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# New connection formulae for some $q$-orthogonal polynomials in $q$-Askey scheme

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**Abstract**

New nonlinear connection formulae of the $q$-orthogonal polynomials, such continuous $q$-Laguerre, continuous big $q$-Hermite, $q$-Meixner-Pollaczek and $q$-Gegenbauer polynomials, in terms of their respective classical analogues are obtained using a special realization of the $q$-exponential function as infinite multiplicative series of ordinary exponential function.

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## 1 Introduction and motivation

In modern mathematical physics, hypergeometric and $q$-hypergeometric functions have found their applications in the development of the theory of difference equations and in quantum and non commutative geometry. And in many results, like the theory of lattice integrable models, Bethe ansatz and Toda systems [1]-[3] for instance, they are formulated or realized in connection with these types of mathematical functions. In this context and to illustrate a physical application, we cite ref.[4] where a representation of $q$-hypergeometric functions of one variable was found in terms of correlators of vertex operators made out of free scalar fields propagating on Riemann sphere. Among these basic functions or $q$-functions, there are polynomials which are structured in schemes. An interesting one, so-called Askey-scheme [5] of hypergeometric orthogonal polynomials, consists of all known sets of orthogonal polynomials which can be defined in terms of a hypergeometric function and their interrelations. The $q$-Askey-scheme is the quantum version of the former, however the hypergeometric orthogonal polynomials may admit several $q$-analogues. Only few of these $q$-orthogonal polynomials possess generating functions written in terms of $q$-exponential functions. It is for this fact that we deal with these $q$-polynomials in this paper. Our interest here was motivated by the results of reference [6] where it was established that the two Jackson's $q$-exponentials

$$e_q(z) = \sum_{k\in\mathbb{N}} \frac{1}{(q;q)_k} z^k = \frac{1}{(z;q)_\infty}, \qquad E_q(z) = \sum_{k\in\mathbb{N}} \frac{q^{k(k-1)/2}}{(q;q)_k} z^k = (-z;q)_\infty, \tag{1.1}$$

with $(a;q)_0 = 1$, $(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j)$, and $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$, could be expressed respectively as the exponential of series as follows

$$e_q(z) = \exp\left(\sum_{k\in\mathbb{N}^*} \frac{z^k}{k(1-q^k)}\right) \tag{1.2}$$

and

$$E_q(z) = \exp\left(\sum_{k\in\mathbb{N}^*} \frac{(-1)^{k+1} z^k}{k(1-q^k)}\right) \tag{1.3}$$

Furthermore, in the reference [7], the multiplicative series form of the $q$-exponential were exploited to derive a new nonlinear connection formula between $q$-orthogonal polynomials and their classical versions, namely $q$-Hermite, $q$-Laguerre and $q$-Gegenbauer polynomials. Their results are expressed in compact form and some explicit examples are given. Also, the authors of [7] emphasized the possibility to extend their work for other $q$-orthogonal polynomials such as little $q$-Jacobi ones. In the present work we will take benefit of their idea to compute the connection formula between other $q$-orthogonal polynomials, appearing in the $q$-Askey scheme [5], and their classical counterparts, namely the continuous $q$-Laguerre, the continuous big $q$-Hermite and the $q$-Meixner-Pollaczek polynomials and we give an alternative connection formula for the $q$-Gegenbauer polynomials distinct from the one given in [7]. In our knowledge, these cases have not been treated before.

To proceed, similarly to the work of Chakrabarti *et al* [7], we first give the generating functions of any $q$-polynomials cited above and use, on one side, the series development. On the other side, the Quesne formulae allow us to express the $q$-exponential function as a product series of the classical exponential function. And this leads finally to the connection formula, up to the resolution of a Diophantine partition equation appearing during our computation for any examined cases.

As all the $q$-polynomials constituting the $q$-Askey scheme can be defined in terms of the basic hypergeometric series ${}_r\phi_s$, we recall here their expression (see for example [8]):

$$ {}_r\phi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q; z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n ...(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n...(b_s;q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n \tag{1.4}$$

with $q \neq 0$. The ratio test shows that for generic values of the parameters the radius of convergence is $\infty$, 1 or 0 for $r < s+1$, $r = s+1$ or $r > s+1$ respectively. Since $(q^{-n};q)_k = 0$ for $k = n+1, n+2, ...$, the series ${}_r\phi_s$ terminates if one of the numerator parameters $\{a_i\}$ is of the form $q^{-n}$ with $n = 0, 1, 2, ...$ and $q \neq 0$. The ${}_r\phi_s$ function is the $q$-analogue of the hypergeometric function defined by

$$ {}_rF_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_r)_n}{(b_1)_n(b_2)_n...(b_s)_n} \frac{z^n}{n!} \tag{1.5}$$

where $(a)_n$ denotes the Pochhammer symbol defined by

$$(a)_0 = 1, \text{and} \quad (a)_k = a(a+1)(a+2)...(a+k-1), k = 1, 2, ...$$

When one of the numerator parameters $a_i$ equals $-n$ where $n$ is a nonnegative integer this hypergeometric series is a polynomial in $z$. Otherwise the radius of convergence is $\infty$, 1 or 0 for $r < s+1$, $r = s+1$ or $r > s+1$ respectively.

## 2 Continuous $q$-Laguerre polynomials

The continuous $q$-Laguerre polynomials had manifested their apparition in the rational solutions of the $q$-analogue of Painlevé V differential equation [9], namely as the entries of its associated determinant. They are defined by: [5]

$$P_n^{\alpha}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} {}_3\phi_2\left(q^{-n}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}; q^{\alpha+1}, 0; q; q\right), \quad x = \cos\theta \tag{2.1}$$

$$= \frac{(q^{\frac{1}{2}\alpha+\frac{3}{4}}e^{-i\theta};q)_n}{(q;q)_n} q^{(\frac{1}{2}\alpha+\frac{1}{4})n} e^{in\theta} {}_2\phi_1\left(q^{-n}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}; q^{-\frac{1}{2}\alpha+\frac{1}{4}-n}; q; q^{-\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}\right) \tag{2.2}$$

The generating function of the continuous $q$-Laguerre polynomials is given by [5]

$$\begin{aligned} \mathsf{G}_q^{\alpha}(x;t) &\equiv \frac{(q^{\alpha+\frac{1}{2}}t;q)_\infty (q^{\alpha+1}t;q)_\infty}{(q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}t;q)_\infty (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}t;q)_\infty} \\ &= E_q(-q^{\alpha+\frac{1}{2}}t)E_q(-q^{\alpha+1}t)e_q(q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}t)e_q(q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}t) = \sum_{n\geq 0} P_n^{\alpha}(x|q)t^n. \end{aligned} \tag{2.3}$$

In the $q \to 1$ limit, when $x$ is replaced by $q^{\frac{x}{2}}$ in the above function (2.3), we find the generating function

$$\mathsf{G}^{\alpha}(x,t) \equiv (1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n \tag{2.4}$$

for the classical Laguerre polynomials [10]

$$L_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(-n;\alpha+1;x\right). \tag{2.5}$$

Using (1.2) and (1.3) the left hand side of (2.3) is reformulated as

$$\mathsf{G}_q^{\alpha}(x;t) = \prod_{k\in\mathbb{N}^*} \exp\left(\frac{-q^{(\alpha+\frac{1}{2})k} - q^{(\alpha+1)k} + 2q^{(\frac{1}{2}\alpha+\frac{1}{4})k}\cos k\theta}{k(1-q^k)} t^k\right). \tag{2.6}$$

To express the continuous $q$-Laguerre generating function (2.3) as a multiplicative series of the classical Laguerre generating function, we introduce the following parameters

$$\begin{aligned} x_k &= \frac{-q^{(\alpha+\frac{1}{2})k} - q^{(\alpha+1)k} + 2q^{(\frac{1}{2}\alpha+\frac{1}{4})k}\cos k\theta}{k(1-q^k)} \\ \tau_k &= \frac{t^k}{t^k-1} \end{aligned} \tag{2.7}$$

then we have

$$\begin{aligned} \mathsf{G}_q^{\alpha}(x,t) &= \prod_{k\in\mathbb{N}^*} \left[\mathsf{G}^{\alpha_k}(x_k,\tau_k)(1-\tau_k)^{\alpha_k+1}\right] && (2.8) \\ &= \sum_{\{n_k\}} \prod_{k\in\mathbb{N}^*} \left[L_{n_k}^{\alpha_k}(x_k)\tau_k^{n_k}(1-\tau_k)^{\alpha_k+1}\right] && (2.9) \end{aligned}$$

For instance $\{\alpha_k\}$ is a family of generic parameters. We rewrite

$$\tau_k^{n_k}(1-\tau_k)^{\alpha_k+1} = (-1)^{n_k} \sum_{m_k\geq 0} \frac{(\alpha_k+n_k+1)_{m_k}}{m_k!} t^{k(n_k+m_k)} \tag{2.10}$$

We obtain

$$\mathsf{G}_q^{\alpha}(x,t) = \sum_{\{n_k\}} \sum_{\{m_k\}} \prod_{k\in\mathbb{N}^*} \left[(-1)^{n_k} L_{n_k}^{\alpha_k}(x_k) \frac{(\alpha_k+n_k+1)_{m_k}}{m_k!} t^{k(n_k+m_k)}\right] \tag{2.11}$$

Inserting the series given in (2.3) in the later relation (2.11) and comparing coefficients of equal power in $t$ on both sides, we obtain our connection formula for the continuous $q$-Laguerre polynomials in terms of their classical analogues

$$P_n^{\alpha}(x|q) = \sum_{\{n_k\}} \sum_{\{m_k\}} \prod_{k\in\mathbb{N}^*} \left[(-1)^{n_k} L_{n_k}^{\alpha_k}(x_k) \frac{(\alpha_k+n_k+1)_{m_k}}{m_k!}\right] \delta_{\sum_{k\in\mathbb{N}^*} k(n_k+m_k),n}. \tag{2.12}$$

It's obvious that the family $\{\alpha_k\}$ could be any real parameters and by construction the left hand side of (2.12) must be independent of this family. Each set $\{\alpha_k\}$ provides an expansion of the continuous $q$-Laguerre polynomials. The solutions of the Diophantine partition relation

$$\sum_{k\in\mathbb{N}^*} k(n_k+m_k)=n \tag{2.13}$$

determine the set of classical Laguerre polynomials contributing to the expansion of the continuous $q$-Laguerre polynomial. For an explicit example we have used the connection formula (2.12) to write the $P_4^\alpha(x|q)$ polynomial. This is done after solving the Diophantine partition equation (2.13) for $n=4$. In Table 1 we have listed the corresponding solutions for this case together with their respective classical Laguerre polynomials contributions to the connection formula (2.12).

# 3 Continuous big $q$-Hermite polynomials

The continuous big $q$-Hermite polynomials $H_n(x;a;q)$ appear in many contexts of mathematical physics in particular in [11] where it was shown that they realize a basis for a representation space of an extended $q$-oscillator algebra. They depend on one parameter and are defined by [5]

$$\begin{aligned} H_n(x;a;q) &= a^{-n}{}_3\phi_2\left(q^{-n},ae^{i\theta},ae^{-i\theta};0,0;q;q\right) \\ &= e^{in\theta}{}_2\phi_0\left(q^{-n},ae^{i\theta};-;q;q^ne^{-2i\theta}\right), \quad x=\cos\theta. \end{aligned} \tag{3.1}$$

And their generating function is given by

$$\begin{aligned} \mathrm{G}_q(x,a,t) &= \frac{(at;q)_\infty}{(e^{i\theta t};q)(e^{-i\theta t};q)}, \qquad x=\cos\theta \\ &= E_q(-at)e_q(e^{i\theta}t)e_q(e^{-i\theta}t)=\sum_{n=0}^{\infty}\frac{H_n(x;a;q)}{(q;q)_n}t^n. \end{aligned} \tag{3.2}$$

Let's recall that the classical Hermite polynomials, defined by [10]

$$H_n(x)=(2x)^n{}_2F_0\left(-n/2,-(n-1)/2;-;-\frac{1}{x^2}\right), \tag{3.3}$$

can be obtained from the continuous big $q$-Hermite polynomials by the following limit

$$\lim_{q\to1}\left(\frac{1-q}{2}\right)^{-\frac{n}{2}}H_n\left(x\left(\frac{1-q}{2}\right)^{\frac12};a(2(1-q))^{\frac12};q\right)=H_n(x-a). \tag{3.4}$$

Their generating function is given by

$$\mathrm{G}(x,t)=\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n. \tag{3.5}$$

In similar construction, the two kinds of the $q$-exponential function in the deformed generating function (3.2) are substituted by their expressions given in (1.2) and (1.3) to obtain the following expression

$$\mathrm{G}_q(x, a, t) = \prod_{k \geq 1} \exp\left( \frac{-a^k + 2\cos(k\theta)}{k(1-q^k)} t^k \right) \tag{3.6}$$

For our purpose we set

$$x_k = \frac{-a^k + 2\cos(k\theta)}{2k(1-q^k)} \tag{3.7}$$

then we can write the deformed generating function $\mathrm{G}_q(x, a, t)$ as an infinite product series of the classical Hermite generating function:

$$\mathrm{G}_q(x, a, t) = \prod_{k \geq 1} \left( \mathrm{G}(x_k, t^k) e^{t^{2k}} \right) \tag{3.8}$$

Inserting the series given in rhs of (3.2) and (3.5) and the series of $e^{t^{2k}}$ in relation (3.8); and comparing coefficients of equal power in $t$ on both sides, we obtain our connection formula for the continuous big $q$-Hermite polynomials in terms of their classical analogues

$$\frac{H_n(x; a; q)}{(q; q)_n} = \sum_{\{n_k\}} \sum_{\{m_k\}} \prod_{k \in \mathbb{N}^*} \frac{H_{n_k}(x_k)}{n_k! m_k!} \delta_{\sum_{k \in \mathbb{N}^*} k(n_k + 2m_k), n}. \tag{3.9}$$

Here again all the problem stands in finding the solutions of the Diophantine partition equation

$$\sum_{k \in \mathbb{N}^*} k(n_k + 2m_k) = n. \tag{3.10}$$

To illustrate the connection formula (3.9) we have listed in Table 2 all the possible non-zero solutions of (3.10) for $n = 5$ and their corresponding classical Hermite polynomials involved in the construction of $H_5(x; a; q)$.

**Remark 3.1** *The continuous $q$-Hermite polynomials, (see, for example, [5], section 3.26), can easily be obtained from the continuous big $q$-Hermite polynomials $H_n(x; a; q)$ by replacing $a = 0$, then we can derive their connection formula in terms of the classical Hermite polynomials by taking $a = 0$ in both sides of (3.9).*

# 4 $q$-Meixner-Pollaczek polynomials

In this section we treat the cases of the $q$-Meixner-Pollaczek polynomials [5]

$$\begin{aligned} P_n(x; \lambda; q) &= q^{-n\lambda} e^{-in\phi} \frac{(q^{2\lambda}; q)_n}{(q; q)_n} {}_3\phi_2 \left( q^{-n}, q^\lambda e^{i(\theta+2\phi)}, q^\lambda e^{-i\theta}; q^{2\lambda}, 0; q; q \right), \quad x = \cos(\theta + \phi) \\ &= \frac{(q^\lambda e^{-i\theta}; q)_n}{(q; q)_n} e^{in(\theta+\phi)} {}_2\phi_1 \left( q^{-n}, q^\lambda e^{i\theta}; q^{1-\lambda-n} e^{i\theta}; q; q^{1-\lambda} e^{-i(\theta+2\phi)} \right). \end{aligned} \tag{4.1}$$

These are the $q$-analogue of the classical Meixner-Pollaczek polynomials defined by [10]

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left(-n, \lambda + ix; 2\lambda; 1 - e^{-2i\phi}\right), \quad \lambda > 0, \quad 0 < \phi < \pi. \tag{4.2}$$

It's obvious from the last two expressions that the following limit is true:

$$\lim_{q\to 1} P_n(\cos(\ln q^{-x} + \phi); \lambda; q) = P_n^{(\lambda)}(x; -\phi) \tag{4.3}$$

The generating function of the $q$-Meixner-Pollaczek polynomials is given by

$$\begin{aligned} \mathbf{G}_q^\lambda(x,t) &= \left| \frac{(q^\lambda e^{i\phi} t; q)_\infty}{(e^{i(\theta+\phi)} t; q)_\infty} \right|^2 = \frac{E_q(-q^\lambda e^{i\phi} t) E_q(-q^\lambda e^{-i\phi} t)}{E_q(-e^{i(\theta+\phi)} t) E_q(-e^{-i(\theta+\phi)} t)} \\ &= \sum_{n=0}^{\infty} P_n(x; \lambda; q) t^n, \qquad x = \cos(\theta + \phi). \end{aligned} \tag{4.4}$$

Which can be written, after using (1.3), as

$$\mathbf{G}_q^\lambda(x,t) = \prod_{k\geq 1} \exp\left( \frac{2}{k} \frac{-q^{k\lambda} \cos k\phi + \cos k(\theta+\phi)}{1 - q^k} t^k \right) \tag{4.5}$$

We set

$$x_k = \frac{2}{k} \frac{-q^{k\lambda} \cos k\phi + \cos k(\theta+\phi)}{1 - q^k} \tag{4.6}$$

then we can write $\mathbf{G}_q^\lambda(x,t)$ as

$$\mathbf{G}_q^\lambda(x,t) = \sum_{\{n_k\}} \prod_{k\geq 1} \left[ \frac{1}{n_k!} x_k^{n_k} t^{k n_k} \right] \tag{4.7}$$

On the other hand, for any $x \in \mathbb{R}$ and $m \in \mathbb{N}$, we can expand $x^m$ with respect of the classical Meixner-Pollaczek polynomials in the following way

$$x^m = \sum_{l=0}^{m} A_{l,m}^{\lambda,\phi} P_l^{(\lambda)}(x;\phi) \tag{4.8}$$

where the $A_{l,m}^{\lambda,\phi}$ satisfy, for $l = 0, ..., m$ , the following recursion relation

$$\begin{cases} A_{0,0}^{\lambda,\phi} = 1 \\ 2\sin\phi A_{l,m+1}^{\lambda,\phi} = (l+2\lambda) A_{l+1,m}^{\lambda,\phi} - 2(l+\lambda)\cos\phi A_{l,m}^{\lambda,\phi} + l A_{l-1,m}^{\lambda,\phi}, \end{cases} \tag{4.9}$$

and for $l > m$, $A_{l,m}^{\lambda,\phi} = 0$. Using this expansion in the rhs of (4.7) to express $x_k^{n_k}$ in term of the classical Meixner-Pollaczek polynomials and comparing coefficients of equal power in $t$

on both sides, we obtain our connection formula for the $q$-Meixner-Pollaczek polynomials in terms of their classical partners of lower dimensions:

$$P_n^{(\lambda)}(x;\phi;q) = \sum_{n_1,n_2,\ldots=0}^{\infty} \sum_{\substack{0\le l_1\le n_1,\\ 0\le l_2\le n_2,\\ \ldots}} \prod_{k\ge1} \left[\frac{1}{n_k!} A_{l_k,n_k}^{\lambda_k,\phi_k} P_{l_k}^{(\lambda_k)}(x_k;\phi_k)\right] \delta_{\sum_{k\ge1} kn_k,n} \tag{4.10}$$

Here, the same observation made above for the $\{\alpha_k\}$ family of the continuous $q$-Laguerre polynomials occurs for the $\{\lambda_k\}$ and $\{\phi_k\}$ families in (4.10) i.e. the later connection formula remains independent of the $\lambda_k$ and $\phi_k$ parameters. As example we list in below a few calculus of $q$-Meixner-Pollaczek polynomials in terms of their classical counterparts, after solving the partition equation $\sum_{k\ge1} kn_k = n$ in each cases.

$$\begin{aligned}
P_0^{(\lambda)}(x;\phi;q) &= P_0^{(\lambda)}(x;\phi) = 1\\
P_1^{(\lambda)}(x;\phi;q) &= \frac{1}{2\sin\phi_1}\left[P_1^{(\lambda_1)}(x_1;\phi_1) - 2\lambda_1\cos\phi_1 P_0^{(\lambda_1)}(x_1;\phi_1)\right]\\
P_2^{(\lambda)}(x;\phi;q) &= \frac{1}{4\sin^2\phi_1}\left[P_2^{(\lambda_1)}(x_1;\phi_1) - (2\lambda_1+1)\cos\phi_1 P_1^{(\lambda_1)}(x_1;\phi_1)\right.\\
&+ \left.\lambda_1(2\lambda_1\cos^2\phi_1+1)P_0^{(\lambda_1)}(x_1;\phi_1)\right]\\
&+ \frac{1}{2\sin\phi_2}\left[P_1^{(\lambda_2)}(x_2;\phi_2) - 2\lambda_2\cos\phi_2 P_0^{(\lambda_2)}(x_2;\phi_2)\right]\\
P_3^{(\lambda)}(x;\phi;q) &= \frac{1}{24\sin^3\phi_1}\left[3P_3^{(\lambda_1)}(x_1;\phi_1) - 6(\lambda_1+1)\cos\phi_1 P_2^{(\lambda_1)}(x_1;\phi_1)\right.\\
&+ ((3\lambda_1+1) + 2(3\lambda_1^2+3\lambda_1+1)\cos^2\phi)P_1^{(\lambda_1)}(x_1;\phi_1)\\
&- \left.2\,\lambda_1\cos\phi_1(3\lambda_1+1+2\lambda_1^2\cos^2\phi_1)P_0^{(\lambda_1)}(x_1;\phi_1)\right]\\
&+ \frac{1}{4\sin\phi_1\sin\phi_2}\left[P_1^{(\lambda_1)}(x_1;\phi_1) - 2\lambda_1\cos\phi_1 P_0^{(\lambda_1)}(x_1;\phi_1)\right]\times\\
&\times \left[P_1^{(\lambda_2)}(x_2;\phi_2) - 2\lambda_2\cos\phi_2 P_0^{(\lambda_2)}(x_2;\phi_2)\right]\\
&+ \frac{1}{2\sin\phi_3}\left[P_1^{(\lambda_3)}(x_3;\phi_3) - 2\lambda_3\cos\phi_3 P_0^{(\lambda_3)}(x_3;\phi_3)\right]
\end{aligned} \tag{4.11}$$

# 5 $q$-Gegenbauer polynomials

The $q$-Gegenbauer (or continuous $q$-ultraspherical or Rogers) polynomials are given by [8]

$$\begin{aligned}
C_n^{(\lambda)}(x;q) &= \frac{(q^{2\lambda};q)_n}{(q;q)_n} q^{-\frac{n\lambda}{2}} {}_4\phi_3\left(q^{-n}, q^{2\lambda+n}, q^{\frac{\lambda}{2}}e^{i\theta}, q^{\frac{\lambda}{2}}e^{-i\theta}; q^{\lambda+\frac{1}{2}}, -q^{\lambda}, -q^{\lambda+\frac{1}{2}}; q; q\right) \qquad (5.1)\\
&= \frac{(q^{2\lambda};q)_n}{(q;q)_n} q^{-n\lambda} e^{-in\theta} {}_3\phi_2\left(q^{-n}, q^{\lambda}, q^{\lambda}e^{2i\theta}; q^{2\lambda}, 0; q; q\right)\\
&= \frac{(q^{\lambda};q)_n}{(q;q)_n} e^{in\theta} {}_2\phi_1\left(q^{-n}, q^{\lambda}; q^{1-n-\lambda}, q; e^{-2i\theta}\right), \qquad x = \cos\theta.
\end{aligned}$$

These polynomials can also be written as

$$C_n^{(\lambda)}(x;q)=\sum_{k=0}^{n}\frac{(q^\lambda;q)_k(q^\lambda;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}e^{i(n-2k)\theta},\qquad x=\cos\theta. \tag{5.2}$$

which are the $q$-analogues of the classical Gegenbauer (or ultraspherical) polynomials [10]

$$\begin{aligned} C_n^{(\lambda)}(x) &= \frac{(2\lambda)_n}{n!}{}_2F_1\left(-n,n+2\lambda;\lambda+\frac{1}{2};\frac{1-x}{2}\right),\quad \lambda\neq 0 &(5.3)\\ &= \sum_{k=0}^{n}\frac{(\lambda)_k(\lambda)_{n-k}}{k!(n-k)!}e^{i(n-2k)\theta},\qquad x=\cos\theta. &(5.4)\end{aligned}$$

The generating function of the $q$-Gegenbauer polynomials is given by

$$\begin{aligned} \mathtt{G}_q^\lambda(x;t) &\equiv \frac{(q^\lambda e^{i\theta}t;q)_\infty(q^\lambda e^{-i\theta}t;q)_\infty}{(e^{i\theta}t;q)_\infty(e^{-i\theta}t;q)_\infty},\qquad x=\cos\theta.\\ &= \frac{E_q(-q^\lambda e^{i\theta}t)E_q(-q^\lambda e^{-i\theta}t)}{E_q(-e^{i\theta}t)E_q(-e^{-i\theta}t)}=\sum_{n\geq 0}C_n^{(\lambda)}(x;q)t^n &(5.5)\end{aligned}$$

Again by the mean of (1.3) the last generating function (5.5) take the following form

$$\mathtt{G}_q^\lambda(x,t) = \exp\left(\sum_{k\geq 1}\frac{2}{k}\frac{1-q^{k\lambda}}{1-q^k}\cos(k\theta)t^k\right) \tag{5.6}$$

With the parametrization $x_k=\cos(k\theta)$ and $[\lambda]_{q^k}=\frac{1-q^{k\lambda}}{1-q^k}$ the deformed generating function reads

$$\mathtt{G}_q^\lambda(x,t)=\sum_{n_k\geq 0}\prod_{k\geq 1}\left[\frac{1}{n_k!}\left(\frac{2}{k}[\lambda]_{q^k}\right)^{n_k}x_k^{n_k}t^{n_k}\right]. \tag{5.7}$$

Recall that for any $|x|<1$ and $m\in\mathbb{N}$, we can expand $x^m$ with respect of the classical Gegenbauer polynomials as

$$x^m=\frac{m!}{2^m}\sum_{l=0}^{[\frac{m}{2}]}a_{l,m}^\lambda C_{2l+s}^{(\lambda)}(x) \tag{5.8}$$

with

$$a_{l,m}^\lambda=\frac{\Gamma(\lambda)(2l+s+\lambda)}{\Gamma([\frac{m}{2}]+l+s\lambda+1)([\frac{m}{2}]-l)!} \tag{5.9}$$

and $s=0$ ( resp. $s=1$) for $m$ even ( resp. $m$ odd). $[\frac{m}{2}]$ denotes the largest integer smaller than or equal to $\frac{m}{2}$. Using this expansion in the rhs of (5.7) to rewrite $x_k^{n_k}$ in terms of the classical Gegenbauer polynomials, and comparing coefficients of equal power in $t$ on both sides, we obtain a new connection formula for the $q$-Gegenbauer polynomials in terms of their classical analogues, which is more general than the one found in [7]

$$C_n^{(\lambda)}(x;q) = \sum_{n_1,n_2,\ldots=0}^{\infty} \sum_{\substack{0\le l_1\le n_1,\\ 0\le l_2\le n_2,\\ \ldots}} \prod_{k\ge1} \left[\left(\frac{1}{k}[\lambda]_{q^k}\right)^{n_k} a^{\lambda_k}_{l_k,n_k} C^{(\lambda_k)}_{2l_k+s_k}(x_k)\right] \delta_{\sum_{k\ge1} kn_k,n} \tag{5.10}$$

Here also the connection formula (5.10) remains independent of the $\lambda_k$ parameters. Each real set $\{\lambda_k\}$ provides an expansion of the $q$-Gegenbauer polynomials. As an illustration we here list the calculus of the first six $q$-Gegenbauer polynomials in terms of their classical counterparts.

$$\begin{aligned}
C_0^{(\lambda)}(x;q) &= C_0^{(\lambda)}(x) = 1.\\
C_1^{(\lambda)}(x;q) &= [\lambda]_q \frac{1}{\lambda_1} C_1^{(\lambda_1)}(x_1).\\
C_2^{(\lambda)}(x;q) &= [\lambda]_q^2 \frac{1}{\lambda_1+1}\left(C_0^{(\lambda_1)}(x_1) + \frac{1}{\lambda_1}C_2^{(\lambda_1)}(x_1)\right) + [\lambda]_{q^2}\frac{1}{2\lambda_2}C_1^{(\lambda_2)}(x_2).\\
C_3^{(\lambda)}(x;q) &= [\lambda]_q^3 \frac{1}{\lambda_1+2}\left(\frac{1}{\lambda_1}C_1^{(\lambda_1)}(x_1) + \frac{1}{\lambda_1(\lambda_1+1)}C_3^{(\lambda_1)}(x_1)\right)\\
&+ \frac{1}{2}[\lambda]_q[\lambda]_{q^2}\frac{1}{\lambda_1\lambda_2}C_1^{(\lambda_1)}(x_1)C_1^{(\lambda_2)}(x_2) + \frac{1}{3}[\lambda]_{q^3}\frac{1}{\lambda_3}C_1^{(\lambda_3)}(x_3).\\
C_4^{(\lambda)}(x;q) &= [\lambda]_q^4 \frac{1}{\lambda_1+1}\left(\frac{1}{(2\lambda_1+2)}C_0^{(\lambda_1)}(x_1) + \frac{1}{\lambda_1(\lambda_1+3)}C_2^{(\lambda_1)}(x_1)\right.\\
&+ \left.\frac{1}{\lambda_1(\lambda_1+2)(\lambda_1+3)}C_4^{(\lambda_1)}(x_1)\right)\\
&+ \frac{1}{2}[\lambda]_{q^2}[\lambda]_q^2\frac{1}{\lambda_2(\lambda_1+1)}\left(C_0^{(\lambda_1)}(x_1) + \frac{1}{\lambda_1}C_2^{(\lambda_1)}(x_1)\right)C_1^{(\lambda_2)}(x_2)\\
&+ \frac{1}{3}[\lambda]_q[\lambda]_{q^3}\frac{1}{\lambda_1\lambda_3}C_1^{(\lambda_1)}(x_1)C_1^{(\lambda_3)}(x_3) + \frac{1}{4}[\lambda]_{q^4}\frac{1}{\lambda_4}C_1^{(\lambda_4)}(x_4).
\end{aligned}$$

$$\begin{aligned}
C_5^{(\lambda)}(x;q) &= [\lambda]_q^5 \frac{1}{\lambda_1(\lambda_1+2)}\left(\frac{1}{2(\lambda_1+3)}C_1^{(\lambda_1)}(x_1) + \frac{1}{(\lambda_1+1)(\lambda_1+4)}C_3^{(\lambda_1)}(x_1)\right.\\
&+ \left.\frac{1}{(\lambda_1+1)(\lambda_1+3)(\lambda_1+4)}C_5^{(\lambda_1)}(x_1)\right)\\
&+ \frac{1}{2}[\lambda]_q^3[\lambda]_{q^2}\frac{1}{\lambda_1\lambda_2(\lambda_1+2)}\left(C_0^{(\lambda_1)}(x_1) + \frac{1}{\lambda_1+1}C_3^{(\lambda_1)}(x_1)\right)C_1^{(\lambda_2)}(x_2)\\
&+ \frac{1}{3}[\lambda]_q^2[\lambda]_{q^3}\frac{1}{\lambda_3(\lambda_1+1)}\left(C_0^{(\lambda_1)}(x_1) + \frac{1}{\lambda_1}C_2^{(\lambda_1)}(x_1)\right)C_1^{(\lambda_3)}(x_3)\\
&+ [\lambda]_q[\lambda]_{q^4}\frac{1}{\lambda_1\lambda_4}C_1^{(\lambda_1)}(x_1)C_1^{(\lambda_4)}(x_4) + \frac{1}{6}[\lambda]_{q^2}[\lambda]_{q^3}\frac{1}{\lambda_2\lambda_3}C_1^{(\lambda_2)}(x_2)C_1^{(\lambda_3)}(x_3)\\
&+ \frac{1}{4}[\lambda]_q[\lambda]_{q^2}^2\frac{1}{\lambda_1(\lambda_2+1)}\left(C_0^{(\lambda_2)}(x_2) + \frac{1}{\lambda_2}C_2^{(\lambda_2)}(x_2)\right)C_1^{(\lambda_1)}(x_1) + \frac{1}{5}[\lambda]_{q^5}\frac{1}{\lambda_5}C_1^{(\lambda_5)}(x_5).
\end{aligned} \tag{5.11}$$

**Remark 5.1** *Recall that the continuous q-Legendre polynomials, denoted $P_n(x|q)$ (see [5] subsection 3.10.2 ), are related to the q-Gegenbauer polynomials by*

$$P_n(x|q) = q^{\frac{n}{4}} C_n^{(\frac{1}{2})}(x;q) \tag{5.12}$$

*and the classical Legendre polynomials can be obtained from the classical Gegenbauer polynomials by replacing $\lambda = \frac{1}{2}$, then we can derive a connection formula between continuous q-Legendre and classical Legendre polynomials by taking $\lambda = \lambda_k = \frac{1}{2}$, $\forall k \in \mathbb{N}^*$ in (5.10), and multiplying both sides by $q^{\frac{n}{4}}$.*

# 6 Conclusion and discussion

In this work, we had successfully written the connection formulae of some $q$-orthogonal polynomials appearing in the Askey scheme [5]. The first one was the continuous $q$-Laguerre polynomials, which are representing others $q$-analogues of the classical Laguerre polynomials. An explicit example $P_4^{\alpha}(x|q)$ was given. It follows from these results that the solutions of the Diophantine equation fix the finite dependence structure between classical polynomials $L_n^{\alpha}(x)$ and deformed polynomials $P_n^{\alpha}(x|q)$ for any fixed $n$. The obtention of the connection formulae was possible only because the generating function of continuous $q$-Laguerre polynomials are the product of Jackson's $q$-exponentials which could be expressed in more useful forms found by C. Quesne [6]. Our second sample in the Askey scheme was the continuous big $q$-Hermite. In this case, we had used the same arguments and method as in the precedent example and our connection formula was supported by an explicit example. The third polynomials in the list of this work were the $q$-Meixner-Pollaczek ones. The uses of relations (1.2) and (1.3) obtained by [6] and series expansion allowed us to write a well defined connection formula relating the deformed polynomials to their classical counterparts. Several examples were given. In the last section, it wasn't difficult to give the connection formula of $q$-Gegenbauer polynomials in more general form than the one given in [7]. In all cases, except in the big $q$-Hermite polynomials cases, the generic family parameters which appear in computation process, drop out by construction in the final results. This means that quantum deformation of such orthogonal polynomials is not bijective. However, for the others polynomials in the Askey scheme possessing generating functions not expressed in product of $q$-exponentials, the above prescription stops working. The case of Bessel functions, which are not orthogonal polynomials, could be a good candidate to write connection formulae since their generating function uses Jackson's $q$-exponentials; but this is not an easy task because the derived Diophantine partition equation couldn't be solved so easily, even in the simplest cases. However, our results may be useful in finding the relations of matrix elements of the unitary co-representations of the some quantum group associated with $q$-orthogonal polynomials.

For instance, we can notice from [12] the existence of simple relations between the matrix elements of the metaplectic representation of $su_q(1,1)$ and $q$-generalization of the Gegenbauer polynomials which are slightly different from (5.2) i.e. the expressions (22) and (23). For these polynomials, we can easily compute their associated connection formula and use it, after setting $\lambda = -(n+m), -(n+m+1)$ and $\lambda_k = -(n_k+m_k), -(n_k+m_k+1)$, to get a new relation which links the quantum matrix elements associated with $SU_q(1,1)$ to their classical analogous; constituting then an infinite dimensional representation of $SU_q(1,1)$. In some way, this relation may be viewed as a kind of realization map of the standard deformation of the group from its non deformed form. But what is more interesting is when one takes a generic value of $\lambda$; this provides a new continuous representation more general than the precedent ones. Whether this representation fits or not with actual known representations of $SU(1,1)$ is now the question.

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| Solutions of (2.13) for $n=4$ | Contributions to $P_4^\alpha(x|q)$ |
|---|---|
| $n_1=4$ | $L_4^{\alpha_1}(x_1)$ |
| $m_1=4$ | $\frac{(\alpha_1+1)_4}{4!}$ |
| $n_1=3$, $m_1=1$ | $-(\alpha_1+4)L_3^{\alpha_1}(x_1)$ |
| $n_1=1$, $m_1=3$ | $-\frac{(\alpha_1+2)_3}{3!}L_1^{\alpha_1}(x_1)$ |
| $n_1=2$, $m_1=2$ | $\frac{(\alpha_1+3)_2}{2}L_2^{\alpha_1}(x_1)$ |
| $n_1=1, m_1=1, n_2=1$ | $(\alpha_1+2)L_1^{\alpha_1}(x_1)L_1^{\alpha_2}(x_2)$ |
| $n_1=1, m_1=1, m_2=1$ | $-(\alpha_1+2)(\alpha_2+1)L_1^{\alpha_1}(x_1)$ |
| $n_1=2, n_2=1$ | $-L_2^{\alpha_1}(x_1)L_1^{\alpha_2}(x_2)$ |
| $n_1=2, m_2=1$ | $(\alpha_2+1)L_2^{\alpha_1}(x_1)$ |
| $m_1=2, n_2=1$ | $-\frac{(\alpha_1+1)_2}{2}L_1^{\alpha_2}(x_2)$ |
| $m_1=2, m_2=1$ | $\frac{1}{2}(\alpha_1+1)_2(\alpha_2+1)$ |
| $n_2=2$ | $L_2^{\alpha_2}(x_2)$ |
| $m_2=2$ | $\frac{(\alpha_2+1)_2}{2}$ |
| $n_2=1$, $m_2=1$ | $-(\alpha_2+2)L_1^{\alpha_2}(x_2)$ |
| $n_1=1, n_3=1$ | $L_1^{\alpha_1}(x_1)L_1^{\alpha_3}(x_3)$ |
| $m_1=1, n_3=1$ | $-(\alpha_1+1)L_1^{\alpha_3}(x_3)$ |
| $n_1=1, m_3=1$ | $-(\alpha_3+1)L_1^{\alpha_1}(x_1)$ |
| $m_1=1, m_3=1$ | $(\alpha_1+1)(\alpha_3+1)$ |
| $n_4=1$ | $-L_1^{\alpha_4}(x_4)$ |
| $m_4=1$ | $(\alpha_4+1)$ |

Table 1: Contributions to $P_4^\alpha(x|q)$.

| Solutions of (3.10) for $n=5$ | Contributions to $H_5(x,a;q)$ |
|---|---|
| $n_1=5$ | $\frac{(q;q)_5}{5!}H_5(x_1)$ |
| $n_1=1$, $m_1=2$ | $\frac{(q;q)_5}{2}H_1(x_1)$ |
| $n_1=1$, $n_2=2$ | $\frac{(q;q)_5}{2}H_1(x_1)H_2(x_2)$ |
| $n_1=1$, $n_2=1$, $m_1=1$ | $(q;q)_5H_1(x_1)H_1(x_2)$ |
| $n_1=2$, $n_3=1$ | $\frac{(q;q)_5}{2}H_1(x_3)H_2(x_1)$ |
| $n_1=1$, $m_2=1$ | $(q;q)_5H_1(x_1)$ |
| $n_3=1$, $m_1=1$ | $(q;q)_5H_1(x_3)$ |
| $n_2=1$, $n_3=1$ | $(q;q)_5H_1(x_2)H_1(x_3)$ |
| $n_1=1$, $n_4=1$ | $(q;q)_5H_1(x_1)H_1(x_4)$ |
| $n_5=1$ | $(q;q)_5H_1(x_5)$ |

Table 2: Contributions to $H_5(x,a;q)$.