Journal of Difference Equations and Applications, 2014
http://dx.doi.org/10.1080/10236198.2014.924930

A characterization of an Askey–Wilson difference equation

Daniel J. Galiffa* and Boon W. Ong

Department of Mathematics, Penn State Erie, The Behrend College, Erie, PA 16563, USA

(*Received 8 March 2013; accepted 12 May 2014*)

We determine the q-orthogonal polynomial solutions to the difference equation $\mathcal{D}_q P_n(x) = \gamma_n P_{n-1}(x)$, where $\mathcal{D}_q$ is the Askey–Wilson divided-difference operator, using an approach that does not appear in the literature. To accomplish this, we construct a polynomial expansion via a Chebyshev basis, which ultimately allows explicit formulas to be derived for the recurrence coefficients of $P_n(x)$ above. From there, we obtain our solutions and discuss some future research.

Keywords: Askey–Wilson operator; Chebyshev polynomials; q-difference equations; q-orthogonal polynomials; q-series

AMS Subject Classification: 33D45; 33C47; 42C05

1. Introduction

In this paper, we determine all of the q-orthogonal polynomial solutions $\{P_n(x)\}_{n=0}^{\infty}$ to the difference equation:

$$\mathcal{D}_q P_n(x) = \gamma_n P_{n-1}(x), \tag{1}$$

where γ_n is a function of n that is independent of x and $\mathcal{D}_q$ is the Askey–Wilson degree-lowering, divided-difference, linear operator defined by

$$\mathcal{D}_q f(x) := \frac{\breve{f}\left(q^{1/2}z\right) - \breve{f}\left(q^{-(1/2)}z\right)}{\breve{e}\left(q^{1/2}z\right) - \breve{e}\left(q^{-(1/2)}z\right)}, \tag{2}$$

with $z = \mathrm{e}^{i\theta}, \breve{f}(z) = f(x) = f(\cos\theta)$, for any function f and $e(x) = x$. For details regarding the development and vast applicability of this operator, consider [1,3,4,11], as well as [8] and the additional references therein.

For the remainder of this section, we first discuss the motivations behind analysing (1) and then outline the details of this paper.

In [2], all classical orthogonal polynomial solutions were obtained from the differential equation

$$\pi(x)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \quad n = 1, 2, 3, \ldots, \tag{3}$$

where $\pi(x)$ is a polynomial of degree at most 2. A unified derivation of (3) appears in [7, p. 167], and is accredited to Tricomi. The paper [2] describes all sequences of monic orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ that solve (3) and satisfy a three-term recurrence

*Corresponding author. Email: djg34@psu.edu

relation (a necessary and sufficient condition for orthogonality) of the form

$$\begin{aligned} P_{n+1}(x) &= (x+B_n)P_n(x) - C_nP_{n-1}(x), \quad n=0,1,2,\ldots, \\ P_{-1}(x) &= 0, \quad P_0(x)=1, \quad C_n \neq 0, \quad n=1,2,3,\ldots. \end{aligned} \tag{4}$$

Since $\pi(x) = ax^2 + bx + c$ is at most quadratic, (3) only needed to be analysed for the following cases:

$$\pi(x) = \begin{cases} 1, \\ x, \\ x^2 + c. \end{cases} \tag{5}$$

Any other form of $\pi(x)$ can be achieved via a linear change-of-variables.

The methodology used to obtain the orthogonal polynomial solutions for each case is essentially as follows. First, differentiate (4) and multiply the result by $\pi(x)$. Then, use (3) to eliminate $\pi(x)P'_{n-1}(x)$, $\pi(x)P'_n(x)$ and $\pi(x)P'_{n+1}(x)$. From there, utilize (4) to remove $P_{n+1}(x)$. This results in an equation relating the coefficients of $\pi(x)$ and recursion coefficients in (3) and (4). Therefore, several difference equations can be obtained from which expressions for the recursion coefficients B_n and C_n can be determined. These expressions contain arbitrary parameters, which when chosen judiciously lead to the sought-after classical orthogonal polynomial solutions.

Upon completing this for each case of $\pi(x)$ in (5), which required a wealth of clever algebraic manipulations, it was determined that the Hermite, Laguerre and Jacobi polynomials are, respectively, the only orthogonal polynomial solutions. For each of these polynomial solutions, orthogonality was defined on the real line with respect to a non-decreasing real function. When considering polynomials orthogonal on the real line with respect to a function of bounded variation, the generalized Bessel polynomials [12] were also solutions in the limiting case $c \to 0$ for $\pi(x) = x^2 + c$.

In 2006, Datta and Griffin [6] discovered all q-orthogonal polynomial solutions to the difference equation

$$\pi(x)D_qP_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \tag{6}$$

where $\pi(x)$ is a polynomial of degree at most 2, with the q-degree-lowering, divided-difference, linear operator D_q, cf. [8], defined as

$$(D_q f)(x) := \frac{f(x) - f(qx)}{x - qx}.$$

Their work was the q-analogue of [2] because the differential operator $\mathrm{d}/\mathrm{d}x$ in (3) was replaced by D_q above. Datta and Griffin determined all of the q-orthogonal polynomial solutions to (6) using essentially the same methodology as in [2], taking into account that (6) does not remain invariant under the linear transformation $x \to ax + b$. Therefore, the following cases and sub-cases were considered:

$$\pi(x) = \begin{cases} 1, \\ x, \quad x + c, \\ x^2, \quad x^2 + s, \quad x^2 + rx, \quad x^2 + rx + s. \end{cases} \tag{7}$$

The Al-Salam–Carlitz I, the discrete q-Hermite I, the big and little q-Laguerre polynomials and the big and little q-Jacobi polynomials were obtained – the q-Bessel polynomials were achieved by taking appropriate limits.

We also mention that the recently submitted manuscript 'q-Orthogonal polynomial solutions to a class of difference equations', by D.J. Galiffa and S.J. Johnston, presents an analysis of the difference equation

$$\pi(x)D_{q^{-1}}P_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x)$$

with

$$(D_{q^{-1}}f)(x) := \frac{f(x) - f(x/q)}{x - x/q}.$$

In this work, the authors obtained the Al-Salam–Carlitz II, the discrete q-Hermite II, the q-Laguerre and the Stieltjes–Wigert polynomials, as well as q-orthogonal polynomials that are currently not fully characterized, as solutions.

Indeed, much analysis has been conducted with regard to determining the classical and quantum orthogonal polynomial solutions to the equation

$$\pi(x)T(P_n(x)) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \quad n = 1, 2, 3, \ldots \tag{8}$$

for the operators $T = \mathrm{d}/\mathrm{d}x, D_q, D_{q^{-1}}$, but not for the case when $T = \mathcal{D}_q$. Thus, our main motivation for studying (1) is to develop a method analogous to those utilized in [2,6] in order to conduct a preliminary analysis of (8), with $T = \mathcal{D}_q$. Namely, we consider Case 1 of (7) ($\pi(x) = 1$), from which it follows that $\alpha_n = \beta_n = 0$ and hence (1). Cases 2 and 3 (with $T = \mathcal{D}_q$) do not appear in the literature.

Next, we mention that characterizing Case 1 of [2] is actually equivalent to determining which Appell sets are also orthogonal. Similarly, characterizing Case 1 of [6] is the same as determining the q-Appell orthogonal sets. Therefore, the secondary motivation for this paper is to determine the $\mathcal{D}_q$-Appell orthogonal sets in a way that is entirely different from what has been established previously, i.e. [1,9].

Lastly, it is also important to address the following. There are two types of theorems in this area: direct theorems and inverse theorems. For the classical orthogonal polynomials, the direct theorems are illustrated by formula (3). For the Jacobi polynomials, (3) follows from the contiguous relations discovered by Gauss (with earlier relations due to Euler and a few others). The inverse problems are usually restricted to orthogonal polynomials, but some are likely to hold in a more general context.

This paper is organized as follows. In Section 2, we alleviate the difficulty of directly applying (2) to (1) by expanding our polynomials via a Chebyshev basis. These expansions are needed for all of our subsequent results. We conclude Section 2 with a few preliminary definitions that we use throughout this work. The crux of this paper is presented in Section 3, where we develop explicit conditions that the recursion coefficients in (4) must satisfy in order for $P_n(x)$ to solve (1). Finally, in Section 4 we determine all of our q-orthogonal polynomial solutions and discuss some future research problems.

2. Expansion via Chebyshev basis polynomials

From the orthogonality relation of the Chebyshev polynomials of the first kind $\{T_n(x)\}_{n=0}^{\infty}$, see [10], we form an orthogonal basis via the weighted inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,\mathrm{d}x$$

with $w(x) = (1-x^2)^{-1/2}$. Thus, for a given set of monic orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$, we have

$$P_n(x) = \sum_{k=0}^{n} f_{n,k} T_k(x), \quad \text{with } f_{n,k} = \frac{\langle P_n, T_k\rangle}{\langle T_k, T_k\rangle}. \tag{9}$$

From (4), we expect $f_{n,k}, B_n$ and C_n to be related. In this regard, we have the following statement.

Lemma 2.1. *For $n = 2, 3, \ldots$, with $f_{n,k}$ as in (9), we have*

$$f_{n+1,n} = \frac{1}{2} f_{n,n-1} + B_n f_{n,n}. \tag{10}$$

For $n = 3, 4, \ldots$, we also have

$$f_{n+1,n-1} = \frac{1}{2}\left(f_{n,n} + f_{n,n-2}\right) + B_n f_{n,n-1} - C_n f_{n-1,n-1} \tag{11}$$

with

$$f_{n,n} = \frac{1}{2^{n-1}} \quad \text{for } n = 1, 2, \ldots. \tag{12}$$

Proof. The Chebyshev polynomials of the first kind satisfy the three-term recurrence relation

$$xT_k(x) = \frac{1}{2} T_{k+1}(x) + \frac{1}{2} T_{k-1}(x) \quad \text{for } k = 1, 2, 3, \ldots.$$

By substituting (9) into (4) and using the above relation for the $xT_k(x)$ term, our result follows from comparing coefficients. □

Next, we discuss a few relationships that are used throughout this paper. We let $\{U_n(x)\}_{n=0}^{\infty}$ be the Chebyshev polynomials of the second kind [10] and recall that the identity function is denoted as $e(x) = x$ via (2). We then see that $\mathcal{D}_q(e) = \mathcal{D}_q(T_1) = 1$. Lastly, we make much use of the definitions

$$\nu_n := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad \text{and} \quad \mu_n := q^{n/2} + q^{-n/2}$$

from which it follows that

$$\mathcal{D}_q(T_k(x)) = \nu_k U_{k-1}(x). \tag{13}$$

Remark 1. It is worth mentioning that μ_k and ν_k are important structures that appear elsewhere in the literature. For example, in [5], $[n]_q$ (called the q-numbers (in symmetric form)) was used to denote our ν_n and $\alpha_q(n)$ was used for our $(1/2)\mu_n$ term. □

3. The recursion coefficients

We begin by obtaining expressions for the recursion coefficients B_n and C_n in terms of the Fourier coefficients as in (9).

LEMMA 3.1. *For* $\{P_n(x)\}_{n=0}^{\infty}$ *to satisfy* (1) *and* (4) *for* $n = 2, 3, 4, \ldots$, *the recursion coefficients* B_n *and* C_n *must satisfy*

$$B_n = \frac{2^{n-2}(\nu_{n+1} - \nu_n)}{\nu_n} f_{n,n-1}, \quad \frac{C_n}{2^{n-2}} = \frac{1}{2^n} + \frac{1}{2} f_{n,n-2} + B_n f_{n,n-1} - f_{n+1,n-1}.$$

Proof. Using (1) and (9) we see that

$$\mathcal{D}_q\left(\sum_{k=0}^{n} f_{n,k} T_k(x)\right) = \gamma_n \sum_{k=0}^{n-1} f_{n-1,k} T_k(x).$$

Applying (13) to the left-hand side of the above expression and using the relation

$$T_k(x) = \frac{1}{2} U_k(x) - \frac{1}{2} U_{k-2}(x), \quad k = 1, 2, 3, \ldots$$

on the right-hand side, we obtain

$$\sum_{k=0}^{n} f_{n,k} \nu_k U_{k-1}(x) = \gamma_n f_{n-1,0} T_0(x) + \frac{1}{2} \gamma_n \sum_{k=1}^{n-1} f_{n-1,k}(U_k(x) - U_{k-2}(x)). \tag{14}$$

Upon comparing the coefficients of $U_{n-1}(x)$ above, we see that

$$f_{n,n} \nu_n = \frac{1}{2} \gamma_n f_{n-1,n-1},$$

which from (12) implies that

$$\gamma_n = \nu_n. \tag{15}$$

Similarly, by comparing coefficients of $U_{n-2}(x)$, we obtain

$$f_{n,n-1} \nu_{n-1} = \frac{1}{2} \gamma_n f_{n-1,n-2}, \quad n = 3, 4, 5, \ldots.$$

From (15), we have

$$f_{n+1,n} = \frac{\nu_{n+1}}{2\nu_n} f_{n,n-1}, \quad n = 2, 3, 4, \ldots. \tag{16}$$

Substituting (16) into (10), we obtain B_n. The expression for C_n is obtained from the expression for B_n and (11). □

We can now derive explicit forms for our recursion coefficients.

Theorem 3.2. *The recursion coefficients as in Lemma* 3.1 *have the following forms:*

$$B_n = (\nu_{n+1} - \nu_n)B_0, \tag{17}$$

which is valid for $n = 1, 2, 3, \ldots$ *and*

$$C_n = \frac{1}{4}\left[1 + \nu_n(\nu_{n+1} - \nu_n)B_0^2 - \nu_{2n}\left(\frac{4G_3}{\nu_2\nu_3} - \sum_{k=2}^{n-2}\frac{1}{\nu_k\nu_{k+1}}\right) + \frac{\nu_{n+1}}{\nu_{n-1}}\right], \tag{18}$$

with $G_3 = \nu_3(B_0B_1 - C_1 + 1/4)$*, which is valid for* $n = 4, 5, 6, \ldots$.

Proof. From $P_1(x)$ in (4) and the definition of $f_{1,0}$, it is clear that $B_0 = f_{1,0}$. For ease of notation, we define $F_n := f_{n,n-1}$ and we see that $F_1 = f_{1,0} = B_0$. Then, from (16), we have the first-order difference equation

$$F_n = \frac{\nu_n}{2\nu_{n-1}}F_{n-1} \quad \text{for } n = 3, 4, 5, \ldots,$$

which has the solution

$$F_n = \frac{\nu_n}{2^{n-2}\nu_2}F_2. \tag{19}$$

Substituting this into (10), we have

$$\frac{\nu_{n+1}}{2^{n-1}\nu_2}F_2 = \frac{\nu_n}{2^{n-1}\nu_2}F_2 + \frac{1}{2^{n-1}}B_n.$$

From considering (1) for $n = 2$, it follows that

$$F_2 = \nu_2 B_0 \tag{20}$$

and therefore,

$$B_n = (\nu_{n+1} - \nu_n)B_0.$$

We next obtain C_n. Recalling (15), we compare the coefficients of $U_{n-3}(x)$ in (14) and obtain

$$f_{n,n-2}\nu_{n-2} = \frac{1}{2}\nu_n\left(f_{n-1,n-3} - f_{n-1,n-1}\right), \quad n = 4, 5, 6, \ldots. \tag{21}$$

Defining

$$G_n := f_{n,n-2}, \quad n = 2, 3, 4, \ldots$$

we achieve the first-order difference equation in G_n as follows:

$$G_n = \frac{\nu_n}{2\nu_{n-2}}\left(G_{n-1} - \frac{1}{2^{n-2}}\right), \quad n = 4, 5, 6, \ldots.$$

Upon iterating this result, we see that the solution is

$$G_n = \frac{\nu_n \nu_{n-1}}{2^{n-1}} \left(\frac{4G_3}{\nu_2 \nu_3} - \sum_{k=2}^{n-2} \frac{1}{\nu_k \nu_{k+1}} \right), \quad n = 4, 5, 6, \ldots. \tag{22}$$

Via Lemma 3.1, for $n = 4, 5, 6, \ldots$ we have

$$\frac{C_n}{2^{n-2}} = \frac{1}{2^n} + B_n F_n + \frac{1}{2} G_n - G_{n+1}.$$

After substituting our expressions for B_n in Lemma 3.1 and F_n in (19), we further obtain

$$\frac{C_n}{2^{n-2}} = \frac{1}{2^n} + \frac{\nu_n(\nu_{n+1} - \nu_n)}{2^{n-2} \nu_2^2} F_2^2 + \frac{1}{2} G_n - G_{n+1}, \quad n = 4, 5, 6, \ldots. \tag{23}$$

We next note that with the identity $\nu_{n+1} - \nu_{n-1} = \mu_n$, $(1/2)G_n - G_{n+1}$ can be further simplified to

$$-\frac{\nu_n \mu_n}{2^n} \left(\frac{4G_3}{\nu_2 \nu_3} - \sum_{k=2}^{n-2} \frac{1}{\nu_k \nu_{k+1}} \right) + \frac{\nu_{n+1}}{2^n \nu_{n-1}}. \tag{24}$$

Thus, we substitute (20) and (24) into (23), which leads to (18).

We now only need to determine the explicit form of G_3. For $n = 3$, we observe that (14) becomes

$$\sum_{k=0}^{3} f_{3,k} \nu_k U_{k-1}(x) = \gamma_3 f_{2,0} T_0 + \frac{1}{2} \gamma_3 \sum_{k=1}^{2} f_{2,k} (U_k(x) - U_{k-2}(x)).$$

Using (15) and $\nu_1 = T_0 = U_0 = 1$, we can compare the constant terms above to achieve

$$f_{3,1} = \nu_3 f_{2,0} - \frac{1}{2} \nu_3 f_{2,2},$$

which gives us the following expression for G_3:

$$G_3 = \nu_3 \left(G_2 - \frac{1}{4} \right).$$

We next derive G_2. By comparing the coefficients of $P_0(x)$, $P_1(x)$ and $P_2(x)$ in (4) and (9), we have

$$F_2 = B_0 + B_1, \quad f_{2,0} - f_{2,2} = B_0 B_1 - C_1.$$

Hence,

$$G_2 = f_{2,0} = B_0 B_1 - C_1 + \frac{1}{2}$$

and we see that

$$G_3 = \nu_3\left(B_0B_1 - C_1 + \frac{1}{4}\right).$$

□

4. Conclusion and future directions

In light of the analysis of Section 3, we have the following statement.

Theorem 4.1. *The continuous* q*-Hermite polynomials of Rogers are the only orthogonal set of polynomials which satisfy the difference equation*

$$\mathcal{D}_q P_n(x) = \gamma_n P_{n-1}(x).$$

Proof. From Theorem 3.2, we see that our recursion coefficients B_n and C_n contain three free parameters: B_0, B_1 and C_1. From Theorem 3.2, we also know that

$$B_n = (\nu_{n+1} - \nu_n)B_0, \quad n = 1, 2, 3, \ldots.$$

Thus, by choosing $B_0 = 0$ we immediately see that

$$B_n \equiv 0, \quad n = 0, 1, 2, \ldots. \tag{25}$$

Next, from selecting $C_1 = (1-q)/4$ and using (25), we observe that (18) becomes

$$C_n = \frac{1}{4}\left[1 - \nu_{2n}\left(\frac{q}{\nu_2} - \sum_{k=2}^{n-2}\frac{1}{\nu_k\nu_{k+1}}\right) + \frac{\nu_{n+1}}{\nu_{n-1}}\right].$$

Now notice that the above sum is telescoping:

$$\begin{aligned}\sum_{k=2}^{n-2}\frac{1}{\nu_k\nu_{k+1}} &= (1-q)\sum_{k=2}^{n-2}\left(\frac{q^{k-(1/2)}}{1-q^k} - \frac{q^{k+(1/2)}}{1-q^{k+1}}\right)\\ &= (1-q)\left(\frac{q^{3/2}}{1-q^2} - \frac{q^{n-3/2}}{1-q^{n-1}}\right).\end{aligned}$$

Taking this into account and using some algebra, we obtain

$$C_n = \frac{1}{4}(1-q^n). \tag{26}$$

Hence, (25) and (26) are the recursion coefficients for the normalized Rogers' q-Hermite polynomials. □

To summarize, Theorem 3.2 establishes the general recursion coefficients (in terms of the arbitrary parameters B_0 and C_1) that a polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ must satisfy in order to solve (1). Consequently, this theorem gives all of the q-orthogonal polynomials that satisfy (1). Later, in Theorem 4.1, we concretely select $B_0 = 0$ and $C_1 = (1-q)/4$, leading to a special case of the general recursion coefficients in Theorem 3.2. We do this because these selections lead to the standard normalized

Rogers' q-Hermite polynomials as they appear in contemporary literature, e.g. the Askey scheme [10].

With regard to future research, we leave open for consideration the determination of the q-orthogonal polynomial solutions to (8) (with $T = \mathcal{D}_q$) for Cases 2 and 3 of (7). As discussed in Section 1, these characterizations do not appear in the literature.

We lastly mention that the Wilson operator, $\mathcal{W}$, see p. 451 of [8], is defined as

$$\breve{f}(y) := f(x) \quad \text{for } x = y^2,$$
$$(\eta_{\pm} f)(x) := \breve{f}\left(y \pm \frac{i}{2}\right),$$
$$(\mathcal{W}f)(x) := \frac{1}{2yi}(\eta_+ f - \eta_- f)(x).$$

This degree-lowering, divided-difference, linear operator is connected with the Bethe Ansatz equations of the Heisenberg XXX spin chain in quantum mechanics – cf. Section 16.5 of [8] for additional details. In addition, $\mathcal{W}$ also solves the Sturm–Liouville problem of the form

$$\Pi(x)\mathcal{W}^2 f(x) + \Phi(x)(A\mathcal{W}f)(x) = r(x)f(x),$$

with

$$\Pi(x) = \frac{1}{w(x)} AP(x) \quad \text{and} \quad \Phi(x) = \frac{1}{w(x)} \mathcal{W}P(x),$$

where $w(x) > 0$ is the weight function corresponding to $\{P_n(x)\}_{n=0}^{\infty}$ and A is the averaging operator

$$(Af)(x) := \frac{1}{2}(\eta_+ f + \eta_- f)(x).$$

For polynomial solutions $f(x)$, it is assumed that $\Pi(x)$, $\Phi(x)$ and $r(x)$ are polynomials of degrees n, $n-1$ and $n-2$, respectively.

Thus, we also leave open for consideration the analysis of (8) with T replaced by $\mathcal{W}$, i.e.

$$\pi(x)\mathcal{W}P_n(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \quad n = 1, 2, 3, \ldots.$$

Acknowledgement

We thank the referee for the valuable feedback and suggestions.

References

[1] W.A. Al-Salam, *A characterization of the Rogers q-Hermite polynomials*, Int. J. Math. Math. Sci. 18 (1995), pp. 641–647.

[2] W.A. Al-Salam and T.S. Chihara, *Another characterization of the classical orthogonal polynomials*, SIAM J. Math. Anal. 3 (1972), pp. 65–70.

[3] R. Askey and J.A. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 54 (1985), no. 319, pp. iv+55.

[4] B.M. Brown, W.D. Evans, and M.E.H. Ismail, *The Askey–Wilson polynomials and q-Sturm–Liouville problems*, Math. Proc. Cambridge Philos. Soc. 119 (1996), pp. 1–16.

[5] R.S. Costas-Santos and F. Marcellán, *q-Classical orthogonal polynomials: A general difference calculus approach*, Acta Appl. Math. 111 (2010), pp. 107–128.

[6] S. Datta and J. Griffin, *A characterization of some q-orthogonal polynomials*, Ramanujan J. 12 (2006), pp. 425–437.

[7] A. Eldélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.

[8] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in one Variable*, Encyclopedia of Mathematics and Its Applications, Vol. 98, Cambridge University Press, Cambridge, 2005.

[9] M.E.H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Adv. Math. 109 (1994), pp. 1–33.

[10] R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, 1998, Reports of the Faculty of Technical Mathematics and Information, No. 98-17, Delft University of Technology. Available at http://aw.twi.tudelft.nl/~koekoek/askey/index.html

[11] A.P. Magnus, *Associated Askey–Wilson polynomials as Laguerre–Hahn orthogonal polynomials*, in *Orthogonal Polynomials and Their Applications*, M. Alfaro, et al., eds., Lecture Notes in Mathematics, Vol. 1329, Springer-Verlag, Berlin, 1988, pp. 261–278.

[12] E.D. Rainville, *Special Functions*, The Macmillan Co., New York, 1960.