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On Askey-scheme and d-orthogonality, I: A characterization theorem

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ABSTRACT

This is the first in a series of papers dealing with generalized hypergeometric d-orthogonal polynomials extending the polynomial families in the Askey-scheme. In this paper, we give a characterization theorem to introduce new examples of generalized hypergeometric d-orthogonal polynomials to be studied in the forthcoming works. For $d = 1$, we obtain an unification of some known characterization theorems in the orthogonal polynomials theory.

1. Introduction

The generalized hypergeometric functions ${}_pF_q(z)$ with p numerator and q denominator parameters are defined by ([1], for instance)

$$ {}_pF_q\left(\begin{matrix}(a_p)\\(b_q)\end{matrix}; z\right) := \sum_{m=0}^{\infty} \frac{[a_p]_m}{[b_q]_m}\frac{z^m}{m!}, \tag{1.1}$$

where (a_p) designates the set $\{a_1, a_2, \ldots, a_p\}$, $[a_r]_p = \prod_{i=1}^{r}(a_i)_p$ and $(a)_p = \frac{\Gamma(a+p)}{\Gamma(a)}$. z being a variable in $\mathbb{C}$ the set of complex numbers.

A generalized hypergeometric function ${}_pF_q(z)$ is reduced to a polynomial of degree n called generalized hypergeometric polynomial if m numerator parameters take the forme $\Delta(m, -n)$, where $\Delta(r, \alpha)$ abbreviates the array of the r parameters: $\frac{\alpha+j-1}{r}, j = 1, \ldots, r$.

Around 1980 the Askey-scheme of generalized hypergeometric polynomials [2] becomes widely known as a convenient graphical way to see the hierarchy of hypergeometric orthogonal polynomials. The Askey-scheme soon got a q-analogue, which was made possible by the discovery of the Askey–Wilson polynomials. Recently, other similar tables to the Askey-scheme [3,4] were coming after the study of the so-called Krall-type orthogonal polynomials (perturbations of linear functional via the addition of Dirac delta functions) and the multiple orthogonal polynomials. On other hand, in the last years, a generalization of the notion of orthogonality, the so-called d-orthogonality, have been intensively studied. It is then natural to look for a similar table to the Askey-scheme in the context of the d-orthogonality notion. This notion was introduced in [5] and completed in [6] as follows. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integer n. Let d be a

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positive integer. We say that the PS $\{P_n\}_{n\geq 0}$ is d-orthogonal (d-OPS, for shorter) with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following orthogonality relations:

$$\begin{cases} \langle \Gamma_k, P_r P_n \rangle = 0, & r > nd + k,\ n \in \mathbb{N} = \{0, 1, 2, \ldots\}, \\ \langle \Gamma_k, P_n P_{nd+k} \rangle \neq 0, & n \in \mathbb{N}, \end{cases} \tag{1.2}$$

for each integer k belonging to $\{0, 1, \ldots, d-1\}$.

In the finite case where P_n is defined for $n = 0, \ldots, n_0$; $n_0 \in \mathbb{N}$; the polynomial sequence $\{P_n\}_{n=0}^{n=n_0}$ is a PS if $\deg P_n = n$ for $n = 0, \ldots, n_0$. Furthermore, we say that the PS $\{P_n\}_{n=0}^{n=n_0}$ is a d-OPS with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies (1.2) with $n = 0, \ldots, n_0$.

For $d = 1$, the d-orthogonality is reduced to the orthogonality in the general sense defined in [7,8]. The d-orthogonality conditions (1.2) are equivalent to the fact that the polynomials P_n, $n \geq 0$, satisfy a $(d+1)$-order recurrence relation of the type [6]:

$$xP_n(x) = \sum_{k=0}^{d+1} \alpha_{k,d}(n) P_{n-d+k}(x), \quad n \in \mathbb{N}, \tag{1.3}$$

with the regularity conditions $\alpha_{d+1,d}(n)\alpha_{0,d}(n) \neq 0, n \geq d,$ and the convention, $P_{-n} = 0,\ n \geq 1$.

This result, for $d = 1$, is reduced to the so-called Favard Theorem [8].

The d-orthogonality notion seems to appear rather naturally in some approximation problems, in particular in simultaneous rational approximation of several functions defined by their series expansions. This new concept of orthogonality appears as a special case of the general multiple orthogonality (see, for instance, [9–16]). This generalization has received much attention these past years. That, many polynomials generalizing known orthogonal ones were introduced by solving certain characterization problems in the context of the d-orthogonality (see, for instance, [17–29]).

In this work, our interest is to give a characterization theorem for classes of generalized hypergeometric polynomials containing, as particular cases, all polynomials belonging to the Askey-scheme. That allows us:

(i) To introduce new examples of generalized hypergeometric d-orthogonal polynomials which are useful to construct similar table to the Askey-scheme in the context of d-orthogonality.
(ii) To derive new characterization theorems, based on generalized hypergeometric representations, for some known d-orthogonal polynomials as Gould–Hopper polynomials [30], Humbert polynomials [31], a generalization of Laguerre polynomials studied in [22] and a generalization of Charlier polynomials studied in [19]. Notice that the known characterization theorems for these polynomial sets are related to special types of generating functions.
(iii) To unify many known characterization theorems for generalized hypergeometric orthogonal polynomials. That concerns, for instance, Hermite, Laguerre, Jacobi, Bessel, Charlier and Meixner polynomials.
(iv) To state, as for as we known, for the first time characterization theorems for Wilson, Racah, Hahn, Continuous Hahn, Dual Hahn, Continuous dual Hahn, Meixner–Pollaczek and Krawtchouk polynomials.

2. Main result

2.1. A characterization theorem

In the sequel, we use the following notations:

- $m, l, p, q \in \mathbb{N}$, such that $m \geq 2$.
- $c, \mu, a_i, b_j \in \mathbb{C}$; $i = 1, \ldots, p$, $j = 1, \ldots, q$; such that $a_i \neq b_j$ and $\mu \notin -\mathbb{N}$.
- $N(x) = \prod_{i=1}^{p}(x + a_i)$, $D(x) = \prod_{j=1}^{q}(x + b_j)$.
- $\lambda_i(x), i = 1, \ldots, s$ being s polynomials of degree 1 and $\{\lambda_i(x), i = 1, \ldots, s\}$ being a s-separable product set,

i.e. there exists a monic polynomial π of degree s such that: $\prod_{i=1}^{s}(\lambda_i(x) + r) = \lambda(x) + \pi(r)$ with $\lambda(x) = \prod_{i=1}^{s}\lambda_i(x)$.

Let us consider the following classes of generalized hypergeometric PSs.

Class $\mathcal{A}_1$: The set of PS $\{P_n\}_{n=0}^{n=n_0}$ or $\{P_n\}_{n=0}^{\infty}$ defined by:

$$P_n(x; m, (a_p), (b_q)) = x^n \ {}_{m+p}F_q\left(\begin{matrix}\Delta(m, -n), (a_p) \\ (b_q)\end{matrix};\ \frac{1}{x^m}\right),$$

Class $\mathcal{A}_2$: The set of PS $\{P_n\}_{n=0}^{n=n_0}$ or $\{P_n\}_{n=0}^{\infty}$ defined by:

$$P_n^{\mu}(x; l, (a_p), (b_q)) = {}_{l+p+1}F_q\left(\begin{matrix}-n, \Delta(l, n+\mu), (a_p) \\ (b_q)\end{matrix};\ x\right),$$

Class $\mathcal{A}_3$: The set of PS $\{P_n\}_{n=0}^{n=n_0}$ or $\{P_n\}_{n=0}^{\infty}$ defined by:

$$P_n^{\mu}(\lambda(x); l, c, (a_p), (b_q)) = {}_{l+s+p+1}F_q\left(\begin{matrix}-n, \Delta(l, n+\mu), (\lambda_s(x)), (a_p) \\ (b_q)\end{matrix};\ \frac{1}{c}\right),$$

Class $\mathcal{B}$: The set of PS $\{P_n\}_{n=0}^{n=n_0}$ or $\{P_n\}_{n=0}^{\infty}$ defined by:

$$P_n^{\mu}(x; m, (a_p), (b_q)) = (\mu)_n x^n \; {}_{m+p}F_{m+q-1}\left(\begin{matrix} \Delta(m, -n), (a_p) \\ \Delta(m-1, 1-\mu-n), (b_q) \end{matrix}; \frac{1}{x^m}\right),$$

Class $\mathcal{C}$: The set of PS $\{P_n\}_{n=0}^{n=n_0}$ or $\{P_n\}_{n=0}^{\infty}$ defined by:

$$P_n(\lambda(x); c, m, (a_p), (b_q)) = [\lambda_s(x)]_n \; {}_{m+p}F_{sm+q}\left(\begin{matrix} \Delta\,(m, -n)\,,\ (a_p) \\ \Delta\,(m, 1-(\lambda_s(x))-n)\,,\ (b_q) \end{matrix}; \frac{1}{c}\right).$$

Put $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. Every PS in the Askey-scheme belongs to the class $\mathcal{A}$ (see Table 2). The polynomials defined by the class $\mathcal{B}$ are reduced to the Gegenbauer ones [32] for $m = 2$ and $p = q = 0$. The polynomials defined by the class $\mathcal{C}$ are reduced to the Charlier ones [32] for $m = s = 1$ and $p = q = 0$.

Our interest here is to solve the following characterization problem:

P: *Find all d-OPSs in* $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

Such a characterization takes into account the fact that PS which are obtainable from one there by a linear change of variable are assumed equivalent.

A solution of this problem is given by the following.

Theorem 2.1. *The only d-OPSs in* $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ *are given by Table 1 with the following conditions:*

(1) $b_i \notin -\mathbb{N},\ i = 1, \ldots, q$.
(2) $\mu + d(1-b_i) \notin -\mathbb{N},\ i = 1, \ldots, q$.
(3) $\sum_{i=1}^{d+1}(\lambda_i(0) - b_i) \notin \mathbb{N}$.
(4) $c = 1$ and $\mu = -\frac{d+1}{2} - (\sigma + \sigma') + \sum_{k=1}^{d+2} b_k$, where $\lambda(y) = (y+\sigma)(-y+\sigma')$.
(5) $b_1 = -n_0;\ n_0 \in \mathbb{N};\ b_i \neq 0, -1, -2, \ldots, -n_0$ for $i = 2, \ldots, q$, and

$$n_0 \geq \max[d+2,\ q + \max(l, d),\ p + s + \max(l, d)] + 1. \tag{2.1}$$

(6) $\mu + d(1-b_i) \neq 0, -1, \ldots, -n_0 + d + 1$ for $i = 1, \ldots, q$.

Remark 2.2. 1. The obtained d-orthogonal polynomials belonging to the classes $\mathcal{A}_1 \cup \mathcal{B} \cup \mathcal{C}$ and $\mathcal{A}_2$ with $q = d$ and $l = 0$ or d were deeply investigated in [22,33,34]. They showed that these PSs have analogous properties of the one's satisfied by their corresponding PSs in the Askey scheme. That is why the authors use the appellations: PS of Laguerre type, Charlier type, Jacobi type and so on. Here, for the sake of brevity in Table 1, we use instead: d-Laguerre, d-Charlier, d-Jacobi and so on. Then, for convenience, we do so for the new d-OPSs. That will be justified by the type of the corresponding properties to be established in a forthcoming work.

More details concerning the three different d-OPSs generalizing Charlier polynomials may be found in [18,33].

2. For some PSs in Table 1, we add the conditions $d \geq 2$ since they collapse for $d = 1$. The corresponding appellations were suggested by the limit cases.
3. Among the d-OPSs listed in Table 1, the known ones are also solutions of other characterization problems related to some special types of generating functions. That concerns Gould–Hopper polynomials or d-Hermite [26,21], Humbert or d-Gegenbauer polynomials [21], d-Laguerre [35,22,20] and d-Charlier I [18].

2.2. Proof of Theorem 2.1

To prove Theorem 2.1, we need the following four lemmas.

Lemma 2.3. *Let* $l' \in \mathbb{N}$ *such that* $n \geq l'$, $l \leq l'$ *and* $r = 0, 1, \ldots, l'+1$. *For* $k = 1, 2, \ldots, l'+1$ *and* $l \neq 0$, *we have*

$$\begin{aligned}
(-n)_{k-1} &= \frac{(-n)_{l'}}{(-n-1+k)_{l'+1-k}}, \\
(n+\mu)_{l(k-1)} &= \frac{(n+\mu-l'+lk)_{l'-l}(n+\mu-l')_{lk}}{(n+\mu-l')_{l'}}, \\
(-n+d'-r)_k &= \frac{(-n-1+k)_{l'+1-r}\,(-n-r+l')_r}{(-n-1+k)_{l'+1-k}}, \\
(n-l'+r+\mu)_{lk} &= \frac{(n+\mu+lk-l')_r\,(n-l'+\mu)_{kl}}{(n-l'+\mu)_r}.
\end{aligned} \tag{2.2}$$

Table 1
d-OPSs in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

Class	p	m	l	s	q	Conditions	Polynomials
$\mathcal{A}_1$	0	$d+1$	0	0	0		Gould–Hopper
$\mathcal{A}_2$	0	1	0	0	d	(1)	d-Laguerre
	0	1	0	0	d	(5)	Finite d-Laguerre
	0	1	d	0	0		d-Bessel
	0	1	d	0	$1, \dots, d-1$	(1), (2), $d \geq 2$	d-Bessel–Jacobi
	0	1	d	0	$1, \dots, d-1$	(5), (6), $d \geq 2$	Finite d-Bessel–Jacobi
	0	1	d	0	d	(1), (2)	d-Jacobi
	0	1	d	0	d	(5), (6)	Finite d-Jacobi
$\mathcal{A}_3$	0	1	0	d	0		d-Charlier II
	0	1	0	d	$1, \dots, d-1$	(1), $d \geq 2$	d-Charlier–Meixner
	0	1	0	d	$1, \dots, d-1$	(5), $d \geq 2$	Finite d-Charlier–Meixner
	0	1	0	$1, \dots, d-1$	d	(1), $d \geq 2$	d-Laguerre–Meixner
	0	1	0	$1, \dots, d-1$	d	(5), $d \geq 2$	Finite d-Laguerre–Meixner
	0	1	0	d	d	(1), $c \neq 1$	d-Meixner
	0	1	0	d	d	(5), $c \neq 1$	d-Krawtchouk
	0	1	0	d	d	(1), $c \neq 1$	d-Meixner–Pollaczek
	0	1	0	$d+1$	$d+1$	(1), (3), $c = 1$	d-Continuous dual Hahn
	0	1	0	$d+1$	$d+1$	(3), (5), $c = 1$	d-Dual Hahn
	0	1	d	1	$d+1$	(1), $c = 1$	d-Continuous Hahn
	0	1	d	1	$d+1$	(5), (6), $c = 1$	d-Hahn
	0	1	d	2	$d+2$	(1), (4)	d-Wilson
	0	1	d	2	$d+2$	(4), (5), (6)	d-Racah
$\mathcal{B}$	0	$d+1$	0	0	0		Humbert
$\mathcal{C}$	0	d	0	1	0		d-Charlier I
	0	m	0	$\frac{d}{m}$	0	$\frac{d}{m} \in \mathbb{N} \setminus \{0, 1\}, d \geq 2$	d-Charlier III

For $k \geq l' + 1$, we have

$$\begin{aligned}
&(-n)_{k-1} = (-n)_{l'}\,(-n+l')_{k-l'-1},\\
&(n+\mu)_{l(k-1)} = (n+\mu)\,(n+1+\mu)_{lk-l'-1}(n+\mu+lk-l')_{l'-l},\\
&(-n+l'-r)_k = (-n+l'-r)_r\,(-n+l')_{k-1-l'}\,(-n-1+k)_{l'+1-r},\\
&(n-l'+r+\mu)_{lk} = (n-l'+r+\mu)_{l'+1-r}(n+1+\mu)_{lk-l'-1}\,(n+\mu+lk-l')_r,\\
&(n+\mu-l')_{l'}(n+\mu) = (n-l'+\mu)_r\,(n-l'+r+\mu)_{l'+1-r}.
\end{aligned} \tag{2.3}$$

Lemma 2.4. *Let D and π be two polynomials such that:*
$\deg[cxD(x-1) - \pi(x)(x-n-1)(lx+n-l+\mu)_l] \leq l+1$, $n \geq l$. *Then, for $k \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, there exist $l+2$ complex numbers $\alpha'_{r,l}(n)$; $r = 0, 1, \dots, l+1$; such that:*

$$ck(-n)_k(n+\mu)_{l(k-1)}D(k-1) - \pi(k)(-n)_k(n+\mu)_{lk} = \sum_{r=0}^{l+1} \alpha'_{r,l}(n)(-n+l-r)_k\,(n-l+r+\mu)_{lk}, \tag{2.4}$$

$$\sum_{r=0}^{l+1} \alpha'_{r,l}(n) = 0. \tag{2.5}$$

Proof. Let $n \geq l$. Since $\deg[cxD(x-1) - \pi(x)(x-n-1)(lx+n-l+\mu)_l] \leq l+1$ and the set $\{\frac{(n+\mu-l)_l\,(-n+l-r)_r}{(n+\mu-l)_r\,(-n)_l}(-n-1+x)_{l+1-r}(n+\mu+lx-l)_r\}_{0\leq r\leq l+1}$ is a basis in $\mathbb{C}_{l+1}[X]$ the vector space of polynomials with coefficients in $\mathbb{C}$ and of degree less or equal to $(l+1)$, then there exist $l+2$ complex numbers $\alpha'_{r,l}(n)$; $r = 0, 1, \dots, l+1$; such that:

$$\begin{aligned}
&cxD(x-1) - \pi(x)(x-n-1)(lx+n-l+\mu)_l\\
&\quad = \sum_{r=0}^{l+1} \alpha'_{r,l}(n)\,\frac{(n+\mu-l)_l\,(-n+l-r)_r}{(n+\mu-l)_r\,(-n)_l}\,(-n-1+x)_{l+1-r}(n+\mu+lx-l)_r, \quad x \in \mathbb{C}.
\end{aligned} \tag{2.6}$$

Multiplying both side of the identity (2.6) with $x = k$ by

$$\varepsilon(n, l, \mu, k) = \begin{cases} \dfrac{(n-l+\mu)_{kl}}{(k-n-1)_{l+1-k}}, & 1 \leq k \leq l+1,\\ (-n+l)_{k-1-l}\,(n+1+\mu)_{l(k-1)-1}, & k \geq l+1, \end{cases}$$

and using Lemma 2.3 with $l' = l$, we deduce (2.4).

Replacing $x = 0$ in (2.6), we obtain (2.5). $\square$

Now, we use Lemma 2.4 to state the following.

Lemma 2.5. *Let $\mathfrak{B} = \{\mathcal{B}_k\}_{k\geq 0}$ be the basis in $\mathcal{P}$ given by:*

$$\mathcal{B}_0(x) = 1 \quad \text{and} \quad \mathcal{B}_k(x) = \prod_{r=0}^{k-1} (x + \pi(r)), k = 1, 2, \ldots, \tag{2.7}$$

where π is a polynomial such that $\pi(0) = 0$.

Let $\{P_n\}_{n\geq 0}$ be a PS of the form

$$P_n(x) = \sum_{k=0}^{n} \frac{(-n)_k\,(n+\mu)_{lk}}{c^k\,k!}\,\frac{[a_p]_k}{[b_q]_k}\mathcal{B}_k(x). \tag{2.8}$$

The only d-OPSs of the form (2.8) arise for $l = d$, $p = 0$, $\deg\pi \leq 2$, and

$q \leq d$ if $\pi \equiv 0$,

$q = d + 1$ and $c = \sigma d^d$ if $\pi(x) = \sigma x\ (\sigma \neq 0)$,

$q = d + 2$, $c = \sigma_2\, d^d$ and $\mu = \sum_{j=1}^{d+2} b_j - \frac{d+1}{2} - \frac{\sigma_1}{\sigma_2}$ if $\pi(x) = \sigma_2 x^2 + \sigma_1 x$, $\sigma_2 \neq 0$.

Proof. This proof is divided into three steeps. The first one is devoted to show that $l = d$, which will be used in the second step to prove that $p = 0$. Finally, in the third step, we use the obtained results to show that $\deg\pi \leq 2$ and we determinate q.

Since $\{P_n\}_{n\geq 0}$ is d-OPS then it satisfies a $(d+1)$-order recurrence relation of type (1.3). Hence $\{P_n\}_{n\geq 0}$ verifies the following recurrence relation

$$xP_n(x) = \sum_{j=0}^{d'+1} \alpha'_j(n) P_{n-d'+j}(x), \tag{2.9}$$

where $d' = \max(d, l)$ and

$$\begin{aligned} &\alpha'_j(n) = \alpha_{j,d}(n), \quad d \geq l,\ j = 0, 1, \ldots, d+1, \\ &\alpha'_{j+l-d}(n) = \alpha_{j,d}(n), \quad d < l,\ j = 0, 1, \ldots, d+1, \\ &\alpha'_j(n) = 0, \quad d < l,\ j = 0, \ldots, l-1-d. \end{aligned} \tag{2.10}$$

Replacing (2.8) in (2.9) and using the fact that $\mathcal{B}_{k+1}(x) = (x + \pi(k))\mathcal{B}_k(x)$, we obtain:

$$\begin{aligned} &\sum_{k=1}^{n+1} \left[ck(-n)_{k-1}(n+\mu)_{l(k-1)} \frac{D(k-1)}{N(k-1)} - (-n)_k(n+\mu)_{lk}\pi(k) \right] \frac{[a_p]_k}{c^k k! [b_q]_k} \mathcal{B}_k(x) \\ &\quad = \sum_{k=0}^{n+1} \sum_{j=0}^{d'+1} \alpha'_j(n) \frac{(-n+d'-j)_k (n-d'+j+\mu)_{lk}}{c^k k!} \frac{[a_p]_k}{[b_q]_k} \mathcal{B}_k(x). \end{aligned}$$

By identification we get, for $n \geq d'$,

$$\sum_{j=0}^{d'+1} \alpha'_j(n) = 0 \tag{2.11}$$

$$\begin{aligned} &\sum_{j=0}^{d'+1} \alpha'_j(n)(-n+d'-j)_k(n-d'+j+\mu)_{lk} \\ &\quad = ck(-n)_{k-1}\,(n+\mu)_{l(k-1)}\,\frac{D(k-1)}{N(k-1)} - \pi(k)(-n)_k\,(n+\mu)_{lk}, \quad 1 \leq k \leq n+1. \end{aligned} \tag{2.12}$$

Substituting respectively in (2.12) the identities (2.2) and (2.3) with $l' = d'$. Multiplying both side of the obtained expressions by

$$\varepsilon(n, k, d', \mu) = \begin{cases} \dfrac{(-n+k-1)_{d'+1-k}}{(n-d'+\mu)_{kl}}, & 1 \leq k \leq d'+1 \\ \dfrac{1}{(-n+d')_{k-d'-1}\,(n+1+\mu)_{lk-d'-1}}, & d'+1 < k. \end{cases}$$

Then, we get, for $1 \le k \le n+1$,

$$Q_{n,d',\mu}(k) = ck(n+\mu+lk-d')_{d'-l}\,\frac{D(k-1)}{N(k-1)} - \pi(k)(n+\mu+lk-d')_{d'}(-n-1+k), \tag{2.13}$$

where $Q_{n,d',\mu}$ is the polynomial defined by

$$Q_{n,d',\mu}(x) = \sum_{j=0}^{d'+1} \alpha'_j(n)\,\frac{(-n+d'-j)_j(n-d'+\mu)_{d'}}{(-n)_{d'}(n-d'+\mu)_j}(-n-1+x)_{d'+1-j}(n+\mu+lx-d')_j. \tag{2.14}$$

Using (2.13) and (2.14), it is easy to verify that $Q_{n,d'\mu}$ is a polynomial in x of degree less or equal to $d'+1$ and $Q_{n,d',\mu}(0) = 0$. Then the identity (2.13) can be rewritten under the form

$$c\,\frac{D(k-1)}{N(k-1)}(n+\mu+lk-d')_{d'-l} = \frac{\pi(k)}{k}\,(-n-1+k)\,(n+\mu+lk-d')_{d'} + \frac{Q_{n,d',\mu}(k)}{k}. \tag{2.15}$$

Put

$$R_{n,d',\mu}(x) = c(n+\mu+lx-d')_{d'-l}D(x-1) - N(x-1)\left[\frac{\pi(x)}{x}\,(-n-1+x)\,(n+\mu+lx-d')_{d'} + \frac{Q_{n,d',\mu}(x)}{x}\right].$$

Then $R_{n,d',\mu}$ is a polynomial in x of degree less or equal to $\max(q+d'-l, p+\deg\pi+d', p+d')$, not depending on n.

Let $n \ge \max(q+d'-l, p+\deg\pi+d', p+d')$. From (2.15), we deduce that $R_{n,d',\mu}$ has $n+1$ roots, that are $k = 1, \dots, n+1$. Using (2.1) in the finite case and $n \ge d'$ in the general case, we deduce that $R_{n,d',\mu} \equiv 0$ and (2.15) is valid for all x in $\mathbb{C}$. Substituting $k = x_n = -\frac{n+\mu-d'}{l}$ in (2.15), we obtain

$$\alpha'_0(n)(-n-1+x_n)_{d'+1}\,(n-d'+\mu)_{d'} = 0, \quad \text{if } l < d \tag{2.16}$$

$$\alpha'_0(n)(-n-1+x_n)_{l+1}\,(n-l+\mu)_l = cx_n(-n)_l\,\frac{D(x_n-1)}{N(x_n-1)}, \quad \text{if } l \ge d. \tag{2.17}$$

We consider the following two cases:

Case 1: $l < d$. In this case $d = d'$. From the regularity conditions of the recurrence relation given by (1.3) and (2.9), we have $\alpha'_0(n)(-n-1+x_n)_{d+1}\,(n-d+\mu)_{d+1} \ne 0$, which is in contradiction with (2.16).

Case 2: $l > d$. Using (2.10) and (2.17), we obtain $cx_n(-n)_l\frac{D(x_n-1)}{N(x_n-1)} = 0$, $n \ge l$. Therefore $D(x_n-1) = 0$, $n \ge l$. Then the polynomial D have $n-l+1$ roots. That, by virtue of (2.1) in the finite case and $n \ge l$ for the general case, leads to a contradiction with $\deg D = q$. Hence $l = d$.

By letting $l = d$ in (2.17), it is easy to show that, for $n \ge d$, $D(x_n-1) \ne 0$, which leads to $\mu + d(1-b_j) \ne 0, -1, \dots, -n_0+d+1$ for the finite case, and $\mu + d(1-b_j) \notin -\mathbb{N}$ for the general case.

To show that $p = 0$, we put $l = d$ in the identity (2.15). We deduce that $\frac{D}{N}$ is a polynomial. But N and D are coprime. From that $N = 1$. Hence $p = 0$, which we suppose in the sequel.

Next, we prove that $\deg\pi \le 2$ and we determinate q. Let us consider the two following cases.

Case 1: $\pi \equiv 0$. By comparing the degree of polynomials in both sides of (2.15) with $l = d$ and $p = 0$, we deduce $\deg D = q \le d$.

Case 2: $\pi \ne 0$. By comparing the degree of polynomials in both sides of (2.15) with $l = d$ and $p = 0$, we deduce $q = d + \deg\pi$.

Let $\pi(x) = \sum_{j=1}^{q-d}\sigma_j x^j$. Replacing $l = d$ and $p = 0$ in (2.15), we obtain

$$\begin{aligned} Q_{n,d,\mu}(k) = {} & \left[1 - \frac{d^d}{c}\sigma_{q-d}\right]k^{q+1} + \left[\sum_{j=1}^{q} b_j - q - \frac{d^d}{c}\left(\sigma_{q-d}\left[\mu - \frac{d+3}{2}\right] + \sigma_{q-d-1}\right)\right]k^q \\ & + \xi(n)k^{q-1} + S_{n,d,\mu}(k), \end{aligned} \tag{2.18}$$

where $S_{n,d,\mu}$ is a polynomial of degree less then $q-2$, and $\xi(x) = \sigma_{q-d}\,\frac{(d+1)_q}{2c}x^2 + \phi(d)x + \psi(d)$, ψ and ϕ being two polynomials in d.

Using (2.1) in the finite case and $n \ge d$ in the general case, we deduce that there exists an integer n satisfying $\xi(n) \ne 0$. That leads to $\deg Q_{n,d,\mu} \ge q-1$. However, according to (2.14), we have $\deg Q_{n,d,\mu} \le d+1$. Consequently $\deg\pi = q-d \le 2$.

Furthermore, if $\deg\pi = 1$, i.e. $\pi(x) = \sigma_1 x$, $(\sigma_1 \ne 0)$, then the identity (2.18) becomes

$$Q_{n,d,\mu}(k) = \left[1 - \frac{d^d}{c}\sigma_1\right]k^{d+2} + \left[\sum_{j=1}^{d+1} b_j - d - 1 - \frac{d^d}{c}\sigma_1\left[\mu - \frac{d+3}{2}\right]\right]k^{d+1} + \xi(n)k^d + S_{n,d,\mu}(k).$$

Taking into account the fact that $\deg Q_{n,d,\mu} \le d+1$, we deduce that $c = d^d\sigma_1$.

In the case when $\deg \pi = 2$, i.e. $\pi(x) = \sigma_2 x^2 + \sigma_1 x$, $\sigma_2 \neq 0$, the identity (2.18) becomes

$$Q_{n,d,\mu}(k) = \left[1 - \frac{d^d}{c}\sigma_2\right] k^{d+3} + \left[\sum_{j=1}^{d+2} b_j - d - 2 - \frac{d^d}{c}\left(\sigma_2\left[\mu - \frac{d+3}{2}\right] + \sigma_1\right)\right] k^{d+2} + \xi(n) k^{d+1} + S_{n,d,\mu}(k).$$

However $\deg Q_{n,d,\mu} \leq d+1$, then $c = d^d \sigma_2$ and $\mu = \sum_{j=1}^{d+2} b_j - \frac{d+1}{2} - \frac{\sigma_1}{\sigma_2}$.

To prove the converse, we show next that the PS $\{P_n\}_{n\geq 0}$ verifies a $(d+1)$-recurrence relation of type (1.3).

By using the explicit expression of P_n given by (2.8), we get

$$xP_n(x) = \sum_{k=1}^{n} \left[ck(-n)_{k-1}\,(n+\mu)_{d(k-1)}\,D(k-1) - \pi(k)\,(-n)_k\,(n+\mu)_{dk}\right] A(k)\mathcal{B}_k(x),$$

where $A(k)$ is defined by $A(k) = \frac{1}{c^k k!} \prod_{r=0}^{k-1} \frac{1}{D(r)}$ for $k \neq 0$, and $A(0) = 1$. That by virtue of Lemma 2.4, leads to

$$\begin{aligned} xP_n(x) &= \sum_{k=1}^{n+1} \sum_{r=0}^{d+1} \alpha'_{r,d}(n)\,(-n+d-r)_k\,(n-d+r+\mu)_{kd} A(k)\mathcal{B}_k(x) \\ &= \sum_{r=0}^{d+1} \alpha'_{r,d}(n) \sum_{k=0}^{n+1} [(-n+d-r)_k (n-d+r+\mu)_{kd} A(k)\mathcal{B}_k(x)] \\ &= \sum_{r=0}^{d+1} \alpha'_{r,d}(n) P_{n-d+r}(x). \end{aligned} \tag{2.19}$$

On the other hand, Putting, respectively, $k = 1 - \frac{n+\mu}{d}$ and $k = n+1$ in (2.6) with $l = d$, we obtain

$$\alpha'_{0,d}(n) = \frac{-c(-n)_d}{d(n-l+1+\mu)_{d-1}} \frac{D\left(-\frac{n+\mu}{d}\right)}{\left(-\frac{(d+1)n+\mu}{d}\right)_{d+1}} \quad \text{and} \quad \alpha'_{d+1,d}(n) = \frac{-c(n+\mu)}{((d+1)n+\mu)_{d+1}} D(n). \tag{2.20}$$

Moreover $\alpha'_{0,d}(n)\alpha'_{d+1,d}(n) \neq 0$; $n \geq d$; since $b_j, \mu, \ \mu + d(1-b_j) \neq 0, -1, -2, \ldots$. Then, according to (1.3), the PS $\{P_n\}_{n\geq 0}$ is a d-OPS. □

Lemma 2.6. *Every PS belonging to $\mathcal{A}_2 \cup \mathcal{A}_3$ is of the type defined by* (2.8).

Proof. Let $\{P_n\}_{n\geq 0}$ be a PS belonging to the class $\mathcal{A}_2$. Using (1.1), we obtain

$$P_n(x) = \sum_{k=0}^{n} \frac{(-n)_k\,(n+\mu)_{lk}}{l^{lk}\,k!} \frac{[a_p]_k}{[b_q]_k} x^k.$$

Then $\{P_n\}_{n\geq 0}$ verifies (2.8) with $c = l^l$ and $\mathcal{B}_k(x) = x^k$, $(\pi \equiv 0)$.

Let $\{P_n\}_{n\geq 0}$ be a PS belonging to the class $\mathcal{A}_3$. Using (1.1), we obtain

$$P_n(\lambda(x)) = \sum_{k=0}^{n} \frac{(-n)_k\,(n+\mu)_{lk}}{(cl)^{lk}\,k!} \frac{[a_p]_k}{[b_q]_k} [\lambda_s(x)]_k. \tag{2.21}$$

Since $\{\lambda_i(x), i = 1, \ldots, s\}$ is a s-separable product set then there exists a monic polynomial π of degree s such that:

$$[\lambda_s(x)]_k = \prod_{r=0}^{k-1} (\lambda(x) + \pi(r)) = \mathcal{B}_k(\lambda(x)),$$

where $\lambda(x) = \prod_{i=1}^{s} \lambda_i(x)$ and $\{\mathcal{B}_k\}_{k\geq 0}$ is the basis given by (2.7).

Replacing in (2.21), $l^l c$ by c and $[\lambda_s(x)]_k$ by $\mathcal{B}_k(\lambda(x))$, we deduce that $\{P_n\}_{n\geq 0}$ is a PS of the form (2.8). □

Proof of Theorem 2.1. The d-OPSs in the class $\mathcal{A}_1 \cup \mathcal{C}$ were characterized in [33], where they showed that:

- The only d-OPSs in $\mathcal{A}_1$ arise for $p = q = 0$ and $m = d+1$.
- The only d-OPSs in $\mathcal{C}$ arise for $p = q = 0$ and $d = sm$.

The d-OPSs in the class $\mathcal{B}$ were characterized in [34], where they showed that the only d-OPSs in $\mathcal{B}$ arise for $p = q = 0$ and $m = d+1$.

For the class $\mathcal{A}_2 \cup \mathcal{A}_3$ with $l = 0$, the d-OPSs were characterized in [17], where they showed that the only d-OPSs in $\mathcal{A}_2 \cup \mathcal{A}_3$ with $l = 0$ arise for $p = 0$ in the following cases.

Table 2
OPSs in $\mathcal{A}$ (also in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$).

Class	p	m	l	s	q	Conditions	Polynomials
$\mathcal{A}_1$	0	2	$\times$	$\times$	0		Hermite
$\mathcal{A}_2$	0	$\times$	0	$\times$	1	$b_1 \notin -\mathbb{N}$	Laguerre
	0	$\times$	0	$\times$	1	$b_1 \in -\mathbb{N}$	Finite Laguerre
	0	$\times$	1	$\times$	1	$b_1, \mu + 1 - b_1 \notin -\mathbb{N}$	Jacobi
	0	$\times$	1	$\times$	1	$b_1 \in -\mathbb{N}$	Finite Jacobi
	0	$\times$	1	$\times$	0		Bessel
$\mathcal{A}_3$	0	$\times$	0	1	0	$c \neq 1$	Charlier
	0	$\times$	0	1	1	$b_1 \notin -\mathbb{N}, c \neq 1$	Meixner
	0	$\times$	0	1	1	$b_1 \in -\mathbb{N}, c \neq 1$	Krawtchouk
	0	$\times$	0	1	1	$b_1 = 2\lambda \notin -\mathbb{N}, c \neq 1$	Meixner–Pollaczek
	0	$\times$	0	2	2	$b_1, b_2 \notin -\mathbb{N}, c = 1$	Continuous dual Hahn
	0	$\times$	0	2	2	$b_1 \in -\mathbb{N}, c = 1$	Dual Hahn
	0	$\times$	1	1	2	$b_1, \notin -\mathbb{N}, c = 1$	Continuous Hahn
	0	$\times$	1	1	2	$b_1, b_2, \mu + 1 - b_1 \notin -\mathbb{N} \in -\mathbb{N}, c = 1$	Hahn
	0	$\times$	1	2	3	$b_i, \mu + 1 - b_i \notin -\mathbb{N}, i = 1, 2, 3, c = 1$	Wilson
	0	$\times$	1	2	3	$b_1 \in -\mathbb{N}, c = 1$	Racah

Case 1: $s < q = d$.

Case 2: $q < s = d$.

Case 3: $q = s = d, c \neq a_d$.

Case 4: $q = s = d + 1, c = a_{d+1}$ and $\frac{a_d}{c} - b_d \notin \mathbb{N}$.

It follows that, to prove Theorem 2.1, it is sufficient to characterize all d-OPSs in the class $\mathcal{A}_2 \cup \mathcal{A}_3$ with $l \neq 0$.

Using Lemma 2.6, we deduce that the classes $\mathcal{A}_2$ and $\mathcal{A}_3$ belongs to the class of PS $\{P_n\}_{n\geq 0}$ of the form (2.8). According to Lemma 2.5, we deduce that the only d-OPSs belonging to the class $\mathcal{A}_2 \cup \mathcal{A}_3$ with $l \neq 0$ arise for $l = d, p = 0$ and $s \leq 2$, where:

- $q \leq d$ if $s = 0$,
- $q = d + 1$ and $c = 1$ if $s = 1$,
- $q = d + 2, c = 1$ and $\mu = -\frac{d+1}{2} - (\sigma + \sigma') + \sum_{k=1}^{d+2} b_k$, where $\lambda(y) = (y + \sigma)(-y + \sigma')$, if $s = 2$.

All these results are summarized in Table 1. $\square$

2.3. *A characterization theorem for all PSs in the Askey-scheme*

For $d = 1$ Theorem 2.1 is reduced to the following characterization theorem.

Corollary 2.7. *The only OPSs in $\mathcal{A}$ (also in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$) are given by Table 2.*

Remark 2.8. 1. Corollary 2.7 contains as particular cases five characterization theorems given in [36,37], and related to Laguerre, Jacobi, Bessel, Hemite, Charlier and Meixner polynomials.
2. As far as we know, the characterization theorems concerning the other PSs in the Askey-scheme and which may be deduced from Corollary 2.7 seem to be new ones.
3. The introduction of the class $\mathcal{B}$ and $\mathcal{C}$ in this paper allows us to obtain d-orthogonal polynomial sets not belonging to $\mathcal{A}$ for $d \geq 2$. But, for $d = 1$, the d-Charlier I and d-Charlier II polynomials are reduced to Charlier polynomials, and Humbert polynomials are reduced to Gegenbauer polynomials, a particular case of Jacobi polynomials.
4. In this paper, we consider the notion of orthogonality ($d = 1$) in the general sense defined in [7,8]. That explains the presence of the Bessel polynomials in Table 2 and not in the Askey-scheme.
5. The finite Laguerre polynomials given in Table 2 do not exist in the Askey-scheme. More details on these polynomials may be found in [38,39].

3. Concluding remarks

- The limit relations between generalized hypergeometric orthogonal polynomials in the Askey-scheme are well known [2]. It is possible to state analogue ones for the polynomials obtained in this paper and to construct a similar table to the Askey-scheme in the context of the d-orthogonality. That will be the subject of a forthcoming paper.

- The declared aim of the paper in the Introduction is to look for a similar table to the Askey-scheme in the context of the d-orthogonality. Our approach was then to introduce first d-orthogonal generalizations of *all* the PSs belonging to the Askey-scheme. That let us to define the class $\mathcal{A}$ as a class containing all PSs in the Askey-scheme to state a characterization problem. We remark that if we extend this problem to the class $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, we obtain new d-OPSs generalizing PSs in the Askey-scheme and not belonging to $\mathcal{A}$, for $d \geq 2$. The classes $\mathcal{B}$ and $\mathcal{C}$ were suggested by certain transformations concerning the generalized hypergeometric representations of the Charlier polynomials and Gegenbauer polynomials. It is then natural to see one's way to extend the characterization problem to a more general class of PSs containing all PSs in the Askey-scheme.

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