n(x) = T_{n-1}(y(s)), \tag{5.2}$$

where $\mathscr{D}_s$ stands for “discrete derivation” operator which acts on the space of function $f(x)$ as

$$\mathscr{D}_s f(x) \equiv \frac{f(z(s+1)) - f(z(s))}{z(s+1) - z(s)}.$$

For the AW-grid it was known that the operator $\mathscr{D}_s$ satisfies property (5.2). Magnus proved that these grids are *the only* preserving property (5.2).

On the other hand, it was noted in [13] that if polynomials $P_n(x)$ satisfy the AW-equation (3.4) then the new polynomials $T_n(x)$ obtained from $P_n(x)$ by (5.1) also satisfy AW-equation (3.4) but with different coefficients $A(s), C(s)$. This property can be considered as a covariance of the Askey–Wilson equation (3.4) with respect to the discrete Darboux transformation (see, e.g., [14]).

However, property (5.1) cannot be directly derived from Eq. (3.4) if the grid $z(s)$ is not concretized. This is why derivation of the necessity of the AW-grid for Eq. (3.4) is not quite elementary and needs rather involved technique which was demonstrated in the present paper.

Acknowledgment

"The authors thank the referees for their critical readings of this paper, which led to its improvement" before the first sentence. A.Zh. thanks Centre de recherches mathématiques of the Université de Montréal for hospitality.

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