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Some results on power sums and Apostol type polynomials

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Abstract

In this paper, we study the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the power sums with respect to λ. As a result, various identities are established, including the multiplication formulae of the Apostol type polynomials, some symmetric identities and some convolution identities.

Keywords: Apostol-Bernoulli polynomials; Apostol-Euler polynomials; Power sums; Multiplication formulae; Combinatorial identities

1. Introduction

The Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$ are defined by the following generating functions (for example, see [1, Chapter 23]):

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi)\quad\text{and}\quad\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad(|t|<\pi)\,.$$

The Bernoulli numbers B_n and the Euler numbers E_n are given by $B_n:=B_n(0)$ and $E_n:=2^nE_n\left(\frac{1}{2}\right)$. It is well known that the power sums $S_k(n)$ and the alternate power sums $T_k(n)$ are closely related to the Bernoulli polynomials and the Euler polynomials, respectively, as follows (see [1, Eq. (23.1.4)]):

$$S_k(n)=\sum_{i=0}^{n}i^k=\frac{B_{k+1}(n+1)-B_{k+1}}{k+1}\,, \tag{1.1}$$

$$T_k(n)=\sum_{i=0}^{n}(-1)^i i^k=\frac{(-1)^nE_k(n+1)+E_k(0)}{2}\,, \tag{1.2}$$

where n and k are nonnegative integers. Moreover, between the Bernoulli numbers and the power sums, the next relation holds:

$$B_n=\sum_{k=0}^{n}\binom{n}{k}a^{k-1}B_kS_{n-k}(a-1)\,, \tag{1.3}$$

where a is a positive integer and n is a nonnegative integer. This relation was proved by Deeba and Rodriguez [4], Gessel [6] and Howard [7].

Tuenter [14] found that (1.3) is a special case of the following symmetric identity

$$\sum_{k=0}^{n}\binom{n}{k}a^{k-1}B_kb^{n-k}S_{n-k}(a-1)=\sum_{k=0}^{n}\binom{n}{k}b^{k-1}B_ka^{n-k}S_{n-k}(b-1)\,, \tag{1.4}$$

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where a and b are positive integers and n is a nonnegative integer. Most recently, Tuenter's result was generalized to the Bernoulli polynomials and Euler polynomials by Kim [8], to the higher order Bernoulli polynomials by Yang [17] and to the degenerate Bernoulli polynomials by Young [18]. We also studied this problem in [9] and established various identities concerning the (higher order) Bernoulli polynomials, the (higher order) Euler polynomials, the Genocchi polynomials and the higher order degenerate Bernoulli polynomials.

On the other hand, in recent years, the Apostol type polynomials and numbers received wide concern. The Apostol-Bernoulli polynomials $\mathfrak{B}_n(x;\lambda)$ and the Apostol-Bernoulli numbers $\mathfrak{B}_n(\lambda)$ were first defined by Apostol [2] when he studied the Lipschitz-Lerch Zeta functions. Recently, Luo and Srivastava introduced and studied the higher order Apostol-Bernoulli polynomials and the higher order Apostol-Euler polynomials [10–13]. More results on Apostol type polynomials can be found in [3, 5, 15].

Let us give the explicit definitions of the Apostol type polynomials and numbers.

Definition 1.1. The higher order Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ and the higher order Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ are defined by the following generating functions:

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}\,, \quad (|t+\log\lambda|<2\pi)\,, \tag{1.5}$$

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}\,, \quad (|t+\log\lambda|<\pi)\,. \tag{1.6}$$

The Apostol-Bernoulli polynomials $\mathfrak{B}_n(x;\lambda)$ and the Apostol-Euler polynomials $\mathfrak{E}_n(x;\lambda)$ are given by

$$\mathfrak{B}_n(x;\lambda) := \mathfrak{B}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathfrak{E}_n(x;\lambda) := \mathfrak{E}_n^{(1)}(x;\lambda)\,. \tag{1.7}$$

Furthermore, the Apostol-Bernoulli numbers $\mathfrak{B}_n(\lambda)$ and the Apostol-Euler numbers $\mathfrak{E}_n(\lambda)$ are given by

$$\mathfrak{B}_n(\lambda) := \mathfrak{B}_n(0;\lambda) \quad \text{and} \quad \mathfrak{E}_n(\lambda) := 2^n\mathfrak{E}_n\left(\frac{1}{2};\lambda\right)\,. \tag{1.8}$$

Obviously, the substitution $\lambda = 1$ in (1.5) to (1.8) gives us the classical Bernoulli and Euler polynomials and numbers. Additionally, it can be shown that

$$\mathfrak{B}_n(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k(x;\lambda)y^{n-k}\,, \tag{1.9}$$

$$\mathfrak{E}_n(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(x;\lambda)y^{n-k}\,, \tag{1.10}$$

which will be frequently made use of later.

The purpose of this paper is to study the power sums and the Apostol type polynomials. In Section 2, we introduce the power sums with respect to λ, and show the elementary relations between these sums and the Apostol type polynomials. Section 3 gives some symmetric identities which are similar to (1.4). From these identities, the multiplication theorems for the Apostol type polynomials as well as some convolution identities are obtained. In Section 4, making use of two expansions similar to those of the hyperbolic cotangent and hyperbolic tangent, we obtain more results. The symmetric identities satisfied by the higher order Apostol type polynomials are also given there.

When $\lambda = 1$, our results will reduce to the corresponding ones established in [4, 6–9, 14, 17]. It should be noticed that Zhang and Yang [19] also gave some identities on the power sums and the higher order Apostol-Bernoulli polynomials. However, they made use of a strange kind of power sums so that their results can not completely unify those presented before.

2. Power sums and Apostol type polynomials

The power sums and the alternate power sums (with respect to λ) are defined by

$$S_k(n;\lambda) = \sum_{i=0}^{n} \lambda^i i^k \quad \text{and} \quad T_k(n;\lambda) = \sum_{i=0}^{n} (-1)^i \lambda^i i^k , \tag{2.1}$$

and their exponential generating functions are

$$\sum_{k=0}^{\infty} S_k(n;\lambda)\frac{t^k}{k!} = \frac{(\lambda e^t)^{n+1}-1}{\lambda e^t - 1} \quad \text{and} \quad \sum_{k=0}^{\infty} T_k(n;\lambda)\frac{t^k}{k!} = \frac{1-(-\lambda e^t)^{n+1}}{1+\lambda e^t} . \tag{2.2}$$

The following are some special values:

$$\begin{gathered} S_k(0;\lambda) = T_k(0;\lambda) = \delta_{0,k} , \\ S_k(1;\lambda) = 0^k + \lambda \cdot 1^k = \delta_{0,k} + \lambda , \quad T_k(1;\lambda) = 0^k - \lambda \cdot 1^k = \delta_{0,k} - \lambda , \end{gathered} \tag{2.3}$$

where $\delta_{i,j}$ is the Kronecker delta defined by $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$. The sums $T_k(n;\lambda)$ are equal to $S_k(n;-\lambda)$, so in theory, it is not necessary to introduce these kind of sums. The $T_k(n;\lambda)$ are formally defined here for convenience and for consistency with the traditional.

The elementary relations between the power sums (2.1) with respect to λ and the Apostol type polynomials are shown in the following theorem.

Theorem 2.1. *Let n and k be nonnegative integers, then*

$$S_k(n;\lambda) = \sum_{i=0}^{n} \lambda^i i^k = \frac{\lambda^{n+1}\mathfrak{B}_{k+1}(n+1;\lambda) - \mathfrak{B}_{k+1}(\lambda)}{k+1} , \tag{2.4}$$

$$T_k(n;\lambda) = \sum_{i=0}^{n} (-1)^i \lambda^i i^k = \frac{(-1)^n \lambda^{n+1}\mathfrak{E}_k(n+1;\lambda) + \mathfrak{E}_k(0;\lambda)}{2} \tag{2.5}$$

$$= 2^{k+1} S_k\left(\left[\frac{n}{2}\right];\lambda^2\right) - S_k(n;\lambda) . \tag{2.6}$$

Proof. From Definition 1.1 and Eq. (2.2), we have

$$\sum_{k=0}^{\infty} S_k(n;\lambda)\frac{t^{k+1}}{k!} = \lambda^{n+1}\frac{t}{\lambda e^t - 1}e^{(n+1)t} - \frac{t}{\lambda e^t - 1} = \lambda^{n+1}\sum_{k=0}^{\infty} \mathfrak{B}_k(n+1;\lambda)\frac{t^k}{k!} - \sum_{k=0}^{\infty} \mathfrak{B}_k(\lambda)\frac{t^k}{k!} .$$

Hence, by identification of the coefficients of $t^{k+1}/k!$, we obtain (2.4). Analogously, we can obtain (2.5). Now, according to the definition of $T_k(n;\lambda)$, we have

$$T_k(n;\lambda) = 2\sum_{\substack{i=0 \\ 2|i}}^{n} \lambda^i i^k - \sum_{i=0}^{n} \lambda^i i^k = 2^{k+1}\sum_{i=0}^{\left[\frac{n}{2}\right]} (\lambda^2)^i i^k - \sum_{i=0}^{n} \lambda^i i^k .$$

This gives relation (2.6). □

Besides the sums $S_k(n;\lambda)$ and $T_k(n;\lambda)$, we may also consider the sum of the k-th powers of the terms of an arithmetic progression, which is defined by

$$S_k^{(a,b)}(n;\lambda) = \sum_{i=0}^{n} \lambda^i (a+ib)^k = a^k + \lambda(a+b)^k + \lambda^2(a+2b)^k + \cdots + \lambda^n(a+nb)^k . \tag{2.7}$$

Then $S_k^{(0,1)}(n;\lambda)=S_k(n;\lambda)$ and $S_k^{(1,2)}(n;\lambda)=1^k+\lambda 3^k+\cdots+\lambda^n(2n+1)^k$, that is, the sum of the k-th powers of the first $n+1$ odd integers. Formally, let the corresponding alternate sum be

$$T_k^{(a,b)}(n;\lambda)=S_k^{(a,b)}(n;-\lambda)=\sum_{i=0}^{n}(-1)^i\lambda^i(a+ib)^k\,.$$

Similarly to Eqs. (2.4) and (2.5), the sums $S_k^{(a,b)}(n;\lambda)$ and $T_k^{(a,b)}(n;\lambda)$ can be computed by the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials, as follows:

$$\begin{aligned} S_k^{(a,b)}(n;\lambda)&=\frac{b^k}{k+1}\left(\lambda^{n+1}\mathfrak{B}_{k+1}\left(\frac{a}{b}+n+1;\lambda\right)-\mathfrak{B}_{k+1}\left(\frac{a}{b};\lambda\right)\right)\,,\\ T_k^{(a,b)}(n;\lambda)&=\frac{b^k}{2}\left((-1)^n\lambda^{n+1}\mathfrak{E}_k\left(\frac{a}{b}+n+1;\lambda\right)+\mathfrak{E}_k\left(\frac{a}{b};\lambda\right)\right)\,. \end{aligned}$$

3. Identities and multiplication theorems

In this section, we prove some symmetric identities involving the power sums and the Apostol type polynomials. From these identities, we can obtain the multiplication theorems of the Apostol type polynomials and some convolution formulae.

Theorem 3.1. *For integers $n\geq 0$, $a\geq 1$ and $b\geq 1$, we have*

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}a^{k-1}b^{n-k}\mathfrak{B}_k(bx;\lambda^a)S_{n-k}(a-1;\lambda^b)&=\sum_{k=0}^{n}\binom{n}{k}b^{k-1}a^{n-k}\mathfrak{B}_k(ax;\lambda^b)S_{n-k}(b-1;\lambda^a)\\ &=a^{n-1}\sum_{i=0}^{a-1}\lambda^{bi}\mathfrak{B}_n\left(bx+\frac{b}{a}i;\lambda^a\right)=b^{n-1}\sum_{i=0}^{b-1}\lambda^{ai}\mathfrak{B}_n\left(ax+\frac{a}{b}i;\lambda^b\right)\,. \end{aligned}\tag{3.1}$$

Proof. Let

$$f(t)=\frac{te^{abxt}(\lambda^{ab}e^{abt}-1)}{(\lambda^a e^{at}-1)(\lambda^b e^{bt}-1)}=\frac{1}{a}\left(\frac{at}{\lambda^a e^{at}-1}e^{abxt}\right)\left(\frac{\lambda^{ab}e^{abt}-1}{\lambda^b e^{bt}-1}\right)\,.$$

Then $f(t)$ can be expanded in two ways:

$$\begin{aligned} f(t)&=\frac{1}{a}\left(\sum_{n=0}^{\infty}\mathfrak{B}_n(bx;\lambda^a)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}S_n(a-1;\lambda^b)\frac{(bt)^n}{n!}\right)\\ &=\frac{1}{a}\sum_{i=0}^{a-1}\lambda^{bi}\left(\frac{at}{\lambda^a e^{at}-1}e^{\left(bx+\frac{b}{a}i\right)at}\right)=\sum_{n=0}^{\infty}\left(a^{n-1}\sum_{i=0}^{a-1}\lambda^{bi}\mathfrak{B}_n\left(bx+\frac{b}{a}i;\lambda^a\right)\right)\frac{t^n}{n!}\,. \end{aligned}$$

Equating coefficients of $t^n/n!$ and noting that $f(t)$ is symmetric in a and b lead us to the identity of the theorem. □

When $\lambda=1$ and $x=0$, Theorem 3.1 reduces to Tuenter's result (1.4). In addition to this, Theorem 3.1 gives the following corollary.

Corollary 3.2. *For any nonnegative integer n and any positive integer a, we have*

$$\mathfrak{B}_n(ax;\lambda)=\sum_{k=0}^{n}\binom{n}{k}a^{k-1}\mathfrak{B}_k(x;\lambda^a)S_{n-k}(a-1;\lambda)=a^{n-1}\sum_{i=0}^{a-1}\lambda^i\mathfrak{B}_n\left(x+\frac{i}{a};\lambda^a\right)\,,\tag{3.2}$$

$$\begin{aligned} \mathfrak{B}_n(ax;\lambda^2)+\lambda^a\mathfrak{B}_n\left(ax+\frac{a}{2};\lambda^2\right)&=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{2}\right)^{k-1}\mathfrak{B}_k(2x;\lambda^a)S_{n-k}(a-1;\lambda^2)\\ &=\left(\frac{a}{2}\right)^{n-1}\sum_{i=0}^{a-1}\lambda^{2i}\mathfrak{B}_n\left(2x+\frac{2}{a}i;\lambda^a\right)\,. \end{aligned}\tag{3.3}$$

Proof. Putting $b=1$ in identity (3.1) and then making use of (2.3) give us (3.2), which is in fact the multiplication formula of the Apostol-Bernoulli polynomials. To prove (3.3), note that when $b=2$, the second member of identity (3.1) turns into

$$\sum_{k=0}^{n}\binom{n}{k}2^{k-1}a^{n-k}\mathfrak{B}_k(ax;\lambda^2)S_{n-k}(1;\lambda^a) = 2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{2}\right)^{n-k}\mathfrak{B}_k(ax;\lambda^2)(0^{n-k}+\lambda^a 1^{n-k})$$
$$= 2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k(ax;\lambda^2)\left(0^{n-k}+\lambda^a\left(\frac{a}{2}\right)^{n-k}\right).$$

Applying (1.9) and combining the result with the other members of (3.1), we can obtain the desired identity. □

Putting $a=2$ in (3.2), using Eqs. (1.9) and (2.3) again and replacing x by $x/2$, we have

$$\mathfrak{B}_n(x;\lambda) = 2^{n-1}\left(\mathfrak{B}_n\left(\frac{x}{2};\lambda^2\right)+\lambda\mathfrak{B}_n\left(\frac{x+1}{2};\lambda^2\right)\right), \tag{3.4}$$

which reduces to $B_n\left(\frac{1}{2}\right)=(2^{1-n}-1)B_n$ when $\lambda=1$ and $x=0$.

Theorem 3.3. *For integers $n\geq 0$, $a\geq 1$ and $b\geq 1$, if a and b have the same parity, then*

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\mathfrak{E}_k(bx;\lambda^a)T_{n-k}(a-1;\lambda^b) = \sum_{k=0}^{n}\binom{n}{k}b^k a^{n-k}\mathfrak{E}_k(ax;\lambda^b)T_{n-k}(b-1;\lambda^a)$$
$$= a^n\sum_{i=0}^{a-1}(-1)^i\lambda^{bi}\mathfrak{E}_n\left(bx+\frac{b}{a}i;\lambda^a\right) = b^n\sum_{i=0}^{b-1}(-1)^i\lambda^{ai}\mathfrak{E}_n\left(ax+\frac{a}{b}i;\lambda^b\right). \tag{3.5}$$

Proof. Let

$$g(t)=\frac{e^{abxt}(1-(-\lambda^b e^{bt})^a)}{(\lambda^a e^{at}+1)(\lambda^b e^{bt}+1)} = \frac{1}{2}\left(\frac{2}{\lambda^a e^{at}+1}e^{abxt}\right)\left(\frac{1-(-\lambda^b e^{bt})^a}{1+\lambda^b e^{bt}}\right). \tag{3.6}$$

We now expand $g(t)$ in two ways:

$$g(t)=\frac{1}{2}\left(\sum_{n=0}^{\infty}\mathfrak{E}_n(bx;\lambda^a)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(a-1;\lambda^b)\frac{(bt)^n}{n!}\right)$$
$$=\frac{1}{2}\sum_{i=0}^{a-1}(-1)^i\lambda^{bi}\left(\frac{2}{\lambda^a e^{at}+1}e^{\left(bx+\frac{b}{a}i\right)at}\right) = \frac{1}{2}\sum_{i=0}^{a-1}(-1)^i\lambda^{bi}\left(\sum_{n=0}^{\infty}\mathfrak{E}_n\left(bx+\frac{b}{a}i;\lambda^a\right)\frac{(at)^n}{n!}\right).$$

If a and b have the same parity, then $g(t)$ is symmetric in a and b. Equating the coefficients of $t^n/n!$ and taking into account the symmetry yield the final result. □

Corollary 3.4. *For any nonnegative integer n and any positive odd integer a, we have*

$$\mathfrak{E}_n(ax;\lambda)=\sum_{k=0}^{n}\binom{n}{k}a^k\mathfrak{E}_k(x;\lambda^a)T_{n-k}(a-1;\lambda) = a^n\sum_{i=0}^{a-1}(-1)^i\lambda^i\mathfrak{E}_n\left(x+\frac{i}{a};\lambda^a\right). \tag{3.7}$$

For any nonnegative integer n and any positive even integer a, we have

$$\mathfrak{E}_n(ax;\lambda^2)-\lambda^a\mathfrak{E}_n\left(ax+\frac{a}{2};\lambda^2\right) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{2}\right)^k\mathfrak{E}_k(2x;\lambda^a)T_{n-k}(a-1;\lambda^2)$$
$$=\left(\frac{a}{2}\right)^n\sum_{i=0}^{a-1}(-1)^i\lambda^{2i}\mathfrak{E}_n\left(2x+\frac{2}{a}i;\lambda^a\right). \tag{3.8}$$

Proof. The substitutions $b = 1$ and $b = 2$ in (3.5) give us (3.7) and (3.8). Note that (3.7) is one of the two formulae of the multiplication theorem of the Apostol-Euler polynomials. □

Theorem 3.5. *For integers $n \geq 1$, $a \geq 1$ and $b \geq 1$, if a is even, then*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} a^{k-1} b^{n-k} \mathfrak{B}_k(bx; \lambda^a) T_{n-k}(a-1; \lambda^b) \\
&= -\frac{n}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} b^k a^{n-1-k} \mathfrak{E}_k(ax; \lambda^b) S_{n-1-k}(b-1; \lambda^a) \\
&= a^{n-1} \sum_{i=0}^{a-1} (-1)^i \lambda^{bi} \mathfrak{B}_n \left(bx + \frac{b}{a} i; \lambda^a \right) = -\frac{n}{2} b^{n-1} \sum_{i=0}^{b-1} \lambda^{ai} \mathfrak{E}_{n-1} \left(ax + \frac{a}{b} i; \lambda^b \right). \qquad (3.9)
\end{aligned}$$

Proof. Let

$$h(t) = \frac{t e^{abxt} (1 - (-\lambda^b e^{bt})^a)}{(\lambda^a e^{at} - 1)(\lambda^b e^{bt} + 1)} \qquad (3.10)$$

and then expand it. Note that now the integer a is even. □

Corollary 3.6. *For any positive integer n and any positive even integer a, we have*

$$\begin{aligned}
\mathfrak{E}_{n-1}(ax; \lambda) &= -\frac{2}{n} \sum_{k=0}^{n} \binom{n}{k} a^{k-1} \mathfrak{B}_k(x; \lambda^a) T_{n-k}(a-1; \lambda) \\
&= -\frac{2}{n} a^{n-1} \sum_{i=0}^{a-1} (-1)^i \lambda^i \mathfrak{B}_n \left(x + \frac{i}{a}; \lambda^a \right), \qquad (3.11) \\
\mathfrak{E}_{n-1}(x; \lambda) &= \frac{2^n}{n} \left(\lambda \mathfrak{B}_n \left(\frac{x+1}{2}; \lambda^2 \right) - \mathfrak{B}_n \left(\frac{x}{2}; \lambda^2 \right) \right) \qquad (3.12) \\
&= \frac{2}{n} \left(\mathfrak{B}_n(x; \lambda) - 2^n \mathfrak{B}_n \left(\frac{x}{2}; \lambda^2 \right) \right). \qquad (3.13)
\end{aligned}$$

Proof. Putting $b = 1$ in Theorem 3.5 yields (3.11). Putting $a = 2$ in (3.11), using Eqs. (1.9) and (2.3) and replacing x by $x/2$, we have (3.12). Finally, combining (3.4) and (3.12), we can obtain (3.13). Eq. (3.11) is another formula of the multiplication theorem of the Apostol-Euler polynomials (see Eq. (3.7) of Corollary 3.4). Additionally, (3.12) and (3.13) are known relations between Apostol-Bernoulli and Apostol-Euler polynomials [13, Eqs. (37) and (38)]. □

Corollary 3.7. *For any positive integer n and any positive even integer a, we have*

$$\begin{aligned}
\mathfrak{E}_{n-1}(ax; \lambda^2) + \lambda^a \mathfrak{E}_{n-1} \left(ax + \frac{a}{2}; \lambda^2 \right) &= -\frac{2}{n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{2} \right)^{k-1} \mathfrak{B}_k(2x; \lambda^a) T_{n-k}(a-1; \lambda^2) \\
&= -\frac{2}{n} \left(\frac{a}{2} \right)^{n-1} \sum_{i=0}^{a-1} (-1)^i \lambda^{2i} \mathfrak{B}_n \left(2x + \frac{2}{a} i; \lambda^a \right), \qquad (3.14) \\
\lambda \mathfrak{B}_n(x+1; \lambda) - \mathfrak{B}_n(x; \lambda) = \frac{n}{2} (\mathfrak{E}_{n-1}(x; \lambda) + \lambda \mathfrak{E}_{n-1}(x+1; \lambda)). \qquad (3.15)
\end{aligned}$$

Proof. (3.14) can be obtained from Theorem 3.5 by the substitution $b = 2$. Putting $a = 2$ in (3.14) and replacing x by $x/2$ yield (3.15). In fact, (3.15) equals nx^{n-1} and corresponds to the difference formula of the Apostol-Bernoulli polynomials (see [13, Eqs. (23) and (24)]). □

Corollary 3.8. *For any positive integers n and b, we have*

$$\begin{aligned}\mathfrak{B}_n(bx;\lambda^2)-\lambda^b\mathfrak{B}_n\left(bx+\frac{b}{2};\lambda^2\right)&=-\frac{n}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{b}{2}\right)^k\mathfrak{E}_k(2x;\lambda^b)S_{n-1-k}(b-1;\lambda^2)\\&=-\frac{n}{2}\left(\frac{b}{2}\right)^{n-1}\sum_{i=0}^{b-1}\lambda^{2i}\mathfrak{E}_{n-1}\left(2x+\frac{2}{b}i;\lambda^b\right).\end{aligned}\tag{3.16}$$

Proof. This corollary comes from Theorem 3.5 by putting $a=2$. Eqs. (3.12) and (3.15) can also be obtained from this corollary by further putting $b=1$ and $b=2$ respectively. □

4. More identities and some remarks

According to generating functions (1.5) and (1.6), we have

$$\frac{\lambda e^{2z}+1}{\lambda e^{2z}-1}=\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n(\lambda)+\lambda\mathfrak{B}_n(1;\lambda)}{n!}(2z)^{n-1},\tag{4.1}$$

$$\frac{\lambda e^{2z}-1}{\lambda e^{2z}+1}=-\sum_{n=0}^{\infty}\mathfrak{E}_n(0;\lambda)\frac{(2z)^n}{n!}+1,\tag{4.2}$$

which are analogues of the expansions of hyperbolic cotangent and hyperbolic tangent, respectively (see [1, Eq. (23.1.20)] and [16]). By means of these two expansions, we can establish more results.

Theorem 4.1. *For integers $n\ge 0$, $a\ge 1$ and $b\ge 1$, if a is odd, then*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}a^{k-1}b^{n-k}\mathfrak{B}_k(bx;\lambda^a)T_{n-k}(a-1;\lambda^b)\\&=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}(\delta_{n-l,1}+2\mathfrak{B}_{n-l}(\lambda^{ab}))b^{n-l-1}\sum_{k=0}^{l}\binom{l}{k}b^ka^{n-k-1}\mathfrak{E}_k(ax;\lambda^b)S_{l-k}(b-1;\lambda^a)\\&=a^{n-1}\sum_{i=0}^{a-1}(-1)^i\lambda^{bi}\mathfrak{B}_n\left(bx+\frac{b}{a}i;\lambda^a\right)\\&=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}(\delta_{n-l,1}+2\mathfrak{B}_{n-l}(\lambda^{ab}))a^{n-l-1}b^{n-1}\sum_{i=0}^{b-1}\lambda^{ai}\mathfrak{E}_l\left(ax+\frac{a}{b}i;\lambda^b\right).\end{aligned}\tag{4.3}$$

Proof. Since $\lambda\mathfrak{B}_n(x+1;\lambda)-\mathfrak{B}_n(x;\lambda)=nx^{n-1}$ (see [13, Eq. (23)]), then

$$\mathfrak{B}_n(\lambda^{ab})+\lambda^{ab}\mathfrak{B}_n(1;\lambda^{ab})=\delta_{n,1}+2\mathfrak{B}_n(\lambda^{ab}).\tag{4.4}$$

When a is odd, making use of (4.1) and (4.4), we can expand the generating function $h(t)$ given by (3.10) into new forms, which yield the final result. The readers may compare this theorem with Theorem 3.5. □

Corollary 4.2. *For any nonnegative integer n and positive integer b, we have*

$$\begin{aligned}\mathfrak{B}_n(bx;\lambda)&=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}(\delta_{n-l,1}+2\mathfrak{B}_{n-l}(\lambda^b))\sum_{k=0}^{l}\binom{l}{k}b^{n-k-1}\mathfrak{E}_{l-k}(x;\lambda^b)S_k(b-1;\lambda)\\&=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}(\delta_{n-l,1}+2\mathfrak{B}_{n-l}(\lambda^b))b^{n-1}\sum_{i=0}^{b-1}\lambda^i\mathfrak{E}_l\left(x+\frac{i}{b};\lambda^b\right),\end{aligned}\tag{4.5}$$

$$\mathfrak{B}_n(x;\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathfrak{B}_{n-l}(\lambda)\mathfrak{E}_l(x;\lambda)+\frac{1}{2}n\mathfrak{E}_{n-1}(x;\lambda).\tag{4.6}$$

Proof. Identity (4.5) can be derived from Theorem 4.1 by the substitution $a = 1$. The further substitution $b = 1$ in (4.5) yields

$$\mathfrak{B}_n(x;\lambda) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}(\delta_{n-l,1} + 2\mathfrak{B}_{n-l}(\lambda))E_l(x;\lambda)\,, \tag{4.7}$$

which is equivalent to (4.6). Identity (4.6) can also be found in [13, Eq. (51)]. Moreover, putting $b = 2$ in (4.5), applying (4.7) and replacing x by $x/2$, we can obtain (3.4) again. □

We now present some results similar to the multiplication theorems of the Apostol type polynomials.

Theorem 4.3. *For any nonnegative integer n and any positive odd integer a, we have*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}&a^{n-k-1}\mathfrak{B}_{n-k}(\lambda^a)\mathfrak{E}_k(ax;\lambda) + \frac{1}{2}n\mathfrak{E}_{n-1}(ax;\lambda)\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{k-1}\mathfrak{B}_k(x;\lambda^a)T_{n-k}(a-1;\lambda) = a^{n-1}\sum_{i=0}^{a-1}(-1)^i\lambda^i\mathfrak{B}_n\left(x+\frac{i}{a};\lambda^a\right)\,.\end{aligned} \tag{4.8}$$

For any nonnegative integer n and any positive even integer a, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}&a^{n-k}\mathfrak{E}_k(ax;\lambda)\mathfrak{E}_{n-k}(0;\lambda^a) - \mathfrak{E}_n(ax;\lambda)\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{k}\mathfrak{E}_k(x;\lambda^a)T_{n-k}(a-1;\lambda) = a^{n}\sum_{i=0}^{a-1}(-1)^i\lambda^i\mathfrak{E}_n\left(x+\frac{i}{a};\lambda^a\right)\,.\end{aligned} \tag{4.9}$$

Proof. Putting $b = 1$ in Theorem 4.1 gives (4.8). To verify (4.9), let

$$p(t) = \frac{e^{axt}(1-(-\lambda e^t)^a)}{(\lambda^a e^{at}+1)(\lambda e^t+1)}$$

and use (4.2) to expand it. Note that the generating function $p(t)$ is just the $b = 1$ case of $g(t)$ (see Eq. (3.6)). □

Theorem 4.4. *For any nonnegative integer n and any positive integer a, we have*

$$\begin{aligned}-2\sum_{k=0}^{n}\binom{n}{k}&\frac{a^{k+1}}{k+1}\mathfrak{E}_{k+1}(0;\lambda^a)\mathfrak{B}_{n-k}(ax;\lambda) - \frac{2}{n+1}(\mathfrak{E}_0(0;\lambda^a)-1)\mathfrak{B}_{n+1}(ax;\lambda)\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{k}\mathfrak{E}_k(x;\lambda^a)S_{n-k}(a-1;\lambda) = a^{n}\sum_{i=0}^{a-1}\lambda^i\mathfrak{E}_n\left(x+\frac{i}{a};\lambda^a\right)\,.\end{aligned} \tag{4.10}$$

Proof. Identity (4.10) comes from the expansions of

$$q(t) = \frac{2e^{axt}(\lambda^a e^{at}-1)}{(\lambda^a e^{at}+1)(\lambda e^t-1)}\,.$$

To obtain the first member of (4.10), expansion (4.2) is still required. □

For convenience, we list in Table 1 some convolution identities on the Apostol type polynomials and the power sums. Besides these identities, we may also compare the convolution identities (3.3), (3.8), (3.14), (3.16) and the one given in Theorem 4.5.

$\sum \binom{n}{k} a^k \mathfrak{B}_k(x;\lambda^a) S_{n-k}(a-1;\lambda)$		Multiplication Formula	(3.2)
$\sum \binom{n}{k} a^k \mathfrak{E}_k(x;\lambda^a) S_{n-k}(a-1;\lambda)$			(4.10)
$\sum \binom{n}{k} a^k \mathfrak{B}_k(x;\lambda^a) T_{n-k}(a-1;\lambda)$	a odd		(4.8)
$\sum \binom{n}{k} a^k \mathfrak{B}_k(x;\lambda^a) T_{n-k}(a-1;\lambda)$	a even	Multiplication Formula	(3.11)
$\sum \binom{n}{k} a^k \mathfrak{E}_k(x;\lambda^a) T_{n-k}(a-1;\lambda)$	a odd	Multiplication Formula	(3.7)
$\sum \binom{n}{k} a^k \mathfrak{E}_k(x;\lambda^a) T_{n-k}(a-1;\lambda)$	a even		(4.9)

Table 1: Some convolution identities on the Apostol type polynomials and the power sums

Theorem 4.5. *For any nonnegative integer n and any positive odd integer a, we have*

$$\begin{aligned} &\mathfrak{E}_n(ax;\lambda^2) - \lambda^a \mathfrak{E}_n\left(ax + \frac{a}{2};\lambda^2\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{2}\right)^k \mathfrak{E}_k(2x;\lambda^a)(\mathfrak{E}_{n-k}(0;\lambda^2) - T_{n-k}(a-1;\lambda^2)) . \end{aligned} \tag{4.11}$$

Proof. Identity (4.11) can be derived by expanding

$$r(t) = \frac{e^{2axt}(1-\lambda^{2a}e^{2at})}{(\lambda^2 e^{2t}+1)(\lambda^a e^{at}+1)},$$

which comes from $g(t)$ by first setting $a = 2$ and then replacing b by a (see Eq. (3.6)). In the proof, expansion (4.2) and multiplication formula (3.7) should be used. □

For the generating function $g(t)$ given by (3.6), we only discussed the case when integers a and b have the same parity. In fact, for the case when a and b have different parity, we can discuss in an analogous way, using expansions (4.1) and (4.2). However, the results are complex, so we chose not to present them, but gave two special cases (4.9) and (4.11).

At the end of this section, we would like to show two results on the higher order Apostol type polynomials and the power sums. These two results are established by means of the next two generating functions:

$$F(t) = \frac{t^{2m-1}e^{abxt}(\lambda^{ab}e^{abt}-1)e^{abyt}}{(\lambda^a e^{at}-1)^m(\lambda^b e^{bt}-1)^m} \quad \text{and} \quad G(t) = \frac{e^{abxt}(1-(-\lambda^b e^{bt})^a)e^{abyt}}{(\lambda^a e^{at}+1)^m(\lambda^b e^{bt}+1)^m},$$

respectively.

Theorem 4.6. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, we have*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \mathfrak{B}_{n-k}^{(m)}(bx;\lambda^a) \sum_{i=0}^{k} \binom{k}{i} S_i(a-1;\lambda^b) \mathfrak{B}_{k-i}^{(m-1)}(ay;\lambda^b) \\ &= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} \mathfrak{B}_{n-k}^{(m)}(ax;\lambda^b) \sum_{i=0}^{k} \binom{k}{i} S_i(b-1;\lambda^a) \mathfrak{B}_{k-i}^{(m-1)}(by;\lambda^a) \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} \sum_{i=0}^{a-1} \lambda^{bi} \mathfrak{B}_k^{(m)}\left(bx + \frac{b}{a}i;\lambda^a\right) \mathfrak{B}_{n-k}^{(m-1)}(ay;\lambda^b) \\ &= \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k+1} \sum_{i=0}^{b-1} \lambda^{ai} \mathfrak{B}_k^{(m)}\left(ax + \frac{a}{b}i;\lambda^b\right) \mathfrak{B}_{n-k}^{(m-1)}(by;\lambda^a) . \end{aligned}$$

When $m = 1$ and $y = 0$, Theorem 4.6 reduces to Theorem 3.1. When $\lambda = 1$, Theorem 4.6 yields the two main results of Yang [17, Theorems 1 and 2]. The readers may also compare Theorem 4.6 with the main results of Zhang and Yang [19, Theorems 2.1, 2.7 and 2.10].

Theorem 4.7. *For integers $n \geq 0$, $a \geq 1$ and $b \geq 1$, if a and b have the same parity, then*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \mathfrak{E}_{n-k}^{(m)}(bx;\lambda^a) \sum_{i=0}^{k} \binom{k}{i} T_i(a-1;\lambda^b) \mathfrak{E}_{k-i}^{(m-1)}(ay;\lambda^b) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k \mathfrak{E}_{n-k}^{(m)}(ax;\lambda^b) \sum_{i=0}^{k} \binom{k}{i} T_i(b-1;\lambda^a) \mathfrak{E}_{k-i}^{(m-1)}(by;\lambda^a) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sum_{i=0}^{a-1} (-1)^i \lambda^{bi} \mathfrak{E}_k^{(m)} \left(bx + \frac{b}{a} i; \lambda^a \right) \mathfrak{E}_{n-k}^{(m-1)}(ay;\lambda^b) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} \sum_{i=0}^{b-1} (-1)^i \lambda^{ai} \mathfrak{E}_k^{(m)} \left(ax + \frac{a}{b} i; \lambda^b \right) \mathfrak{E}_{n-k}^{(m-1)}(by;\lambda^a) .
\end{aligned}$$

The substitutions $m = 1$ and $y = 0$ in Theorem 4.7 give Theorem 3.3, and the substitution $\lambda = 1$ in Theorem 4.7 gives [9, Theorems 2.1 and 2.10]. More identities can be obtained from the expansions of $G(t)$ under the assumption that a and b have different parity.

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