Elliptic Askey-Wilson functions and associated elliptic Schur functions

Masatoshi NOUMI (Kobe University)

EIS, Leiden (July 15, 2013)

In this talk I discuss a family of elliptic functions that generalize Askey-Wilson polynomials, with emphasis on their difference equations. Also, I investigate a class of multivariable elliptic functions of Schur type built up from them by determinants. This class of functions can be regarded as an elliptic extension of Koornwinder polynomials with $t = q$, and carries various characteristic properties. I will describe in particular difference equations for this class, and an explicit formula for rectangular cases.

目次

1 Askey-Wilson polynomials and Koornwinder polynomials

• Askey-Wilson polynomials

q-Hypergeometric series:

$$p_n(z;a,b,c,d|q) = {}_4\phi_3\left[\begin{matrix} abcdq^{n-1}, q^{-n}, az, a/z \\ ab, ac, ad \end{matrix}; q, q\right]$$

$$= \sum_{k=0}^{n} \frac{(abcdq^{n-1};q)_k (q^{-n};q)_k (az;q)_k (a/z;q)_k}{(q;q)_k (ab;q)_k (ac;q)_k (ad;q)_k} q^k$$

$$(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a) \qquad (k = 0, 1, 2, \ldots)$$

q-Difference equation: $p_n(z) = p_n(z;a,b,c,d|q)$ satisfies

$$\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}(T_{q,z}-1)p_n(z) + \frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)}(T_{q,z}^{-1}-1)p_n(z) = -p_n(z)\,(1-abcdq^{n-1})(1-q^{-n})$$

where $T_{q,z}f(z) = f(qz)$.

Spectral duality: Regarding the values at the reference points $z = aq^k$ $(k = 0, 1, 2, \ldots)$,

$$p_l(aq^k; a, b, c, d|q) = {}_4\phi_3\left[\begin{matrix} abcdq^{l-1}, q^{-l}, a^2q^k, q^{-k} \\ ab,\ ac,\ ad \end{matrix}; q, q\right]$$
$$= {}_4\phi_3\left[\begin{matrix} \alpha^2q^l, q^{-l}, \alpha\beta\gamma\delta q^{k-1}, q^{-k} \\ \alpha\beta,\ \alpha\gamma,\ \alpha\delta \end{matrix}; q, q\right] = p_k(\alpha q^l; \alpha, \beta, \gamma, \delta|q)$$

with the involutive transformation on parameters $(a, b, c, d) \leftrightarrow (\alpha, \beta, \gamma, \delta)$:

$$\alpha = \sqrt{abcd/q}, \quad \beta = \sqrt{abq/cd}, \quad \gamma = \sqrt{acq/bd}, \quad \delta = \sqrt{adq/bc}.$$

Namely,

$$p_l(aq^k; a, b, c, d|q) = p_k(\alpha q^l; \alpha, \beta, \gamma, \delta|q) \qquad (k, l = 0, 1, 2, \ldots).$$

Symmetry with respect to (a, b, c, d): The monic Askey-Wilson polynomials

$$\frac{(ab, ac, ad; q)_l}{a^l(abcdq^{l-1}; q)_l}\, p_l(z; a, b, c, d|q, t) = (z^l + z^{-l}) + \cdots$$

are symmetric with respect to the parameters (a, b, c, d).

$\cdots$ Sears' transformation formula for terminating ${}_4\phi_3$.

• Koornwinder polynomials

Macdonald polynomials of type BC, multivariable Askey-Wilson polynomials
$x = (x_1, \dots, x_m)$: canonical coordinates of $\mathbb{T}^m = (\mathbb{C}^*)^m$
a, b, c, d, q, t: generic complex parameters
$W_m = W(BC_m) = \{\pm 1\}^m \rtimes \mathfrak{S}_m$: Weyl group of type BC_m (hyperoctahedral group)
W_m acts on $(x_1, \dots, x_m)$ through permutations of indices and inversions $x_i \to x_i^{-1}$.

Koorwinder's q-difference operator $\mathcal{D}_x$ is defined by

$$\mathcal{D}_x = \sum_{i=1}^{m} A_{i,+}(x)(T_{q,x_i} - 1) + \sum_{i=1}^{m} A_{i,-}(x)(T_{q,x_i}^{-1} - 1)$$

$$A_{i,+}(x) = \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(abcdq^{-1})^{\frac{1}{2}}(1-x_i^2)(1-qx_i^2)} \prod_{1\le j\le m;\ j\ne i} \frac{(1-tx_i/x_j)(1-tx_ix_j)}{t(1-x_i/x_j)(1-x_ix_j)},$$

$$A_{i,-}(x) = A_{i,+}(x^{-1}) \qquad (i = 1, \dots, m).$$

To make the formula shorter, we use the multiplicative notation $\langle z\rangle = z^{\frac{1}{2}} - z^{-\frac{1}{2}} = -z^{-\frac{1}{2}}(1-z)$ of the sine function:

$$A_{i,+}(x) = \frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle} \prod_{1\le j\le m; j\ne i} \frac{\langle tx_i/x_j\rangle\langle tx_ix_j\rangle}{\langle x_i/x_j\rangle\langle x_ix_j\rangle} \quad (i = 1, \dots, m).$$

For each partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, there exists a unique W_m-invariant Laurent polynomial

$$P_\lambda(x) = P_\lambda(x; a, b, c, d|q, t) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]^{W_m}$$

such that

(1) In terms of orbit sums $m_\mu(x) = \sum_{\nu \in W_m \mu} x^\nu$, $P_\lambda(x)$ is expressed in the form

$$P_\lambda(x) = m_\lambda(x) + \sum_{\mu < \lambda} p_{\lambda,\mu}\, m_\mu(x)$$

with respect to the dominance ordering of partitions, and

(2) $P_\lambda(x)$ satisfies the q-difference equation

$$\mathcal{D}_x\, P_\lambda(x) = P_\lambda(x)\, d_\lambda; \quad d_\lambda = \sum_{i=1}^{m} \langle \alpha t^{m-i} q^{\lambda_i}; \alpha t^{m-i} \rangle, \quad \alpha = (abcdq^{-1})^{\frac{1}{2}},$$

where $\langle z; a \rangle = \langle za \rangle \langle z/a \rangle = z^{-1}(1 - za)(1 - z/a) = z + z^{-1} - a - a^{-1}$.

The Koornwinder polynomials $P_\lambda(x)$ form a $\mathbb{C}$-basis of the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]^{W_m}$ of W_m-invariant Laurent polynomials.

Van Diejen's commuting family of q-difference operators:

For $r = 0, 1, 2, \ldots, m$, define $\mathcal{D}_x^{(r)}$ by

$$\mathcal{D}_x^{(r)} = \sum_{(I;\epsilon);\,|I|\le r} A_{I,\epsilon}^{(r)}(x)\, T_{q,x}^{(I;\epsilon)}, \qquad T_{q,x}^{(I;\epsilon)} = \prod_{i\in I} T_{q,x_i}^{\epsilon_i},$$

where the sum is taken over all pairs $(I;\epsilon)$ of $I \subseteq M = \{1, \ldots, m\}$ and $\epsilon : I \to \{\pm 1\}$ with $|I| \le r$. The coefficients are defined by

$$A_{I,\epsilon}^{(r)}(x) = (-1)^{r-|I|} U(x_I^{\epsilon}; x_{M\setminus I}) \sum_{(I',\epsilon');\, I'\subseteq M\setminus I, |I'|=r-|I|} V(x_{I'}^{\epsilon'}; x_{M\setminus I\cup I'}),$$

$$U(x_I; x_J) = \prod_{i\in I} \frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle} \prod_{i,j\in I; i<j} \frac{\langle tx_ix_j\rangle\langle qtx_ix_j\rangle}{\langle x_ix_j\rangle\langle qx_ix_j\rangle} \prod_{i\in I;\, j\in J} \frac{\langle tx_i/x_j\rangle\langle tx_ix_j\rangle}{\langle x_i/x_j\rangle\langle (x_ix_j\rangle},$$

$$V(x_I; x_J) = \prod_{i\in I} \frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle} \prod_{i,j\in I; i<j} \frac{\langle tx_ix_j\rangle\langle qx_ix_j/t\rangle}{\langle x_ix_j\rangle\langle qx_ix_j\rangle} \prod_{i\in I;\, j\in J} \frac{\langle tx_i/x_j\rangle\langle tx_ix_j\rangle}{\langle x_i/x_j\rangle\langle (x_ix_j\rangle}.$$

These q-difference opeartors $\mathcal{D}_x^{(r)}$ commute with each other: $\mathcal{D}_x^{(r)}\mathcal{D}_x^{(s)} = \mathcal{D}_x^{(s)}\mathcal{D}_x^{(r)}$ $(r, s \in \{1, \ldots, m\})$. The Koornwinder polynomials $P_\lambda(x)$ are the joint eigenfunctions of this commuting family of q-difference operators.

In terms of the generation function,

$$\mathcal{D}_x(u) = \sum_{r=0}^{m} (-1)^{m-r} \langle u;\alpha\rangle_{t,r}\, \mathcal{D}_x^{(r)}, \qquad \langle u;\alpha\rangle_{t,r} = \langle u;\alpha\rangle\langle u;t\alpha\rangle\cdots\langle u;t^{r-1}\alpha\rangle.$$

for any partition $\lambda = (\lambda_1, \dots, \lambda_m)$ we have

$$\mathcal{D}_x(u)\, P_\lambda(x) = P_\lambda(x) \prod_{i=1}^{m} \langle u; \alpha t^{m-i} q^{\lambda_i}\rangle.$$

In the context of affine Hecke algebras, it is known that the q-operators of $\mathcal{D}_x^{(r)}$ arise from the center of the affine Hecke algebra of type C_n through q-Dunkl operators.

Spectral duality:

Renormalize the Koornwinder polynomials by the value at the base point at^ρ:

$$\widetilde{P}_\lambda(x;a,b,c,d|q,t) = \frac{P_\lambda(x;a,b,c,d|q,t)}{P_\lambda(at^\rho;a,b,c,d|q,t)}; \quad at^\rho = (at^{m-1}, at^{m-2}, \dots, a).$$

Then for any pair of partitions λ, μ,

$$\widetilde{P}_\lambda(at^\rho q^\mu;a,b,c,d|q,t) = \widetilde{P}_\mu(\alpha t^\rho q^\lambda;\alpha,\beta,\gamma,\delta|q,t).$$

Okounkov's binomial formula:

The normalized Koornwinder polynomials are expressed as

$$\widetilde{P}_\lambda(x;a,b,c,d|q,t)=\sum_{\mu\subseteq\lambda}c_\mu(a,b,c,d|q,t)\,R_\mu(\alpha\,t^\rho q^\lambda;\alpha|q,t)R_\mu(x;a|q,t)$$

in terms of Okounkov's interpolation polynomials $R_\mu(x;a|q,t)$; $R_\mu(x;a|q,t)$ are characterized as W_m-invariant Laurent polynomials of degree $|\lambda|$ satisfying the interpolation property that

$$R_\mu(at^\rho q^\lambda;a|q,t)=0\quad\text{unless}\quad\mu\subseteq\lambda,\quad R_\mu(at^\rho q^\mu;a|q,t)\neq 0.$$

Koornwinder polynomials with $t=q$:

$P_\lambda(x;a,b,c,d|q,q)$ can be regraded as Macdonald's ninth variation of Schur functions associated with Askey-Wilson polynomials:

$$P_\lambda(x;a,b,c,d|q,q)=\text{const.}\frac{\det\left(p_{m-j+\lambda_j}(x_i;a,b,c,d|q)\right)_{i,j=1}^m}{\det\left(p_{m-j}(x_i;a,b,c,d|q)\right)_{i,j=1}^m}.$$

• Basic questions in view of the extension to elliptic functions:

Q1: How elliptic extension of Askey-Wilson and Koornwinder polynomials should be constructed?
Q2: How they should be related with BC type difference operators of Ruijsenaars-van Diejen?

Elliptic extension of Askey-Wilson and Koornwinder polynomials has been developed succesfully in the framework of

- BC type elliptic interpolation functions ... Coskun-Gustafson (2006), Rains (2006)
- Elliptic biorthogonal functions defined through the binomial formula ... Rains (2006).

It seems, however, that the relationship to difference operators of Ruijsenaars - van Diejen has not been completely clarified.

In this talk, I concentrate on the special special case ($t = q$) where the multivariable elliptic functions can be defined by determinants (of Schur type). Presumably, this class of multivariable elliptic Askey-Wilson functions is related to the hypergeometric solutions of the elliptic Painlevé equation of affine E_8 Weyl group symmetry.

2 Very well-poised elliptic hypergeometric series

• Hermite's theorem

Let $s(z)$ be a nonzero entire function in $z \in \mathbb{C}$, and suppose that $s(z)$ satisfies the *Riemann relation*

$$\begin{gathered} s(z+a)s(z-a)s(a+b)s(a-b) + s(z+b)s(z-b)s(b+c)s(b-c) \\ + s(z+c)s(z-c)s(c+a)s(c-a) = 0 \end{gathered}$$

for any $z, a, b, c \in \mathbb{C}$. Then

(1) $s(z)$ is an odd function and the set $\Omega = \{\omega \in \mathbb{C} \mid s(\omega) = 0\}$ of zeros of $s(z)$ is a closed discrete subgroup of the additive group $\mathbb{C}$ (hence, of rank ≤ 2).

(2) $s(z)$ is quasi-periodic with respect to Ω. Furthermore

(3) $s(z)$ is expressed as follows according to the rank Ω:

Rational case:	$s(z) = e^{az^2+b}\, z,$	$\Omega = 0$
Trigonometric case:	$s(z) = e^{az^2+b}\, \sin(\pi z/\omega),$	$\Omega = \mathbb{Z}\,\omega$
Elliptic case:	$s(z) = e^{az^2+b}\, \sigma(z;\Omega),$	$\Omega = \mathbb{Z}\,\omega_1 \oplus \mathbb{Z}\,\omega_2$

In the elliptic case, Ω is generated by two complex numbers ω_1, ω_2 which are linearly independent over $\mathbb{R}$, and $\sigma(z;\Omega)$ stands for the Weierstrass sigma function associated with the period lattice Ω.

We fix a nonzero entire function $s(z)$ satisfying the Riemann relation, and denote it simply by $[z] = s(z)$ (e-number notation). The Riemann relation can be expressed in various forms:

$$[x \pm u][y \pm v] - [x \pm v][y \pm u] = [x \pm y][u \pm v],$$

$$\frac{[x \pm u]}{[x \pm v]} - \frac{[y \pm u]}{[y \pm v]} = \frac{[x \pm y][u \pm v]}{[x \pm v][y \pm v]},$$

where $[x \pm y] = [x + y][x - y]$.

• Very well-poised elliptic hypergeometric series

Fixing a generic complex number δ, we introduce the notation of δ-shifted factorials and very well-poised hypergeometric series associated with $[z]$.

$$[z]_k = [z]_{\delta,k} = [z][z+\delta]\cdots[z+(k-1)\delta] \quad (k = 0, 1, 2, \ldots)$$

$${}_{r+5}V_{r+4}(a_0; a_1, a_2, \ldots, a_r) = \sum_{k=0}^{\infty} \frac{[a_0 + 2k\delta]}{[a_0]} \frac{[a_0]_k [a_1]_k [a_2]_k \cdots [a_r]_k}{[\delta]_k [b_1]_k [b_2]_k \cdots [b_r]_k}$$

$$a_i + b_i = \delta + a_0 \quad (i = 1, \ldots, r).$$

When we use the notation of the V series, we always assume the termination condition

$$\exists i \in \{0, 1, \ldots, r\} : \quad a_i \equiv -n\delta \mod \Omega, \quad n \in \mathbb{N} = \{0, 1, 2, \ldots\}.$$

so that the V-series becomes a finite sum.

• Frenkel-Turaev sum (1997)

Under the balancing condition and the termination condition

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2a_0 + \delta; \qquad a_5 = -n\delta \quad (n \in \mathbb{N}),$$

$$\begin{aligned} &{}_{10}V_9(a_0;\ a_1, a_2, a_3, a_4, -n\delta) \\ &= \frac{[\delta + a_0]_n[\delta + a_0 - a_1 - a_2]_n[\delta + a_0 - a_1 - a_3]_n[\delta + a_0 - a_2 - a_3]_n}{[\delta + a_0 - a_1]_n[\delta + a_0 - a_2]_n[\delta + a_0 - a_3]_n[\delta + a_0 - a_1 - a_2 - a_3]_n}. \end{aligned}$$

• Elliptic Bailey transformation

Under the balancing condition and the termination condition

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 3a_0 + 2\delta; \qquad a_7 = -n\delta \quad (n \in \mathbb{N}),$$

$$\begin{aligned} &{}_{12}V_{11}(a_0;\ a_1, a_2, a_3, a_4, a_5, a_6, -n\delta) \\ &= \frac{[\delta + a_0]_n[\delta + a_0 - a_4 - a_5]_n[\delta + a_0 - a_4 - a_6]_n[\delta + a_0 - a_5 - a_6]_n}{[\delta + a_0 - a_4]_n[\delta + a_0 - a_5]_n[\delta + a_0 - a_6]_n[\delta + a_0 - a_4 - a_5 - a_6]_n} \\ &\quad \cdot {}_{12}V_{11}(\widetilde{a}_0;\ \widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3, a_4, a_5, a_6, -n\delta). \end{aligned}$$

$$\widetilde{a}_0 = \delta + 2a_0 - a_1 - a_2 - a_3,$$

$$\widetilde{a}_1 = \delta + a_0 - a_2 - a_3, \quad \widetilde{a}_2 = \delta + a_0 - a_1 - a_3, \quad \widetilde{a}_3 = \delta + a_0 - a_1 - a_2.$$

3 Elliptic Askey-Wilson functions

• An elliptic extension of Askey-Wilson polynomials

We introduce a sequence of elliptic hypergeometric series $\Phi_l(z;\boldsymbol{a};b|\delta)$ $(l=0,1,2,\ldots)$ with parameters $\boldsymbol{a}=(a_0,a_1,a_2,a_3)$ and b. Setting $\alpha_0=\frac{1}{2}(a_0+a_1+a_2+a_3-\delta)$, for each $l=0,1,2,\ldots$ we define

$$
\begin{aligned}
&\Phi_l(z;\boldsymbol{a},b|\delta)=\Phi_l(z;a_0,a_1,a_2,a_3,b|\delta)\\
&={}_{12}V_{11}(a_0+b-\delta;\ a_0+z,a_0-z,2\alpha_0+l\delta,-l\delta,b-a_1,b-a_2,b-a_3)\\
&=\sum_{k=0}^{l}\frac{[a_0\pm z]_k}{[b\pm z]_k}\frac{[a_0+b+(2k-1)\delta]}{[a_0+b-\delta]}\frac{[a_0+b-\delta]_k[2\alpha_0+l\delta]_k[-l\delta]_k}{[\delta]_k[a_0+b-2\alpha_0-l\delta]_k[a_0+b+l\delta]_k}\prod_{i=1}^{3}\frac{[b-a_i]_k}{[a_0+a_i]_k}.
\end{aligned}
$$

In the trigonometric case, this corresponds to

$$\begin{aligned}
&\Phi_l(z;a_0,a_1,a_2,a_3,b|q)\\
&= {}_{10}W_9(a_0bq^{-1};a_0z,a_0/z,a_0a_1a_2a_3q^{l-1},q^{-l},b/a_1,b/a_2,b/a_3;q,q)\\
&= \sum_{k=0}^{l} \frac{1-a_0bq^{2k-1}}{1-a_0bq^{-1}} \frac{(a_0bq^{-1})_k}{(q)_k} \frac{(a_0z)_k(a_0/z)_k}{(bz)_k(b/z)_k} \frac{(a_0a_1a_2a_3q^{l-1})_k(q^{-l})_k}{(bq^{1-l}/a_1a_2a_3)_k(a_0bq^l)_k} \frac{(b/a_1)_k(b/a_2)_k(b/a_3)_k}{(a_0a_1)_k(a_0a_2)_k(a_0a_3)_k}\, q^k
\end{aligned}$$

in the multiplicative variables, where $(a)_k = (a;q)_k$.

In the limit $b \to 0$, this series recovers the Askey-Wilson polynomial

$$\begin{aligned}
p_l(z;a_0,a_1,a_2,a_3,a_4|q) &= {}_4\phi_3\left[\begin{matrix} a_0z,\, a_0/z,\, a_0a_1a_2a_3q^{l-1},\, q^{-l} \\ a_0a_1,\ a_0a_2,\ a_0a_3 \end{matrix};\ q,q\right]\\
&= \sum_{k=0}^{l} \frac{(a_0z)_k(a_0/z)_k(a_0a_1a_2a_3q^{l-1})_k(q^{-l})_k}{(q)_k(a_0a_1)_k(a_0a_2)_k(a_0a_3)_k}\, q^k
\end{aligned}$$

• Difference equation

Consider the difference operator

$$L(z, T_z; \boldsymbol{a}, b, u) = A(z; \boldsymbol{a}, b, u)\big(T_z^{\delta} - 1\big) + A(-z; \boldsymbol{a}, b, u)\big(T_z^{-\delta} - 1\big) + \Lambda_0(\boldsymbol{a}, b; u)$$

$$A(z; \boldsymbol{a}, b, u) = \frac{[z-b][z-b+\delta][z+b-\alpha_0 \pm u]\prod_{i=0}^{3}[z+a_i]}{[2z][2z+\delta]}$$

$$\Lambda_0(\boldsymbol{a}, b, u) = [\alpha_0 \pm u]\prod_{i=0}^{3}[b-a_i], \quad \alpha_0 = \tfrac{1}{2}(a_0+a_1+a_2+a_3-\delta),$$

where $T_z^{\delta} f(z) = f(z+\delta)$. Then the elliptic hypergeometric series

$$\Phi_l(z; \boldsymbol{a}, b) = {}_{12}V_{11}(a_0+b-\delta;\ a_0+z, a_0-z, 2\alpha_0+l\delta, -l\delta, b-a_1, b-a_2, b-a_3)$$

satisfies the following difference equation (*almost eigenfunction* with parameter shift in b):

$$L(z, T_z; \boldsymbol{a}, b, u)\Phi_l(z; \boldsymbol{a}, b) = \Phi_l(z; \boldsymbol{a}, b+\delta)\,\Lambda_l(\boldsymbol{a}, b; u) \qquad (l = 0, 1, 2, \ldots)$$

$$\Lambda_l(\boldsymbol{a}, b; u) = \frac{[\alpha_0+l\delta \pm u]}{[\alpha_0+l\delta \pm \beta]}[\alpha_0 \pm \beta]\prod_{i=0}^{3}[b-a_i], \quad \beta = b + \tfrac{1}{2}(a_0-a_1-a_2-a_3+\delta).$$

• The difference operator $L(z, T_z; \boldsymbol{a}, b, u)$ constitutes a 6-parameter subfamily of Ruijsenaars-van Diejen operators of type BC_1. In the trigonometric case, these difference equations recover the q-difference equations for Askey-Wilson polynomials in the limit "$b \to 0$".

• Spectral duality

Regarding a_0 as a distinguished parameter, for the parameters $(\boldsymbol{a}, b) = (a_0, a_1, a_2, a_3, b)$ we define the dual parameters $(\boldsymbol{\alpha}, \beta) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta)$ by

$$\begin{aligned} \alpha_0 &= \tfrac{1}{2}(a_0 + a_1 + a_2 + a_3 - \delta), \\ \alpha_1 &= \tfrac{1}{2}(a_0 + a_1 - a_2 - a_3 + \delta), \\ \alpha_2 &= \tfrac{1}{2}(a_0 - a_1 + a_2 - a_3 + \delta), \\ \alpha_3 &= \tfrac{1}{2}(a_0 - a_1 - a_2 + a_3 + \delta), \end{aligned} \qquad \beta = b + \tfrac{1}{2}(a_0 - a_1 - a_2 - a_3 + \delta).$$

Note that $a_0 + a_i = \alpha_0 + \alpha_i$ $(i = 1, 2, 3)$ and $a_0 + b = \alpha_0 + \beta$. Then the *elliptic Askey-Wilson functions* $\Phi_l(z; \boldsymbol{a}, b)$ $(l = 0, 1, 2, \ldots)$ satisfy the spectral duality:

$$\Phi_l(a_0 + k\delta; \boldsymbol{a}, b) = \Phi_k(\alpha_0 + l\delta; \boldsymbol{\alpha}, \beta) \qquad (k, l \in \mathbb{N}).$$

Through this spectral duality, the difference equation for $\Phi_l(z; \boldsymbol{a}, b)$ mentioned above are translated into three term recurrence relation for $\Phi_l(z; \boldsymbol{a}, b)$.

• Symmetry with respect to $\boldsymbol{a} = (\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3})$

The elliptic functions

$$\frac{[2\alpha_0 - a_0 - b + \delta]}{[a_0 + b]_l} \prod_{i=1}^{3} [a_0 + a_i]_l \; \Phi_l(z; \boldsymbol{a}, b) \quad (l = 0, 1, 2, \ldots)$$

are symmetric with respect to $\boldsymbol{a} = (a_0, a_1, a_2, a_3)$. This fact is equivalent to the elliptic Bailey transformation formula for ${}_{12}V_{11}$.

4 Elliptic Schur functions with two parameters

• Schur functions associated with a sequence of functions

Let $f_k(z)$ $(k = 0, 1, 2, \ldots)$ be a sequence of functions in one variable z. Considering $f_k(z)$ as being "of degree k", for each partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $l(\lambda) \le m$, we define symmetric function $S^{(m)}_\lambda(x; \boldsymbol{f})$ in m variables $x = (x_1, \ldots, x_m)$ by the Weyl formula:

$$S^{(m)}_\lambda(x; \boldsymbol{f}) = \frac{\det\big(f_{m-j+\lambda_j}(x_i)\big)_{i,j=1}^m}{\det\big(f_{m-j}(x_i)\big)_{i,j=1}^m}$$

assuming that the Weyl denominator is nonzero. This function $S^{(m)}_\lambda(x) = S^{(m)}_\lambda(x; \boldsymbol{f})$ is called the *Schur function associated with the sequence of functions* $\boldsymbol{f} = (f_k)_{k\ge 0}$ (Macdonald's ninth variation). It is known that this class of Schur functions carry many nice properties including

- Weyl formula (definition)
- Jacobi-Trudi formula: $S^{(m)}_\lambda(x) = \det\big(h^{(m-j+1)}_{\lambda_i-i+j}(x)\big)_{i,j=1}^m$ $\quad S^{(m)}_\lambda(x) = \det\big(e^{(m+j-1)}_{\lambda'_i-i+j}(x)\big)_{i,j=1}^m$
- Giambelli formula: $S^{(m)}_\lambda(x) = \det\big(h^{(m)}_{p_i,q_j}(x)\big)_{i,j=1}^r$
- Dual Cauchy formula: $\Psi^{(m,n)}(x, y) = \displaystyle\sum_{\lambda\subseteq(n^m)} (-1)^{|\lambda|} S^{(m)}_\lambda(x) S^{(n)}_{\lambda^*}(y)$
- Tableau representation: $S^{(m)}_\lambda(x) = \displaystyle\sum_{T\in\mathrm{SSTab}_m(\lambda)} w^{(m)}_T(x)$ (with weights appropriately defined)

• Elliptic Schur functions with parameters (a, b)

As the sequence of reference functions, we take

$$f_k(z;a,b) = \frac{[a \pm z]_k}{[b \pm z]_k} = \frac{[a+z]_k[a-z]_k}{[b+z]_k[b-z]_k} \qquad (k = 0, 1, 2, \ldots)$$

and denote by

$$S_\lambda^{(m)}(x;a,b|\delta) = \frac{\det\left(f_{m-j+\lambda_j}(x_i)\right)_{i,j=1}^m}{\det\left(f_{m-j}(x_i)\right)_{i,j=1}^m} = \frac{\det\left(\dfrac{[a \pm x_i]_{m-j+\lambda_j}}{[b \pm x_i]_{m-j+\lambda_j}}\right)_{i,j=1}^m}{\det\left(\dfrac{[a \pm x_i]_{m-j}}{[b \pm x_i]_{m-j}}\right)_{i,j=1}^m}$$

the associated Schur functions in m variables $x = (x_1, \ldots, x_m)$. The denominator in this case factorizes thanks to Warnaar's elliptic version of the *Krattenthaler determinant formula*:

$$\det\left(\frac{[a \pm x_i]_{m-j}}{[b \pm x_i]_{m-j}}\right)_{i,j=1}^m = \frac{\prod_{1 \le i < j \le m}[x_i \pm x_j] \prod_{j=1}^{m-1}[a-b]_j[a+b+(j-1)\delta]_j}{\prod_{i=1}^m [b \pm x_i]_{m-1}}$$

This clss of Schur functions is a special case of elliptic interpolation functions due to Coskun-Gustafson and Rains.

• Recursion on the number of variables

$$S_\lambda^{(m)}(x_1,\ldots,x_m;a,b)=\sum_{\mu\subseteq\lambda}S_\mu^{(m-1)}(x_1,\ldots,x_{m-1};a,b+\delta)\,\psi_{\lambda/\mu}^{(m)}(a,b)\,f_{\lambda/\mu}^{(m)}(x_m;a,b)$$

The summation is taken over all horizontal strips λ/μ with $l(\mu)\le m-1$.

$$f_{\lambda/\mu}^{(m)}(z;a,b)=\prod_{i\ge1}\frac{[a+(m-i)\delta\pm z]_{\lambda_i}}{[a+(m-i)\delta\pm z]_{\mu_i}}\frac{[b+(m-i-1)\delta\pm z]_{\mu_i}}{[b+(m-i)\delta\pm z]_{\lambda_i}}$$

$$\psi_{\lambda/\mu}^{(m)}(a,b)=\prod_{i\ge1}\frac{[a+b+2(m-i-1+\mu_i)\delta]}{[a+b+2(m-i-1+\lambda_i)\delta]}$$

• Tableau representation

$$S_\lambda^{(m)}(x;a,b)=\sum_{\phi=\lambda^{(0)}\subseteq\lambda^{(1)}\subseteq\cdots\subseteq\lambda^{(m)}=\lambda}\;\prod_{1\le i\le j\le m}\frac{[a+b+(m+j-2i-1+2\lambda_i^{(j)})\delta]}{[a+b+(m+j-2i-1)\delta]}$$

$$\cdot\prod_{1\le i\le j\le m}\frac{[a+(j-i+\lambda_i^{(j-1)})\delta\pm x_j]_{\lambda_i^{(j)}-\lambda_i^{(j+1)}}[b+(m-i-1)\delta\pm x_j]}{[b+(m-i-1+\lambda_i^{(j-1)})\delta\pm x_j]_{\lambda_i^{(j)}-\lambda_i^{(j+1)}+1}}$$

• Single rows and single columns

$$h_l^{(m)}(x;a,b) = \sum_{\nu_1+\cdots+\nu_m=l} \prod_{j=1}^{m} \frac{[a+b+(m-j-3+2\nu_{\le j})\delta]}{[a+b+(m-j-3)\delta]} \cdot \prod_{j=1}^{m} \frac{[a+(j-1+\nu_{<j})\delta \pm x_j]_{\nu_j}[b+(m-2)\delta \pm x_j]}{[b+(m-2+\nu_{<j})\delta \pm x_j]_{\nu_j+1}}$$

$$e_r^{(m)}(x;a,b) = \sum_{1\le k_1<\cdots<k_r\le m} \prod_{i=1}^{r} \frac{[a+b+(2m-2i-1)\delta]_2}{[a+b+(m+k_i-2i-1)\delta]_2} \frac{[a+(k_i-i)\delta \pm x_{k_i}]}{[b+(m-1)\delta \pm x_{k_i}]} \cdot \prod_{i=1}^{r} \prod_{k_i<j<k_{i+1}} \frac{[b+(m-i-1)\delta \pm x_j]}{[b+(m-1)\delta \pm x_j]}$$

• Dual Cauchy formula

For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$,

$$\sum_{\mu \subseteq (n^m)} (-1)^{|\mu|} S_{\mu}^{(m)}(x; a, b)\, S_{\mu^*}^{(n)}(y; a, b) = \Psi^{(m,n)}(x, y; a, b),$$

where $\mu^* = (m - \mu'_n, \ldots, m - \mu'_1) \subseteq (m^n)$ and

$$\Psi^{(m,n)}(x, y; a, b) = \frac{C^{(m,n)}(a, b) \prod_{i=1}^{m} \prod_{j=1}^{n} [y_j \pm x_i]}{\prod_{i=1}^{m} [b + (m-1)\delta \pm x_i]_n \prod_{j=1}^{n} [b + (n-1)\delta \pm y_j]_m}.$$

• Principal Specialization: Hook length formula

The value at $c + \rho\,\delta = (c + (m-1)\delta, c + (m-2)\delta, \ldots, c)$ is evaluated explicitly as

$$S_{\lambda}^{(m)}(c + \rho\,\delta; a, b) = \prod_{1 \le i < j \le m} \frac{[(j - i + \lambda_i - \lambda_j)\delta]}{[(j-i)\delta]} \frac{[a + b + (2m - i - j - 1 + \lambda_i + \lambda_j)\delta]}{[a + b + (2m - i - j - 1)\delta]}$$
$$\cdot \prod_{i=1}^{m} \frac{[a + c + (m-i)\delta]_{\lambda_i} [a - c + (1-i)\delta]_{\lambda_i}}{[b - c + (m-i)\delta]_{\lambda_i} [b + c + (2m - i - 1)\delta]_{\lambda_i}}$$

$\Longrightarrow$ Summation formulas that generalize Frenkel-Turaev sum

The parameter c can be taken arbitrarily, in contrast to the general case of elliptic interpolation functions.

5 Schur functions associated with elliptic Askey-Wilson functions

We now take our elliptic Askey-Wilson functions

$$\begin{aligned}\Phi_l(z;\boldsymbol{a},b) &= {}_{12}V_{11}(a_0+b-\delta;\, a_0+z, a_0-z, 2\alpha_0+k\delta, -l\delta, b-a_1, b-a_2, b-a_3)\\ &= \sum_{k=0}^{l}\frac{[a_0\pm z]_k}{[b\pm z]_k}\frac{[a_0+b+(2k-1)\delta]}{[a_0+b-\delta]}\frac{[a_0+b-\delta]_k[2\alpha_0+l\delta]_k[-l\delta]_k}{[\delta]_k[a_0+b-2\alpha_0-l\delta]_k[a_0+b+l\delta]_k}\prod_{i=1}^{3}\frac{[b-a_i]_k}{[a_0+a_i]_k}.\end{aligned}$$

with parameters $(\boldsymbol{a},b)=(a_0,a_1,a_2,a_3,b)$ for the reference functions $f_l(z)$ $(l=0,1,2,\ldots)$, and consider the associated Schur functions (multivariable elliptic Askey-Wilson functions)

$$\Phi_\lambda^{(m)}(x;\boldsymbol{a},b)=\frac{\det\left(\Phi_{m-j+\lambda_j}(x_i;\boldsymbol{a},b)\right)_{i,j=1}^{m}}{\det\left(\Phi_{m-j}(x_i;\boldsymbol{a},b)\right)_{i,j=1}^{m}}$$

in m variables $x=(x_1,\ldots,x_m)$; the denominator factorizes similarly to the case of $S_\lambda^{(m)}(x;a,b)$.

• Difference equation

The relevant difference operator, involving an extra parameter u, is expressed as follows:

$$L^{(m)}(x,T_x;\boldsymbol{a},b,u)=\sum_{\epsilon:\{1,\dots,m\}\to\{\pm1,0\}}\prod_{i=1}^{m}A^{\epsilon_i}(x_i;\boldsymbol{a},b,u)\prod_{1\le i<j\le m}\frac{[(x_i+\epsilon_i\delta)\pm(x_j+\epsilon_j)\delta]}{[x_i\pm x_j]}\prod_{i=1}^{m}T_{x_i}^{\epsilon_i\delta}$$

where

$$A^{+}(z;\boldsymbol{a},b,u)=\frac{[z-b-(m-1)\delta][z-b-(m-2)\delta][z+b-\alpha_0\pm u]\prod_{i=0}^{3}[z+a_i]}{[2z][2z+\delta]}$$

$$A^{-}(z;\boldsymbol{a},b,u)=A^{+}(-z;\boldsymbol{a},b,u)$$

$$A^{0}(z;\boldsymbol{a},b,u)=\frac{1}{2}\sum_{r=0}^{3}e(-(\tfrac{1}{2}(\omega_r-\delta)+b)\eta_r)[\tfrac{1}{2}(\omega_r-\delta)+b-\alpha_0\pm u]$$

$$\cdot\prod_{i=0}^{3}[\tfrac{1}{2}(\omega_r-\delta)+a_i]\cdot\frac{[b+(m-1)\delta\pm z]}{[\tfrac{1}{2}(\omega_r-\delta)\pm z]}.$$

- Difference equation (continued)

$$L^{(m)}(x, T_x; \boldsymbol{a}, b, u) = \sum_{\epsilon:\{1,\dots,m\}\to\{\pm 1,0\}} \prod_{i=1}^{m} A^{\epsilon_i}(x_i; \boldsymbol{a}, b, u) \prod_{1\le i<j\le m} \frac{[(x_i+\epsilon_i\delta) \pm (x_j+\epsilon_j)\delta]}{[x_i \pm x_j]} \prod_{i=1}^{m} T_{x_i}^{\epsilon_i\delta}$$

For any partition λ with $l(\lambda) \le m$, the Schur function $\Phi_\lambda^{(m)}(x; \boldsymbol{a}, b)$ associated with the elliptic Askey-Wilson functions satisfies the difference equation

$$\begin{aligned} &L^{(m)}(x, T_x; \boldsymbol{a}, b, u)\Phi_\lambda^{(m)}(x; \boldsymbol{a}, b) \\ &= \Phi_\lambda^{(m)}(x; \boldsymbol{a}, b+\delta)\, [\alpha_0 \pm \beta]_m \prod_{i=0}^{3} [b-a]_m \prod_{j=1}^{m} \frac{[\alpha_0 + (m-j+\lambda_j)\delta \pm u]}{[\alpha_0 + (m-j+\lambda_j)\delta \pm \beta]}. \end{aligned}$$

Our $\Phi^{(m)}(x; \boldsymbol{a}, b)$ is an *almost joint eigenfunction* for the family of difference operators $L^{(m)}(x, T_x; \boldsymbol{a}, b, u)$. In the trigonometeric case, this recovers the q-difference equation for the whole family of van Diejen's operators with $t = q$.

• Principal specialization

The value of $\Phi_\lambda^{(m)}(x;\boldsymbol{a},b)$ at $a_0+\rho\delta=(a_0+(m-1)\delta,a_0+(m-1)\delta,\ldots,a_0)$ is determined explicitly as

$$\Phi_\lambda^{(m)}(a_0+\rho\,\delta;\boldsymbol{a},b)=\prod_{1\le i<j\le m}\frac{[2\alpha_0+(2m-2i+\lambda_i+\lambda_j)\delta]}{[2\alpha_0+(2m-2i)\delta]}\frac{[(j-i+\lambda_i-\lambda_j)\delta]}{[(j-i)\delta]}\cdot\prod_{i=1}^{m}\frac{[\alpha_0+\beta+(m-i)\delta]_{\lambda_i}}{[\alpha_0+\beta+(2m-i-1)\delta]_{\lambda_i}}\frac{[\alpha_0-\beta+(2-i)\delta]_{\lambda_i}}{[\alpha_0-\beta+(m-i+1)\delta]_{\lambda_i}}.$$

• Spectral duality

We renormalize $\Phi_\lambda^{(m)}(x;\boldsymbol{a},b)$ by setting

$$\widetilde{\Phi}_\lambda^{(m)}(x;\boldsymbol{a},b)=\frac{\Phi_\lambda^{(m)}(x;\boldsymbol{a},b)}{\Phi_\lambda^{(m)}(a_0+\rho\,\delta;\boldsymbol{a},b)};\qquad \widetilde{\Phi}_\lambda^{(m)}(a_0+\rho\delta;\boldsymbol{a},b)=1.$$

Then $\widetilde{\Phi}_\lambda^{(m)}(x;\boldsymbol{a},b)$ satisfies the spectral duality

$$\widetilde{\Phi}_\lambda^{(m)}(a_0+(\rho+\mu)\delta;\boldsymbol{a},b)=\widetilde{\Phi}_\mu^{(m)}(\alpha_0+(\rho+\lambda)\delta;\boldsymbol{\alpha},\beta)\tag{5.1}$$

for any partition λ, μ with $l(\lambda)\le m$, $l(\mu)\le m$.

• Binomial formula

$\Phi_\lambda^{(m)}(x;\boldsymbol{a},b)$ is expanded as follows in terms of the elliptic Schur functions $S_\mu^{(m)}(x;a_0,b)$:

$$\widetilde{\Phi}_\lambda^{(m)}(x;\boldsymbol{a},b)=\sum_{\mu\subseteq\lambda} d_\mu^{(m)}(\boldsymbol{a},b)\, S_\mu^{(m)}(\alpha_0+(\rho+\lambda)\delta;\alpha_0,\beta)\, S_\mu^{(m)}(x;a_0,b),$$

$$d_\mu^{(m)}(\boldsymbol{a},b)=\prod_{j=1}^{m}\frac{[a+b+(2m-2j-1+2\mu_j)\delta]}{[a+b+(2m-2j-1)\delta]}\frac{[a+b+(m-j-1)\delta]_{\mu_j}}{[(m-j+1)\delta]_{\mu_j}}$$
$$\cdot\prod_{j=1}^{m}\prod_{i=1}^{3}\frac{[b-a_i+(m-j)\delta]_{\mu_j}}{[a_0+a_i+(m-j)\delta]_{\mu_j}}.$$

For rectangular partitions $\lambda=(n^m)$ $(n=0,1,2,\ldots)$, the coefficients $S_\mu^{(m)}(\alpha_0+n\delta+\rho\delta;\alpha_0,\beta)$ are explicitly evaluated in factorized forms (principal specialization of elliptic Schur functions). Hence the multivariable elliptic Askey-Wilson function $\Phi_{(n^m)}^{(m)}(x;\boldsymbol{a},b)$ $(n=0,1,2,\ldots)$ is expressed as a linear combination of elliptic Schur functions with factorized coefficients.

• Rectangular cases : Hypergeometric functions

$$\widetilde{\Phi}^{(m)}_{(n^m)}(x;\boldsymbol{a},b)=\sum_{\mu\subseteq(n^m)}c^{(m,n)}_{\mu}(\boldsymbol{a},b)\,S^{(m)}_{\mu}(x;\boldsymbol{a},b)\qquad(n=0,1,2,\ldots)$$

$$\begin{aligned}
&c^{(m,n)}_{\mu}(\boldsymbol{a},b)\\
&=\prod_{1\le i<j\le m}\frac{[a_0+b+(2m-i-j-1+\mu_i+\mu_j)\delta]}{[a_0+b+(2m-i-j-1)\delta]}\frac{[(j-i+\mu_i-\mu_j)\delta]}{[(j-i)\delta]}\\
&\quad\cdot\prod_{i=1}^{m}\frac{[a_0+b+(2m-2i-1+2\mu_i)\delta]}{[a_0+b+(2m-2i-1)\delta]}\frac{[a_0+b+(m-i-1)\delta]_{\mu_i}}{[(m-i+1)\delta]_{\mu_i}}\\
&\quad\cdot\prod_{i=1}^{m}\frac{[a_0+a_1+a_2+a_3+(m-i-1+n)\delta]_{\mu_i}}{[b-a_1-a_2-a_3+(m-i+1-n)\delta]_{\mu_i}}\frac{[(1-i-n)\delta]_{\mu_i}}{[a_0+b+(2m-i-1+n)\delta]_{\mu_i}}\\
&\quad\cdot\prod_{i=1}^{m}\frac{[b-a_1+(m-i)\delta]_{\mu_i}}{[a_0+a_1+(m-i)\delta]_{\mu_i}}\frac{[b-a_2+(m-i)\delta]_{\mu_i}}{[a_0+a_2+(m-i)\delta]_{\mu_i}}\frac{[b-a_3+(m-i)\delta]_{\mu_i}}{[a_0+a_3+(m-i)\delta]_{\mu_i}}.
\end{aligned}$$

This expression can be regarded as a multiple extension of very well-poised elliptic hypergeometric series representing the elliptic Askey-Wilson function:

$$\Phi_n(z;\boldsymbol{a},b)={}_{12}V_{11}(a_0+b-\delta;a_0+z,a_0-z,2\alpha_0+n\delta,-n\delta,b-a_1,b-a_2,b-a_3).$$