A q-UMBRAL APPROACH TO q-APPELL POLYNOMIALS

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ABSTRACT. In this paper we aim to specify some characteristics of the so called family of q-Appell Polynomials by using q-Umbral calculus. Next in our study, we focus on q-Genocchi numbers and polynomials as a famous member of this family. To do this, firstly we show that any arbitrary polynomial can be written based on a linear combination of q-Genocchi polynomials. Finally, we approach to the point that similar properties can be found for the other members of the class of q-Appell polynomials.

1. Introduction and preliminaries

1.1. q-**Calculus.** Throughout this work we consider the notation $\mathbb{N}$ as the set of natural numbers, $\mathbb{N}_0$ as the set of positive integers and $\mathbb{C}$ as the set of complex numbers. We refer the readers to [1] for all the following q-standard notations. The q-shifted factorial is defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1-q^j a), \quad n \in \mathbb{N}, (a;q)_\infty = \prod_{j=0}^{\infty}(1-q^j a), \quad |q| < 1, a \in \mathbb{C}.$$

The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \quad [0]! = 1; \quad [n]_q! = [1]_q[2]_q\ldots[n]_q, \quad [2n]_q!! = [2n]_q[2n-2]_{q\ldots}[2]_q, \quad n \in \mathbb{N}, a \in \mathbb{C},$$

respectively. The q-polynomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-analogue of the function $(x+y)^n$ is defined by

$$(x+y)_q^n := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{1/2k(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0. \tag{1}$$

The q-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1}(1-q^j a) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{1/2k(k-1)}(-1)^k a^k. \tag{2}$$

In the standard approach to the q-calculus, one of the q-analogues of the exponential function is defined as

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \ |z| < \frac{1}{|1-q|}, z \in \mathbb{C}. \tag{3}$$

The q-derivative of a function f at point $0 \neq z \in \mathbb{C}$, is defined as

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1. \tag{4}$$

From this we easily see that

$$D_q e_q(z) = e_q(z).$$

Moreover, Jackson definite integral of an arbitrary function $f(x)$ is defined as [2]

$$(5) \qquad \int_0^x f(x) d_q x = (1-q) \sum_{n=0}^{\infty} x q^j f(x q^j), \quad 0 < q < 1.$$

Noting to the definitions of q-derivative and q-integral of a function $f(x)$ in (4) and (5), it is clear that

$$(6) \qquad D_q \int_0^x f(x) d_q x = f(x), \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

According to Carlitz's extension of the classical Bernoulli and Euler polynomials, [3], [4], [5], q-Bernoulli and q-Euler polynomials are defined by means of the following generating functions

$$(7) \qquad \frac{t}{e_q(t) - 1} e_q(tx) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{z^n}{[n]_q!},$$

$$(8) \qquad \frac{2}{e_q(t) + 1} e_q(tx) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{z^n}{[n]_q!},$$

respectively. In a similar way, according to Kim, q-Genocchi polynomials can be defined by means of the following generating function, [6]

$$(9) \qquad \frac{2t}{e_q(t) + 1} e_q(tx) = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{z^n}{[n]_q!}.$$

For $x = 0$, $\quad B_{n,q}(0) = B_{n,q}$, $\quad E_{n,q}(0) = E_{n,q}$, and $G_{n,q}(0) = G_{n,q}$, are called the n-th q-Bernoulli, q-Euler, and q-Genocchi numbers, respectively.

The research on the above mentioned polynomials is vast. The interested readers are referred to [7]-[30] to see various extensions and relations regarding these numbers and polynomials.

The class of Appell polynomials for the first time attracted Appell's note in 1880, [31]. In his studies, Appell, characterized this family of polynomials completely. Later, the research done by Throne [32], Sheffer [33], and Varma [34] from different points of views, developed the aforementioned class of polynomials. Sheffer, also, showed that how the properties of Appell polynomials hold well for his generalization. In 1954, Sharma and Chak, for the first time, introduced a q-analogue for the family of q-Appell polynomials and called this sequence of polynomials as q-Harmonic, [35]. In the light of their works, Al-Salam, in 1967, reintroduced the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$, and studied some of its properties, [36]. According to his definition, the n-degree polynomials $A_{n,q}(x)$, are called q-Appell provided that any $A_{n,q}(x)$ holds the following q-differential equation

$$(10) \qquad D_{q,x}(A_{n,q}(x)) = [n]_q A_{n-1,q}(x), \quad \text{for } n = 0, 1, 2, \dots.$$

This is equivalent to define this family of polynomials by means of the following generating function $A_q(t)$, as follows

$$(11) \qquad A_q(x,t) := A_q(t) e_q(tx) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_q!}, \quad 0 < q < 1,$$

where

$$(12) \qquad A_q(t) := \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!}, \quad A_q(t) \neq 0,$$

is an analytic function at $t = 0$, and $A_{n,q}(x) := A_{n,q}(0)$. The formal power series $A_q(t)$, in the above definition, is called the determining function of the class of q-Appell polynomials $\{A_{n,q}(x)\}$. In his researches, algebraically, Al-Salam showed that the class of all q-Appell polynomials is a maximal commutative subgroup of the group of all polynomial sets. Later, in 1982, Srivastava's studies on the family of q-Appell polynomials led to more characterization and clarification of these type of polynomials, [37]. During the past few decades, the class of q-Appell polynomials has been studied from different aspects, [39], [40], and using different thechniques, [38]. Recently, the q-difference equations as well as recurrence relations satisfied by sequence of q-Appell polynomials are derived by Mahmudov, [41].

1.2. q-**Umbral Calculus.** In 1978, Roman and Rota viewed the classical umbral calculus from a new perspective and proposed an interesting approach based on a simple but innovative indication for the effect of linear functionals on polynomials, which Roman later called it the modern classical umbral calculus, [46], [47]. Using this new umbral calculus, they defined the sequence of Sheffer polynomials whose their characteristics proved that this new proposed family of polynomials is equivalent to the family of polynomials of type zero which was previously introduced by Sheffer, [50]. Roman, also, proposed a similar umbral approach under the area of nonclassical umbral calculus which is called q-umbral calculus, [47], [48], [49]. Inspired by his work, in the following, we recast the results of q-umbral calculus for q-Appell polynomials.

Let $\mathbb{C}$ be the field of complex numbers and $\mathcal{F}$ set of all formal power q-series in the variable t over $\mathbb{C}$. In other words, $f(t)$ is an element of $\mathcal{F}$ if

$$(13) \qquad f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k,$$

where a_k is in $\mathbb{C}$.

Let $\mathcal{P}$ be the algebra of all polynomials in variable x over $\mathbb{C}$. Let $\mathcal{P}^*$ be the vector space of all linear functionals on $\mathcal{P}$. The action of a linear functional L on an arbitrary polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. We remind that the vector space addition and scalar multiplication operations on $\mathcal{P}^*$ are defined by $\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, and $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, for any constant $c \in \mathbb{C}$.

The formal power q-series in (13) defines the following functional on $\mathcal{P}$

$$(14) \qquad \langle f(x)|x^n\rangle = a_n,$$

for all $n \geq 0$.

Particularly, according to (13) and (14) we have

$$(15) \qquad \langle t^k|x^n\rangle = [n]_q!\delta_{n,k} \quad n, k \geq 0,$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Assume that $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{[k]_q!} t^k$. Since $\langle f_L(t)|x^n\rangle = \langle L|x^k\rangle$, so $f_L(t) = L$. Hence, it is clear that the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Therefore, $\mathcal{F}$ not only can be considered as the algebra of all formal power q-series in variable t, but also it is the vector space of all linear functionals on $\mathcal{P}$. This follows the fact that each member of $\mathcal{F}$ can be assumed as both a formal power $q$series and a linear functional. $\mathcal{F}$ is called the q-umbral algebra and studying its properties is called q-umbral calculus.

Remark 1. *For the q-exponential function $e_q(t)$, defined in (3), it can be easily observed that $\langle e_q(yt)|x^n \rangle = y^n$ and consequently*

$$\langle e_q(yt)|p(x) \rangle = p(y),$$

and

$$\langle e_q(yt) \pm 1 | p(x) \rangle = p(y) \pm p(0).$$

Remark 2. *For $f(t)$ in $\mathcal{F}$ we have*

$$(16) \qquad f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k \rangle}{[k]_q!} t^k,$$

and for all polynomials $p(x)$ in $\mathcal{P}$ we have

$$(17) \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x) \rangle}{[k]_q!} x^k.$$

Proposition 3. *For $f(t)$ and $g(t) \in \mathcal{F}$ we have*

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle.$$

Proposition 4. *For $f(t)$ and $g(t) \in \mathcal{F}$ we have*

$$\langle f(t)g(t)|x^n \rangle = \sum_{k=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right]_q \langle f(t)|x^k \rangle \langle g(t)|x^{n-k} \rangle.$$

Proposition 5. *For $f_1(t), f_2(t), \ldots, f_n(t) \in \mathcal{F}$ we have*

$$(18) \quad \langle f(t)_1 f_2(t) \ldots f_k(t)|x^n \rangle =$$

$$\sum_{i_1+i_2+\ldots+i_k=n} \left[\begin{array}{c} n \\ i_1, i_2, \ldots, i_k \end{array} \right]_q \langle f_1(t)|x_1^i \rangle \langle f_2(t)|x_2^i \rangle \ldots \langle f_k(t)|x_k^i \rangle,$$

where $\left[\begin{array}{c} n \\ i_1, i_2, \ldots, i_k \end{array} \right]_q = \frac{[n]_q!}{[i_1]_q![i_2]_q!\ldots[i_k]_q!}.$

We use the notation t^k for the k-th q-derivative operator, D_q^k, on $\mathcal{P}$ as follows

$$(19) \qquad t^k x^n = \begin{cases} \frac{[n]_q!}{[k]_q!} x^{n-k}, & k \leq n, \\ 0, & k > n. \end{cases}$$

Consequently, using the notation above, each arbitrary function in the form of (13) can be considered as a linear operator on $\mathcal{P}$ defined by

$$(20) \qquad f(t)x^n = \sum_{k=0}^{\infty} \left[\begin{array}{c} n \\ k \end{array} \right]_q a_k x^{n-k}.$$

Now, consider an arbitrary polynomials $p(x) \in \mathcal{P}$. Then, according to the relation (17) for its k-th q-derivative we have

$$(21) \qquad D_q^k p(x) = p^{(k)}(x) = \sum_{j=k}^{\infty} \frac{\langle t^j|p(x) \rangle}{[j]_q!} [j]_q[j-1]_q \ldots [j-k+1]_q x^{j-k}.$$

As the result of the fact above we obtain

$$(22) \qquad t^k p(x) = D_q^k p(x) = p^{(k)}(x),$$

and, also,

$$(23) \qquad p^{(k)}(0) = \langle t^k|p(x) \rangle = \langle 1|p^{(k)}(x) \rangle.$$

The immediate conclusion of the relations (13), (14) and (20) is that each member of $\mathcal{F}$ plays three roles in the q-umbral calculus; a formal power q-series, a linear functional and a linear operator.

The order of a non-zero power q-series $f(t)$ in (13) is denoted by $O(f(t))$ and is defined as the smallest integer k for which the coefficient of t^k is non-zero, that is $a_k \neq 0$. A q-series $f(t)$ with $O(f(t)) = 0$ is called invertible and in case that $O(f(t)) = 1$ it is called a delta q-series.

Theorem 6. *Let $f(t)$ be a delta q-series and $g(t)$ be an invertible series. Then there exists a unique sequence $S_{n,q}(x)$ of q-polynomials satisfying the following conditions*

$$\langle g(t)f(t)^k | S_{n,q}(x) \rangle = [n]_q! \delta_{n,k},$$

for all $n, k \geq 0$.

Definition 7. *In Theorem (6), $\{S_{n,q}(x)\}_{n=0}^{\infty}$ is called the q-Sheffer sequence for the pair $(g(t), f(t))$. Moreover, the q-Sheffer sequences for $(g(t), t)$ is the q-Appell sequence for $g(t)$.*

Theorem 8. *Let $A_{n,q}(x)$ be q-Appell for $g(t)$. Then*

a) *(The Expansion Theorem) for any $h(t)$ in $\mathcal{F}$*

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | A_{k,q}(x) \rangle}{[k]_q!} g(t) t^k,$$

b) *(The Polynomial Expansion Theorem) for any $p(x)$ in $\mathcal{P}$ we have*

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) t^k | p(x) \rangle}{[k]_q!} A_{k,q}(x).$$

Theorem 9. *The following facts are equivalent*

a) $A_{n,q}(x)$ *is q-Appell for $g(t)$.*

b) $t A_{n,q}(x) = [n]_q A_{n-1,q}(x)$, *where $t A_{n,q}(x) = D_q(A_{n,q}(x))$.*

c) *For all $y \in \mathbb{C}$* $\quad \frac{1}{g(t)} e_q(tx) = \sum_{k=0}^{\infty} \frac{A_{k,q}(x)}{[k]_q!} t^k.$

d) $A_{n,q}(x) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle g^{-1}(t) | x^{n-k} \rangle x^k.$

e) $A_{n,q}(x) = g^{-1}(t) x^n.$

Remark 10. *Based on different selections for $g(t)$ in part (c) of Theorem (9), we obtain various families of q-Appell polynomials. For instance, it is clear from relations (7), (8) and (9) that taking $g(t)$ as $\frac{e_q(t)-1}{t}$, $\frac{e_q(t)+1}{2}$ or $\frac{e_q(t)+1}{2t}$, leads to construct the families of q-Bernoulli, q-Euler or q-Genocchi polynomials, respectively.*

Theorem 11. *(The Recurrence Formula for q-Appell Sequences) Suppose that $A_{n,q}(x)$ is q-Appell for $g(t)$. Then we have*

$$A_{n+1,q}(qx) = \left[qx - q^n \frac{D_{q,t}g(t)}{g(qt)} \right] A_{n,q}(x).$$

Proof. We prove this theorem in the light of the technique which is applied in the proof of Theorem 2 in [41]. Since $A_{n,q}(x)$ is q-Appell for $g(t)$ we can write

$$(24) \qquad \frac{1}{g(t)}e_q(tqx) = \sum_{n=0}^{\infty} A_{n,q}(qx)\frac{t^n}{[n]_q!}.$$

Take $\frac{1}{g(t)} = A_q(t)$. According to (12), $A_q(t)$ is analytic. So, differentiating equation (24) and multiplying both sides of the obtained equality by t, we get

$$(25) \qquad \sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = A_q(qt)e_q(tqx)\Big[t\frac{D_q A_q(t)}{A_q(qt)} + tqx\Big],$$

so it follows that

$$(26) \qquad \sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}q^n A_{n,q}(x)\frac{t^n}{[n]_q!}\Big[t\frac{D_q A_q(t)}{A_q(qt)} + tqx\Big].$$

This means that

$$(27) \qquad \sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty}\Big[q^{n-1}A_{n-1,q}(x)\frac{D_q A_q(t)}{A_q(qt)} + qxA_{n-1,q}(x)\Big]\frac{t^n}{[n]_q!},$$

which is equivalent to write

$$(28) \qquad \sum_{n=0}^{\infty}[n]_q A_{n,q}(qx)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty}\Big[q^{n-1}\frac{D_q A_q(t)}{A_q(qt)} + qx\Big]A_{n-1,q}(x)\frac{t^n}{[n]_q!}.$$

Comparing both sides of identity(28), we have

$$(29) \qquad A_{n,q}(qx) = \Big[q^{n-1}\frac{D_q A_q(t)}{A_q(qt)} + qx\Big]A_{n-1,q}(x),$$

whence the result. $\qquad \square$

2. q-UMBRAL PERSPECTIVE OF q-GENOCCHI NUMBERS AND POLYNOMIALS, AN EXAMPLE OF q-APPELL SEQUENCES

Over the pas decades, many results have been derived using Umbral as well as q-Umbral methods for different members of the family of Appell and q-Appell polynomials. In this section, we look at the characteristics and properties of q-Genocchi numbers and polynomials, as an example of the family of q-Appell polynomials, from q-umbral point of view. Indeed, it is possible to derive similar results to the obtained results here for the q-Bernoulli and q-Euler polynomials. The interested readers may see, for instance [42]-[45].

2.1. **Various results regarding q-Genocchi polynomials.** According to relation(9), the sequence of q-Genocchi polynomials $\{G_{n,q}(x)\}_{n=0}^{\infty}$ is q-Appell for $g(t) = \frac{e_q(t)+1}{2t}$. Therefore, relation (6) for the sequence of q-Genocchi polynomials, $\{G_{n,q}(x)\}$, can be expressed as follows

$$(30) \qquad \Big\langle \frac{e_q(t)+1}{2t}t^k|G_{n,q}(x)\Big\rangle = [n]_q!\delta_{n,k}, \quad n,k \geq 0.$$

Remark 12. *As direct corollaries of Theorems (9) and (11) we have*

    a) $tG_{n,q}(x) = D_q G_{n,q}(x) = [n]_q G_{n-1,q}(x),$

    b) $G_{n,q}(x) = \sum_{k=0}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}_q \Big\langle \frac{2t}{e_q(t)+1}\Big|x^{n-k}\Big\rangle x^k,$

c) $G_{n,q}(x) = \frac{2t}{e_q(t)+1}x^n$,

d) $G_{n+1,q}(qx) = \left[qx - q^{n-1}\left(\frac{e_q(t)(t-1)+1}{2t^2}\right)\right]G_{n,q}(x)$.

Proposition 13. *For $n \in \mathbb{N}$ we have*

$$G_{0,q} = 1, \quad \sum_{k=1}^{n}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q G_{n-k,q} = -[n+1]_q(1+G_{n,q}).$$

Proof. According to the relations (4), (15) and (30) we can write

$$\left\langle\frac{e_q(t)+1}{2t}\bigg|x^n\right\rangle = \frac{1}{2[n+1]_q}\left\langle\frac{e_q(t)+1}{t}\bigg|tx^{n+1}\right\rangle$$

$$= \frac{1}{2[n+1]_q} = \frac{1}{2}\int_0^1 x^n d_qx.$$

Therefore, for an arbitrary polynomial $p(x) \in \mathcal{P}$ we can conclude

(31) $$\left\langle\frac{e_q(t)+1}{2t}\bigg|p(x)\right\rangle = \frac{1}{2}\left(\int_0^1 p(x)d_qx + p(0)\right).$$

Now, from one hand if we take $p(x) = G_{n,q}(x)$, then we have

$$\frac{1}{2}\left(\int_0^1 G_{n,q}(x)d_qx + G_{n,q}(0)\right) = \left\langle\frac{e_q(t)+1}{2t}\bigg|G_{n,q}(x)\right\rangle$$

(32) $$= \left\langle 1\bigg|\frac{e_q(t)+1}{2t}G_{n,q}(x)\right\rangle = \left\langle t^0\big|x^n\right\rangle = [n]_q!\delta_{n,0}.$$

From another hand, considering the fact that

(33) $$G_{n,q}(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q G_{n-k,q}x^k,$$

we can conclude that

(34) $$\int_0^1 G_{n,q}(x)d_qx = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q G_{n-k,q}\int_0^1 x^k d_qx = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{G_{n-k,q}(x)}{[k+1]_q}.$$

Comparing identity (32) with (34), we obtain

(35) $$\int_0^1 G_{n,q}(x)d_qx = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{G_{n-k,q}(x)}{[k+1]_q} = \begin{cases} 2 - G_{0,q}(0) & n = 0 \\ -G_{0,q}(0) & n \neq 0 \end{cases},$$

whence the result. $\square$

Remark 14. *According to part(b) of Theorem (8), for an arbitrary polynomial $p(x) \in \mathcal{P}$ we can write*

$$p(x) = \sum_{k=0}^{\infty}\langle\frac{e_q(t)+1}{2t}t^k|p(x)\rangle\frac{G_{k,q}(x)}{[k]_q!}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\langle\frac{e_q(t)+1}{t}|t^kp(x)\rangle\frac{G_{k,q}(x)}{[k]_q!} \quad = \frac{1}{2}\sum_{k=0}^{\infty}\frac{G_{k,q}(x)}{[k]_q!}\left(\int_0^1 t^kp(x)d_qx + t^kp(0)\right).$$

Remark 15. *We know that*
$$\langle e_q(t)t^k|(x-1)_q^n\rangle = [n]_q!\delta_{n,k}.$$
Therefore, according to part(b) of Theorem (8), for $G_{n,q}(x)$ as a polynomial chosen from $\mathcal{P}$ we can obtain

$$G_{n,q}(x) = \sum_{k=0}^{n}\langle e_q(t)|t^k G_{n,q}(x)\rangle\frac{(x-1)_q^n}{[k]_q!}$$

$$= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q G_{n-k,q}(1)(x-1)_q^n.$$

Proposition 16. *For $n \in \mathbb{N}$ we have*
$$(x-1)_q^n =$$
$$\frac{1}{2}\left(\sum_{k=0}^{n}\sum_{l=0}^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q\begin{bmatrix} n-k \\ l \end{bmatrix}_q\frac{1}{[m+1]_q}G_{k,q}(x)(-1)^{n-k-l}q^{\frac{l(l-1)}{2}} + \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}(x)\right).$$

Proof. From the binomial relation(2), we obtain

(36) $$(x-1)_q^n = \sum_{l=0}^{n}(-1)^{n-l}q^{\frac{l(l-1)}{2}}x^l.$$

Now, taking k-th q-derivative from both sides of identity(36), we have

(37) $$t^k(x-1)_q^n = \sum l = k^n\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!}(x-1)_q^{n-k}$$

According to part(b) of Theorem (8), we can write

(38) $$(x-1)_q^n = \sum_{k=0}^{n}\frac{1}{[k]!}\left\langle\frac{e_q(t)+1}{2t}t^k|(x-1)_q^n\right\rangle G_{n,q}(x)$$

$$= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q G_{n,q}(x)\left\langle\frac{e_q(t)+1}{2t}|(x-1)_q^{n-k}\right\rangle$$

$$= \sum_{k=0}^{n}G_{n,q}(x)\left(\int_0^1(x-1)_q^{n-k}d_qx + 1\right)$$

$$= \frac{1}{2}\left(\sum_{k=0}^{n}\sum_{l=0}^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}_q\begin{bmatrix} n-k \\ l \end{bmatrix}_q\frac{1}{[m+1]_q}G_{k,q}(x)(-1)^{n-k-l}q^{\frac{l(l-1)}{2}} + \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}(x)\right).$$
$$\square$$

Theorem 17. *Let $\mathcal{P}_n = \{p(x) \in \mathcal{P}|deg(p(x)) \leq n\}$. Then for an arbitrary $p(x) \in \mathcal{P}_n$ and a constant $c_{n,q}$, we may assume that $p(x) = \sum_{i=0}^{n}c_{i,q}G_{i,q}(x)$. Then for any constant k, the coefficient $c_{k,q}$ is equal to $\frac{1}{[k]_q!}\left\langle\frac{e_q(t)+1}{2t}|p^{(k)}(x)\right\rangle$, and it can be obtained from the following relation*

$$c_{k,q} = \frac{1}{2[k]_q!}\left(\int_0^1 p^{(k)}(x)d_qx + p^{(k)}(0)\right),$$

where $p^{(k)}(x) = D_q^k p(x)$.

Proof. For any polynomial $p(x) = \sum_{i=0}^{n}c_{i,q}G_{i,q}(x)$ in $\mathcal{P}_n$, we may write

(39) $$\left\langle\frac{e_q(t)+1}{2t}t^k|p(x)\right\rangle = \sum_{i=0}^{n}c_{i,q}\left\langle\frac{e_q(t)+1}{2t}t^k|G_{i,q}(x)\right\rangle.$$

So, according to the relation (30), we obtain

$$(40) \qquad = \sum_{i=0}^{n} c_{i,q} [i]_q! \delta_{i,k} = [k]_q! c_{k,q},$$

which means that

$$(41) \qquad c_{k,q} = \frac{1}{[k]_q!} \Big\langle \frac{e_q(t) + 1}{2t} t^k | p(x) \Big\rangle.$$

According to the relation (22), this is equivalent to write

$$(42) \qquad c_{k,q} = \frac{1}{[k]_q!} \Big\langle \frac{e_q(t) + 1}{2t} | t^k p(x) \Big\rangle = \frac{1}{[k]_q!} \Big\langle \frac{e_q(t) + 1}{2t} | p^{(k)}(x) \Big\rangle.$$

finally, using the relation (31), we obtain

$$(43) \qquad c_{k,q} = \frac{1}{2[k]_q!} \Big(\int_0^1 p^{(k)}(x) d_q x + p^{(k)}(0) \Big).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2. **Some results regarding q-Genocchi polynomials of higher order.** Let $q \in \mathbb{C}, m \in \mathbb{N}$ and $0 < |q| < 1$. The q-Genocchi polynomials $G_{n,q}^{[m]}(x)$ in x, of order m, in a suitable neighborhood of $t = 0$, are defined by means of the following generating function, [11]

$$(44) \qquad \Big(\frac{2t}{e_q(t) + 1} \Big)^m e_q(tx) = \sum_{n=0}^{\infty} G_{n,q}^{[m]}(x) \frac{t^n}{[n]_q!}.$$

In case that $x = 0$, $G_{n,q}^{[m]}(0) = G_{n,q}^{[m]}$ is called the n-th q-Genocchi number of order m.

From the above definition, it is clear that the class of q-Genocchi polynomials, $\{G_{n,q}^{[m]}(x)\}_{n=0}^{\infty}$, of order m is q-Appell for $g(t) = \Big(\frac{e_q(t)+1}{2t} \Big)^m$. Thus, according to the relation (6), for the sequence of q-Genocchi polynomials, $G_{n,q}^{[m]}(x)$, of order m, we can write

$$(45) \qquad \Big\langle \Big(\frac{e_q(t) + 1}{2t} \Big)^m t^k | G_{n,q}^{[m]}(x) \Big\rangle = [n]_q! \delta_{n,k}, \quad n, k \geq 0.$$

Lemma 18. *For any $n \in \mathbb{N}_0$, the following identity holds for the n-th q-Genocchi number of order m*

$$G_{n,q}^{[m]} = \sum_{i_1 + i_2 + \ldots + i_m = n} \begin{bmatrix} n \\ i_1, i_2, \ldots, i_m \end{bmatrix}_q G_{i_1,q} G_{i_2,q} \ldots G_{i_m,q}$$

Proof. From one hand, according to the relation (44), it is obvious that

$$(46) \qquad \Big\langle \Big(\frac{2t}{e_q(t) + 1} \Big)^m t^k | x^n \Big\rangle = \sum_{k=0}^{\infty} \frac{G_{n,q}^{[m]}}{[k]_q!} \langle t^k | x^n \rangle = G_{n,q}^{[m]}.$$

From another hand, according to the Proposition (5), we have

$$(47) \quad G_{n,q}^{[m]} = \sum_{i_1 + i_2 + \ldots + i_m = n} \begin{bmatrix} n \\ i_1, i_2, \ldots, i_m \end{bmatrix}_q \Big\langle \frac{2t}{e_q(t) + 1} | x^{i_1} \Big\rangle \Big\langle \frac{2t}{e_q(t) + 1} | x^{i_2} \Big\rangle \ldots \Big\langle \frac{2t}{e_q(t) + 1} | x^{i_m} \Big\rangle.$$

Based on the relations (9) and (15) for each $\langle \frac{2t}{e_q(t)+1} | x^{i_l} \rangle$, $l \in \{1, 2, \ldots, m\}$ we can write

$$(48) \qquad \Big\langle \frac{2t}{e_q(t) + 1} | x^{i_l} \Big\rangle = \sum_{k=0}^{\infty} \frac{G_{i_l,q}}{[k]!} \langle t^k | x^{i_l} \rangle = G_{i_l,q},$$

whence the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 19. *For any $n \in \mathbb{N}_0$, the following identity holds for the n-th q-Genocchi polynomial of order m*

$$G_{n,q}^{[m]}(x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left\langle \frac{e_q(t)+1}{2t} | G_{n-k,q}^{[m]}(x) \right\rangle G_{k,q}(x) = \frac{1}{2^{m-1}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q G_{n-k,q}^{[m-1]} G_{k,q}(x).$$

Proof. According to the relation (44), it is clear that

$$(49) \qquad\qquad\qquad G_{n,q}^{[m]}(x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q G_{n-k,q}^{[m]} x^k.$$

Therefore, we may assume that $G_{n,q}^{[m]}(x) = \sum_{k=0}^{n} c_{k,q} G_{k,q}(x)$ is a polynomial with degree n in $\mathcal{P}_n$. Since $G_{n,q}^{[m]}(x)$ is a q-Appell polynomial, according to part(b) of Theorem (9) for its k-th q-derivative we can write

$$(50) \qquad D_q^k G_{n,q}^{[m]}(x) = [n]_q [n-1]_q \ldots [n-k+1]_q G_{n-k,q}^{[m]}(x) = \frac{[n]_q!}{[n-k]_q!} G_{n-k,q}^{[m]}(x).$$

Now, according to the relation (42), we may continue as

$$c_{k,q} = \frac{1}{[k]_q!} \left\langle \frac{e_q(t)+1}{2t} | t^k G_{n,q}^{[m]}(x) \right\rangle$$

$$(51) \qquad\qquad\qquad = \frac{1}{[k]_q!} \left\langle \frac{e_q(t)+1}{2t} | D_q^k G_{n,q}^{[m]}(x) \right\rangle = \left[\begin{array}{c} n \\ k \end{array} \right]_q \left\langle \frac{e_q(t)+1}{2t} | G_{n-k,q}^{[m]}(x) \right\rangle.$$

According to part(e) of Theorem (9), it is clear that the q-Appell polynomial $G_{n-k,q}^{[m]}(x)$ is equal to $\left(\frac{e_q(t)+1}{2t} \right)^m x^{n-k}$. As the result of this fact and noting to the relation (23), we obtain from the last identity in (51)

$$(52) \qquad c_{k,q} = \left[\begin{array}{c} n \\ k \end{array} \right]_q \left\langle t^0 | \frac{2t}{e_q(t)+1} \left(\frac{e_q(t)+1}{2t} \right)^m x^{n-k} \right\rangle = \frac{1}{2^{m-1}} \left[\begin{array}{c} n \\ k \end{array} \right]_q G_{n-k,q}^{[m-1]},$$

whence the result. □

Theorem 20. *For any arbitrary polynomial $p(x) \in \mathcal{P}_n$ the following identity holds*

$$p(x) = \sum_{k=0}^{n} \left\langle \left(\frac{e_q(t)+1}{2t} \right)^m t^k | p(x) \right\rangle \frac{G_{k,q}^{[m]}(x)}{[k]_q!}.$$

Proof. Assume that $p(x) = \sum_{i=0}^{n} c_{i,q} G_{i,q}^{[m]}(x)$. Therefore, noting to the relation (45) for the q-Appell polynomial $G_{i,q}^{[m]}(x)$, we may conclude that

$$(53) \quad \left\langle \left(\frac{e_q(t)+1}{2t} \right)^m t^k | p(x) \right\rangle = \sum_{i=0}^{n} c_{i,q} \left\langle \left(\frac{e_q(t)+1}{2t} \right)^m t^k | G_{i,q}^{[m]}(x) \right\rangle = \sum_{i=0}^{n} c_{i,q} [i]_q! \delta_{i,k} = c_{k,q} [k]_q!.$$

Thus,

$$(54) \qquad\qquad\qquad c_{k,q} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)+1}{2t} \right)^m t^k | p(x) \right\rangle.$$

Substituting $c_{k,q}$ in the summation assumed in the beginning of the proof, leads to obtain the desired result. □

Theorem 21. *For any $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}$, the n-th q-Genocchi polynomial can be expressed based on the following relation*

$$G_{n,q}(x) = \sum_{k=0}^{m-1} \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \times$$

$$\left\{ \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q \sum_{l=0}^{n+m-k} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q G_{n+m-k-l,q} \right\} G_{k,q}^{[m]}(x)$$

$$+ \sum_{k=m}^{n} \frac{\begin{bmatrix} n \\ k-m \end{bmatrix}_q}{2^m [k]_q! \begin{bmatrix} k \\ m \end{bmatrix}_q} \times$$

$$\left\{ \sum_{i=0}^{m} \sum_{l=0}^{n-k+m} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1, l_2, \ldots, l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q G_{n-k+m-l,q} \right\} G_{k,q}^{[m]}(x)$$

Proof. In Theorem (20), take $p(x)$ to be the n-th q-Genocchi polynomial $G_{n,q}(x)$, that is

(55)
$$G_{n,q}(x) = \sum_{k=0}^{n} c_{k,q} G_{k,q}^{[m]}(x),$$

where

(56)
$$c_{k,q} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)+1}{2t} \right)^m t^k \Big| G_{n,q}(x) \right\rangle.$$

Then, for $k < m$, we have

$$c_{k,q} = \frac{1}{2^m [k]_q!} \left\langle \frac{(e_q(t)+1)^m}{t^{m-k}} \Big| G_{n,q}(x) \right\rangle$$

$$= \frac{1}{2^m [k]_q!} \cdot \frac{1}{[n+m-k]_q! \ldots [n+1]_q!} \left\langle (e_q(t)+1)^m \left(\frac{1}{t} \right)^{m-k} \Big| t^{m-k} G_{n+m-k,q}(x) \right\rangle$$

$$= \frac{[m]_q!}{2^m [k]_q! [m-k]_q!} \cdot \frac{[m-k]_q!}{[n+m-k]_q! \ldots [n+1]_q!} \left\langle (e_q(t)+1)^m \Big| G_{n+m-k,q}(x) \right\rangle$$

$$= \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m} \cdot \frac{[m-k]_q!}{[m]_q! [n+m-k]_q! \ldots [n+1]_q!} \left\langle \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q (e_q(t))^m \Big| G_{n+m-k,q}(x) \right\rangle$$

$$= \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \left\langle \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q (e_q(t))^m \Big| G_{n+m-k,q}(x) \right\rangle.$$

Applying relation (33) to $G_{n+m-k,q}(x)$, we may continue as

$$c_{k,q} = \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \left\langle \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q (e_q(t))^m \Big| \sum_{l=0}^{n+m-k} \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q G_{n+m-k-l,q} x^l \right\rangle.$$

Using Proposition (5) and considering Remark (1), we obtain
(57)
$$c_{k,q} = \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q \sum_{l=0}^{n+m-k} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1,l_2,\ldots,l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q G_{n+m-k-l,q}.$$

Now, assume that $k \geq m$. Then starting from the relation (56), we have

$$c_{k,q} = \frac{1}{[k]_q!} \langle (\frac{e_q(t)+1}{2t})^m t^k | G_{n,q}(x) \rangle$$

$$= \frac{1}{2^m [k]_q!} \langle (e_q(t)+1)^m t^{k-m} | G_{n,q}(x) \rangle = \frac{1}{2^m [k]_q!} \langle (e_q(t)+1)^m | t^{k-m} G_{n,q}(x) \rangle$$

$$= \frac{1}{2^m [k]_q!} \cdot \frac{1}{[n+m-k]_q! \ldots [n+1]_q!} \langle (e_q(t)+1)^m | G_{n-k+m,q}(x) \rangle$$

$$= \frac{1}{2^m [k]_q!} \cdot \frac{[n]_q! [k-m]_q!}{[n-k-m]_q! [k-m]_q!} \langle (e_q(t)+1)^m | G_{n-k+m,q}(x) \rangle$$

$$= \frac{[k-m]_q!}{2^m [k]_q!} \begin{bmatrix} n \\ k-m \end{bmatrix}_q \sum_{i=0}^{m} \langle (e_q(t)+1)^i | G_{n-k+m,q}(x) \rangle.$$

Finally, we obtain
(58)
$$c_{k,q} = \frac{\begin{bmatrix} n \\ k-m \end{bmatrix}_q}{2^m [k]_q! \begin{bmatrix} k \\ m \end{bmatrix}_q} \sum_{i=0}^{m} \sum_{l=0}^{n-k+m} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1,l_2,\ldots,l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q G_{n-k+m-l,q}.$$

Replacing identities (57) and (58) in the assumed sum in (55), completes the proof. $\square$

Remark 22. *According to the proof of Theorem (21), for any $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}$, the n-th q-Appell polynomial, $A_{n,q}(x)$, can be expressed based on the following relation*

$$A_{n,q}(x) = \sum_{k=0}^{m-1} \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q}{2^m [m]_q! \begin{bmatrix} n+m-k \\ m-k \end{bmatrix}_q} \times$$

$$\left\{ \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q \sum_{l=0}^{n+m-k} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1,l_2,\ldots,l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q A_{n+m-k-l,q} \right\} G_{k,q}^{[m]}(x)$$

$$+ \sum_{k=m}^{n} \frac{\begin{bmatrix} n \\ k-m \end{bmatrix}_q}{2^m [k]_q! \begin{bmatrix} k \\ m \end{bmatrix}_q} \times$$

$$\left\{ \sum_{i=0}^{m} \sum_{l=0}^{n-k+m} \sum_{l_1+l_2+\ldots+l_i=l} \begin{bmatrix} l \\ l_1,l_2,\ldots,l_i \end{bmatrix}_q \begin{bmatrix} n+m-k \\ l \end{bmatrix}_q A_{n-k+m-l,q} \right\} G_{k,q}^{[m]}(x)$$

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