

Some results on the Apostol–Bernoulli and Apostol–Euler polynomials

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Abstract

The main object of this paper is to investigate the Apostol–Bernoulli polynomials and the Apostol–Euler polynomials. We first establish two relationships between the generalized Apostol–Bernoulli and Apostol–Euler polynomials. It can be found that many results obtained before are special cases of these two relationships. Moreover, we have a study on the sums of products of the Apostol–Bernoulli polynomials and of the Apostol–Euler polynomials.

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1. Introduction

For a real or complex parameter α , the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α , are defined by the following generating functions (for details, see [1, Section 2.8] and [2, Section 1.6]):

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi), \quad (1.1)$$

$$\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi). \quad (1.2)$$

Clearly, the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are given by

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x), \quad (n \in \mathbb{N}_0), \quad (1.3)$$

respectively, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N} := \{1, 2, 3, \dots\}$. Moreover, the classical Bernoulli numbers B_n and the classical Euler numbers E_n are given by

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$$B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n \left(\frac{1}{2} \right), \quad (n \in \mathbb{N}_0), \tag{1.4}$$

respectively.

These polynomials and numbers have numerous important applications in combinatorics, number theory and numerical analysis. They have therefore been studied extensively over the last two centuries.

It is the purpose of this paper to consider the so called generalized Apostol–Bernoulli and Apostol–Euler polynomials, which are natural generalizations of $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$, respectively. These polynomials are defined as follows [3–5].

Definition 1.1. For arbitrary real or complex parameters α and λ , the generalized Apostol–Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol–Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ are defined by the following generating functions:

$$\left(\frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t + \log \lambda| < 2\pi), \tag{1.5}$$

$$\left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t + \log \lambda| < \pi). \tag{1.6}$$

The so called Apostol–Bernoulli polynomials $\mathfrak{B}_n(x; \lambda)$ and the so called Apostol–Euler polynomials $\mathfrak{E}_n(x; \lambda)$ are given by

$$\mathfrak{B}_n(x; \lambda) := \mathfrak{B}_n^{(1)}(x; \lambda) \quad \text{and} \quad \mathfrak{E}_n(x; \lambda) := \mathfrak{E}_n^{(1)}(x; \lambda), \quad (n \in \mathbb{N}_0), \tag{1.7}$$

respectively. Furthermore, the Apostol–Bernoulli numbers $\mathfrak{B}_n(\lambda)$ and the Apostol–Euler numbers $\mathfrak{E}_n(\lambda)$ are given by

$$\mathfrak{B}_n(\lambda) := \mathfrak{B}_n(0; \lambda) \quad \text{and} \quad \mathfrak{E}_n(\lambda) := 2^n \mathfrak{E}_n \left(\frac{1}{2}; \lambda \right), \quad (n \in \mathbb{N}_0), \tag{1.8}$$

respectively. Obviously, when $\lambda = 1$ in (1.5)–(1.8), we obtain the corresponding well known forms given by (1.1)–(1.4).

The Apostol–Bernoulli polynomials $\mathfrak{B}_n(x; \lambda)$ and the Apostol–Bernoulli numbers $\mathfrak{B}_n(\lambda)$ were first defined by Apostol [6] when he studied the Lipschitz–Lerch Zeta functions. Recently, Luo and Srivastava introduced the generalized Apostol–Bernoulli and Apostol–Euler polynomials. They also studied these polynomials systematically (see [3–5,7,8]).

From the works referred to above, we can see that the (generalized) Apostol–Bernoulli polynomials and the (generalized) Apostol–Euler polynomials have many interesting and useful properties, and they deserve further study.

This paper is organized as follows. Some basic properties for $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ will be listed below. Section 2 is devoted to the general relationships involving these polynomials. It can be found that many results established before (see [5,9–11]) are special cases of these relationships. Finally, in Section 3, we will present some identities and give the explicit expressions for the sums of the products of the Apostol–Bernoulli polynomials and of the Apostol–Euler polynomials.

Now, let us give a brief review of the properties satisfied by $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$.

It is easily observed from the generating functions (1.5) and (1.6) that

$$\mathfrak{B}_n^{(\alpha+\beta)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(x; \lambda) \mathfrak{B}_{n-k}^{(\beta)}(y; \lambda), \tag{1.9}$$

$$\mathfrak{E}_n^{(\alpha+\beta)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(x; \lambda) \mathfrak{E}_{n-k}^{(\beta)}(y; \lambda). \tag{1.10}$$

From (1.5) and (1.6), it follows also that

$$\lambda \mathfrak{B}_n^{(\alpha)}(x + 1; \lambda) - \mathfrak{B}_n^{(\alpha)}(x; \lambda) = n \mathfrak{B}_{n-1}^{(\alpha-1)}(x; \lambda), \tag{1.11}$$

$$\lambda \mathfrak{E}_n^{(\alpha)}(x + 1; \lambda) + \mathfrak{E}_n^{(\alpha)}(x; \lambda) = 2\mathfrak{E}_n^{(\alpha-1)}(x; \lambda). \tag{1.12}$$

Moreover, since

$$\mathfrak{B}_n^{(0)}(x; \lambda) = \mathfrak{E}_n^{(0)}(x; \lambda) = x^n,$$

upon setting $\beta = 0$ in (1.9) and (1.10), and interchanging x and y , we get

$$\mathfrak{B}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k^{(\alpha)}(y; \lambda) x^{n-k}, \tag{1.13}$$

$$\mathfrak{E}_n^{(\alpha)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(y; \lambda) x^{n-k}. \tag{1.14}$$

The properties (1.9)–(1.14) and some other ones can be found in [3–5]. Here, we will only present two more theorems for $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ which have not appeared before.

Theorem 1.2. *The generalized Apostol–Bernoulli polynomials satisfy*

$$\mathfrak{B}_n^{(\alpha+1)}(x; \lambda) = \left(1 - \frac{n}{\alpha}\right) \mathfrak{B}_n^{(\alpha)}(x; \lambda) + (x - \alpha) \frac{n}{\alpha} \mathfrak{B}_{n-1}^{(\alpha)}(x; \lambda). \tag{1.15}$$

Theorem 1.3. *The generalized Apostol–Euler polynomials satisfy*

$$\frac{\alpha\lambda}{2} \mathfrak{E}_n^{(\alpha+1)}(x + 1; \lambda) = x \mathfrak{E}_n^{(\alpha)}(x; \lambda) - \mathfrak{E}_{n+1}^{(\alpha)}(x; \lambda), \tag{1.16}$$

$$\mathfrak{E}_n^{(\alpha+1)}(x; \lambda) = \frac{2}{\alpha} \mathfrak{E}_{n+1}^{(\alpha)}(x; \lambda) - (x - \alpha) \frac{2}{\alpha} \mathfrak{E}_n^{(\alpha)}(x; \lambda). \tag{1.17}$$

These two theorems can be readily obtained by computing the generating functions; hence we chose not to prove them here. It is worth noticing that all the properties given above could reduce to the corresponding ones for the generalized Bernoulli and Euler polynomials by setting $\lambda = 1$.

2. Relations between $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$

In 2003, Cheon [9] rederived several known properties and relations involving the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ by making use of some standard techniques based upon series rearrangement as well as matrix representation.

Srivastava and Pintér [10] followed Cheon’s work [9] and established two relations involving the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$. More recently, Luo and Srivastava [5] extended the results in [10] to the generalized Apostol–Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol–Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$.

We also presented two relations between $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ with matrix representation [11]. In this section, we will study further the relations between $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ and $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ with the methods of generating function and series rearrangement. As a consequence, it can be found that the relationships demonstrated here are in fact common generalizations of the works [5,9–11].

Theorem 2.1. *For $\alpha, \beta, \lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have the relationship*

$$\mathfrak{B}_n^{(\alpha)}(x + y; \lambda) = \frac{1}{2^\beta} \sum_{k=0}^n \binom{n}{k} \left(\sum_{m \geq 0} \binom{\beta}{m} \lambda^m \mathfrak{B}_{n-k}^{(\alpha)}(y + m; \lambda) \right) \mathfrak{E}_k^{(\beta)}(x; \lambda) \tag{2.1}$$

between the generalized Apostol–Bernoulli and Apostol–Euler polynomials.

Proof. Let us compute the generating functions for both sides of (2.1). By Definition 1.1, the right side gives

$$\begin{aligned} & \sum_{n \geq 0} \frac{1}{2^\beta} \sum_{k=0}^n \binom{n}{k} \left(\sum_{m \geq 0} \binom{\beta}{m} \lambda^m \mathfrak{B}_{n-k}^{(\alpha)}(y+m; \lambda) \right) \mathfrak{E}_k^{(\beta)}(x; \lambda) \frac{t^n}{n!} \\ &= \sum_{m \geq 0} \frac{1}{2^\beta} \binom{\beta}{m} \lambda^m \sum_{k \geq 0} \mathfrak{E}_k^{(\beta)}(x; \lambda) \frac{t^k}{k!} \sum_{n \geq k} \mathfrak{B}_{n-k}^{(\alpha)}(y+m; \lambda) \frac{t^{n-k}}{(n-k)!} \\ &= \left(\frac{2}{\lambda e^t + 1} \right)^\beta e^{xt} \left(\frac{t}{\lambda e^t - 1} \right)^\alpha e^{yt} \frac{1}{2^\beta} \sum_{m \geq 0} \binom{\beta}{m} \lambda^m e^{tm} = \left(\frac{t}{\lambda e^t - 1} \right)^\alpha e^{(x+y)t}, \end{aligned}$$

which coincides with the generating function of the left side. \square

Corollary 2.2. For $\alpha, \beta, \lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have the relationships:

$$\mathfrak{B}_n^{(\alpha)}(x+y; \lambda) = \sum_{k=0}^n \binom{n}{k} \left(\mathfrak{B}_k^{(\alpha)}(y; \lambda) + \frac{k}{2} \mathfrak{B}_{k-1}^{(\alpha-1)}(y; \lambda) \right) \mathfrak{E}_{n-k}(x; \lambda), \tag{2.2}$$

$$B_n^{(\alpha)}(x+y) = \frac{1}{2^\beta} \sum_{k=0}^n \binom{n}{k} \left(\sum_{m \geq 0} \binom{\beta}{m} B_{n-k}^{(\alpha)}(y+m) \right) E_k^{(\beta)}(x). \tag{2.3}$$

Proof. By setting $\beta = 1$ in Theorem 2.1, we have

$$\mathfrak{B}_n^{(\alpha)}(x+y; \lambda) = \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \left(\mathfrak{B}_{n-k}^{(\alpha)}(y; \lambda) + \lambda \mathfrak{B}_{n-k}^{(\alpha)}(y+1; \lambda) \right) \mathfrak{E}_k(x; \lambda),$$

which, in light of the recurrence relation (1.11), leads us at once to (2.2). Next, by setting $\lambda = 1$ in Theorem 2.1, the relationship (2.3) can also be obtained. \square

Remark. (2.2) and (2.3) are main results of [5,11], respectively (see [5, Section 3, Theorem 1] and [11, Section 3, Theorem 1]). A common special case of these two identities is

$$B_n^{(\alpha)}(x+y) = \sum_{k=0}^n \binom{n}{k} \left(B_k^{(\alpha)}(y) + \frac{k}{2} B_{k-1}^{(\alpha-1)}(y) \right) E_{n-k}(x),$$

which is one of the main results of [10] (see [10, Section 3, Theorem 1]).

It should be noticed that the identity demonstrated in [11, Section 3, Theorem 1] is not in the simplest form: one of the inner sums can be computed.

Corollary 2.3. For $\beta, \lambda \in \mathbb{C}$ and $n, j \in \mathbb{N}_0$, we have the relationships:

$$x^n = \frac{1}{2^\beta} \sum_{m \geq 0} \binom{\beta}{m} \lambda^m \mathfrak{E}_n^{(\beta)}(x+m; \lambda), \tag{2.4}$$

$$(n)_j x^{n-j} = \sum_{m \geq 0} (-1)^{j-m} \binom{j}{m} \lambda^m \mathfrak{B}_n^{(j)}(x+m; \lambda), \tag{2.5}$$

where $(n)_j = n(n-1) \cdots (n-j+1)$.

Proof. The special case of Theorem 2.1 when $\alpha = 0$ is

$$\begin{aligned} (x+y)^n &= \frac{1}{2^\beta} \sum_{k=0}^n \binom{n}{k} \left(\sum_{m \geq 0} \binom{\beta}{m} \lambda^m (y+m)^{n-k} \right) \mathfrak{E}_k^{(\beta)}(x; \lambda) \\ &= \frac{1}{2^\beta} \sum_{m \geq 0} \binom{\beta}{m} \lambda^m \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\beta)}(x; \lambda) (y+m)^{n-k} \end{aligned}$$

$$= \frac{1}{2^\beta} \sum_{m \geq 0} \binom{\beta}{m} \lambda^m \mathfrak{E}_n^{(\beta)}(x + y + m; \lambda),$$

which, by setting $y = 0$, yields (2.4). Eq. (2.5) is an analogue of (2.4). To prove it, let us compute the generating function again:

$$\sum_{n \geq 0} \sum_{m \geq 0} (-1)^{j-m} \binom{j}{m} \lambda^m \mathfrak{B}_n^{(j)}(x + m; \lambda) \frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1} \right)^j e^{xt} \sum_{m \geq 0} \binom{j}{m} (-1)^{j-m} \lambda^m e^{tm} = t^j e^{xt}.$$

Now, it suffices to identify the coefficients of $t^n/n!$ in the first and last members of the equation above. \square

When $\lambda = 1$, Corollary 2.3 gives two identities for the generalized Bernoulli polynomials and the generalized Euler polynomials:

$$x^n = \frac{1}{2^\beta} \sum_{m \geq 0} \binom{\beta}{m} E_n^{(\beta)}(x + m),$$

$$(n)_j x^{n-j} = \sum_{m \geq 0} (-1)^{j-m} \binom{j}{m} B_n^{(j)}(x + m).$$

Next, we will show that the generalized Apostol–Euler polynomials can be expressed by the generalized Apostol–Bernoulli polynomials. For convenience, we introduce a lemma.

Lemma 2.4. Define the numbers $S(k, j; \lambda)$ by

$$S(k, j; \lambda) = \frac{1}{j!} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} \lambda^m m^k, \quad (\lambda \in \mathbb{C}, k, j \in \mathbb{N}_0).$$

Then the $S(k, j; \lambda)$ have the generating function

$$\sum_{k \geq 0} S(k, j; \lambda) \frac{t^k}{k!} = \frac{1}{j!} (\lambda e^t - 1)^j. \tag{2.6}$$

Proof. We have

$$\begin{aligned} \sum_{k \geq 0} S(k, j; \lambda) \frac{t^k}{k!} &= \frac{1}{j!} \sum_{k \geq 0} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} \lambda^m m^k \frac{t^k}{k!} = \frac{1}{j!} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} \lambda^m \sum_{k \geq 0} \frac{m^k t^k}{k!} \\ &= \frac{1}{j!} \sum_{m=0}^j (-1)^{j-m} \binom{j}{m} \lambda^m e^{tm} = \frac{1}{j!} (\lambda e^t - 1)^j. \end{aligned}$$

This completes the proof. \square

According to the definition of the numbers $S(k, j; \lambda)$, we can readily see that $S(k, j; 1) = S(k, j)$, where $S(k, j)$ are the famous Stirling numbers of the second kind (see, e.g., [12, p. 204]). Moreover, in view of the generating function (2.6), we have

$$\sum_{l \geq 0} j! S(l + j, j; \lambda) \frac{t^{l+j}}{(l + j)!} = \sum_{k \geq j} j! S(k, j; \lambda) \frac{t^k}{k!} = (\lambda e^t - 1)^j - \sum_{l=-j}^{-1} j! S(l + j, j; \lambda) \frac{t^{l+j}}{(l + j)!}, \tag{2.7}$$

which will be used in the proof of the theorem below.

Theorem 2.5. For $\alpha, \lambda \in \mathbb{C}$ and $n, j \in \mathbb{N}_0$, we have the relationship

$$\mathfrak{E}_n^{(\alpha)}(x + y; \lambda) = \sum_{l=-j}^n \sum_{k=0}^{n-l} \frac{n! j!}{k!(l + j)!(n - k - l)!} S(l + j, j; \lambda) \mathfrak{E}_{n-k-l}^{(\alpha)}(y; \lambda) \mathfrak{B}_k^{(j)}(x; \lambda) \tag{2.8}$$

between the generalized Apostol–Euler and Apostol–Bernoulli polynomials.

Proof. Let $F(n, k, l)$ be the summand in relation (2.8); then the double sum of (2.8) can be rewritten as

$$\sum_{l=-j}^n \sum_{k=0}^{n-l} F(n, k, l) = \left\{ \sum_{l=0}^n \sum_{k=0}^{n-l} + \sum_{l=-j}^{-1} \sum_{k=0}^n + \sum_{l=-j}^{-1} \sum_{k=n+1}^{n-l} \right\} F(n, k, l). \tag{2.9}$$

Let us further define

$$\begin{aligned} \mathcal{A} &:= \left(\frac{1}{\lambda e^t - 1} \right)^j \left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{(x+y)t} \sum_{l=-j}^{-1} j! S(l + j, j; \lambda) \frac{t^{l+j}}{(l + j)!}, \\ \mathcal{B} &:= \left(\frac{t}{\lambda e^t - 1} \right)^j e^{xt} \sum_{l=-j}^{-1} \sum_{i=0}^{-l-1} \frac{j!}{(l + j)!} S(l + j, j; \lambda) \mathfrak{E}_i^{(\alpha)}(y; \lambda) \frac{t^{i+l}}{i!}. \end{aligned}$$

With this notation, we now give the proof.

We first consider the case when $j \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{1\}$. In this case, by Definition 1.1 and Eq. (2.7), the first part of (2.9) gives

$$\begin{aligned} \sum_{n \geq 0} \sum_{l=0}^n \sum_{k=0}^{n-l} F(n, k, l) \frac{t^n}{n!} &= \sum_{k \geq 0} \sum_{n \geq k} \sum_{l=0}^{n-k} F(n, k, l) \frac{t^n}{n!} = \sum_{k \geq 0} \sum_{l \geq 0} \sum_{i \geq l} F(k + i, k, l) \frac{t^{k+i}}{(k + i)!} \\ &= \sum_{l \geq 0} j! S(l + j, j; \lambda) \frac{t^l}{(l + j)!} \sum_{k \geq 0} \mathfrak{B}_k^{(j)}(x; \lambda) \frac{t^k}{k!} \sum_{i \geq l} \mathfrak{E}_{i-l}^{(\alpha)}(y; \lambda) \frac{t^{i-l}}{(i - l)!} \\ &= \left(\frac{t}{\lambda e^t - 1} \right)^j e^{xt} \left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{yt} \frac{1}{t^j} \left((\lambda e^t - 1)^j - \sum_{l=-j}^{-1} j! S(l + j, j; \lambda) \frac{t^{l+j}}{(l + j)!} \right) \\ &= \left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{(x+y)t} - \mathcal{A}. \end{aligned} \tag{2.10}$$

The generating function for the second part of (2.9) is

$$\begin{aligned} \sum_{n \geq 0} \sum_{l=-j}^{-1} \sum_{k=0}^n F(n, k, l) \frac{t^n}{n!} &= \sum_{l=-j}^{-1} \sum_{k \geq 0} \sum_{n \geq k} F(n, k, l) \frac{t^n}{n!} = \sum_{l=-j}^{-1} \sum_{k \geq 0} \sum_{i \geq 0} F(k + i, k, l) \frac{t^{k+i}}{(k + i)!} \\ &= \sum_{l=-j}^{-1} j! S(l + j, j; \lambda) \frac{t^l}{(l + j)!} \sum_{k \geq 0} \mathfrak{B}_k^{(j)}(x; \lambda) \frac{t^k}{k!} \sum_{i \geq 0} \mathfrak{E}_{i-l}^{(\alpha)}(y; \lambda) \frac{t^{i-l}}{(i - l)!} \\ &= \left(\frac{t}{\lambda e^t - 1} \right)^j e^{xt} \sum_{l=-j}^{-1} j! S(l + j, j; \lambda) \frac{t^l}{(l + j)!} \left(\left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{yt} - \sum_{i=0}^{-l-1} \mathfrak{E}_i^{(\alpha)}(y; \lambda) \frac{t^i}{i!} \right) \\ &= \mathcal{A} - \mathcal{B}. \end{aligned}$$

The generating function for the third part of (2.9) is

$$\begin{aligned} \sum_{n \geq 0} \sum_{l=-j}^{-1} \sum_{k=n+1}^{n-l} F(n, k, l) \frac{t^n}{n!} &= \sum_{l=-j}^{-1} \sum_{k \geq 1} \sum_{n=\max\{k+l, 0\}}^{k-1} F(n, k, l) \frac{t^n}{n!} \\ &= \sum_{l=-j}^{-1} \sum_{k=1}^{-l-1} \sum_{n=0}^{k-1} F(n, k, l) \frac{t^n}{n!} + \sum_{l=-j}^{-1} \sum_{k \geq -l} \sum_{n=k+l}^{k-1} F(n, k, l) \frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

According to (1.5), when $j \in \mathbb{N}$ and $\lambda \neq 1$, $\mathfrak{B}_k^{(j)}(x; \lambda) = 0$ for $0 \leq k \leq j - 1$. Then the first term of (2.11) vanishes, and this generating function reduces to

$$\sum_{l=-j}^{-1} \sum_{k \geq -l} \sum_{i=0}^{-l-1} F(k + l + i, k, l) \frac{t^{k+l+i}}{(k + l + i)!}$$

$$\begin{aligned}
 &= \sum_{l=-j}^{-1} \frac{j!}{(l+j)!} S(l+j, j; \lambda) \sum_{k \geq -l} \mathfrak{B}_k^{(j)}(x; \lambda) \frac{t^k}{k!} \sum_{i=0}^{-l-1} \mathfrak{E}_i^{(\alpha)}(y; \lambda) \frac{t^{i+l}}{i!} \\
 &= \sum_{l=-j}^{-1} \sum_{i=0}^{-l-1} \frac{j!}{(l+j)!} S(l+j, j; \lambda) \mathfrak{E}_i^{(\alpha)}(y; \lambda) \frac{t^{i+l}}{i!} \sum_{k \geq 0} \mathfrak{B}_k^{(j)}(x; \lambda) \frac{t^k}{k!} = \mathcal{B}.
 \end{aligned}$$

Combined with the computation above, we have

$$\sum_{n \geq 0} \sum_{l=-j}^n \sum_{k=0}^{n-l} F(n, k, l) \frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1} \right)^\alpha e^{(x+y)t}, \tag{2.12}$$

as desired.

When $j \in \mathbb{N}$ and $\lambda = 1$, the numbers $S(k, j; 1)$ turn out to be $S(k, j)$, where $S(k, j)$ are the Stirling numbers of the second kind. Moreover, $S(k, j) = 0$ for $0 \leq k \leq j - 1$. Then the second and third parts of formula (2.9) vanish, and

$$\sum_{l=-j}^n \sum_{k=0}^{n-l} F(n, k, l) = \sum_{l=0}^n \sum_{k=0}^{n-l} F(n, k, l).$$

In addition to these, \mathcal{A} also equals zero. Therefore, Eq. (2.12) holds.

When $j = 0$, (2.9) reduces to $\sum_{l=0}^n \sum_{k=0}^{n-l} F(n, k, l)$. Since \mathcal{A} still vanishes here, Eq. (2.10) indicates that (2.12) still holds. This completes the proof. \square

Corollary 2.6. For $\alpha, \lambda \in \mathbb{C}$ and $n, j \in \mathbb{N}_0$, we have the relationships:

$$\begin{aligned}
 \mathfrak{E}_n^{(\alpha)}(x+y; \lambda) &= \sum_{k=0}^n \frac{2}{k+1} \binom{n}{k} \left(\mathfrak{E}_{k+1}^{(\alpha-1)}(y; \lambda) - \mathfrak{E}_{k+1}^{(\alpha)}(y; \lambda) \right) \mathfrak{B}_{n-k}(x; \lambda) \\
 &\quad + \frac{\lambda-1}{n+1} \mathfrak{E}_0^{(\alpha)}(y; \lambda) \mathfrak{B}_{n+1}(x; \lambda),
 \end{aligned} \tag{2.13}$$

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^n \binom{n}{k} \left(\sum_{l=0}^{n-k} \binom{n-k}{l} \binom{l+j}{j}^{-1} S(l+j, j) E_{n-k-l}^{(\alpha)}(y) \right) B_k^{(j)}(x). \tag{2.14}$$

Proof. (2.14) is an immediate consequence of Theorem 2.5 by setting $\lambda = 1$. To get (2.13), set $j = 1$ in (2.9). Then those three parts equal

$$\begin{aligned}
 &\sum_{k=0}^n \frac{\lambda}{n-k+1} \binom{n}{k} \mathfrak{B}_k(x; \lambda) \left(\mathfrak{E}_{n-k+1}^{(\alpha)}(y+1; \lambda) - \mathfrak{E}_{n-k+1}^{(\alpha)}(y; \lambda) \right), \\
 &\sum_{k=0}^n \frac{\lambda-1}{n-k+1} \binom{n}{k} \mathfrak{B}_k(x; \lambda) \mathfrak{E}_{n-k+1}^{(\alpha)}(y; \lambda), \\
 &\frac{\lambda-1}{n+1} \mathfrak{E}_0^{(\alpha)}(y; \lambda) \mathfrak{B}_{n+1}(x; \lambda),
 \end{aligned}$$

respectively. Therefore,

$$\begin{aligned}
 \mathfrak{E}_n^{(\alpha)}(x+y; \lambda) &= \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} \left(\lambda \mathfrak{E}_{n-k+1}^{(\alpha)}(y+1; \lambda) - \mathfrak{E}_{n-k+1}^{(\alpha)}(y; \lambda) \right) \mathfrak{B}_k(x; \lambda) \\
 &\quad + \frac{\lambda-1}{n+1} \mathfrak{E}_0^{(\alpha)}(y; \lambda) \mathfrak{B}_{n+1}(x; \lambda).
 \end{aligned}$$

In view of the recurrence relation (1.12), assertion (2.13) holds. \square

Remark. Similar to Corollary 2.2, the two relationships given by Corollary 2.6 are the main results of [5] and [11], respectively (see [5, Section 3, Theorem 2] and [11, Section 3, Theorem 2]). However, the representations there both

have problems. One is lacking a term (i.e., the second term $\frac{\lambda-1}{n+1} \mathfrak{E}_0^{(\alpha)}(y; \lambda) \mathfrak{B}_{n+1}(x; \lambda)$ on the right side of (2.13)), and the other is not in the neatest form. The common special case of (2.13) and (2.14) is

$$E_n^{(\alpha)}(x + y) = \sum_{k=0}^n \frac{2}{k+1} \binom{n}{k} \left(E_{k+1}^{(\alpha-1)}(y) - E_{k+1}^{(\alpha)}(y) \right) B_{n-k}(x),$$

which has already been obtained in [10, Section 3, Theorem 2].

Corollary 2.7. For $\lambda \in \mathbb{C}$ and $n, j \in \mathbb{N}_0$, we have the relationships:

$$x^n = \sum_{l=-j}^n \frac{n!j!}{(l+j)!(n-l)!} S(l+j, j; \lambda) \mathfrak{B}_{n-l}^{(j)}(x; \lambda), \tag{2.15}$$

$$x^n = \sum_{l=0}^n \binom{n}{l} \binom{l+j}{j}^{-1} S(l+j, j) B_{n-l}^{(j)}(x). \tag{2.16}$$

Proof. The special case of Theorem 2.5 when $\alpha = 0$ is

$$(x + y)^n = \sum_{l=-j}^n \sum_{k=0}^{n-l} \frac{n!j!}{k!(l+j)!(n-k-l)!} S(l+j, j; \lambda) y^{n-k-l} \mathfrak{B}_k^{(j)}(x; \lambda).$$

With the substitution $y = 0$ in the last equation, we get (2.15). Eq. (2.16) is a further special case of (2.15) when $\lambda = 1$. \square

Many other identities can be obtained from Theorems 2.1 and 2.5. For example, when $\beta = 0$ and $j = 0$, Theorems 2.1 and 2.5 will reduce to (1.13) and (1.14) respectively. Other special cases can be found in the references [5,9–11].

3. Explicit expressions for sums of products

One of the most remarkable identities for the Bernoulli numbers is the convolution identity

$$\sum_{j=0}^n \binom{n}{j} B_j B_{n-j} = -n B_{n-1} - (n-1) B_n, \quad (n \geq 1),$$

which is equivalent to the form

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1) B_{2n}, \quad (n \geq 2).$$

These two identities have been generalized in many works (see [13] or [14] for a review on this subject). Particularly, in [13], explicit expressions are obtained for sums of products of arbitrarily many Bernoulli numbers. Corresponding results are also derived for Bernoulli polynomials, and for Euler numbers and polynomials.

In this section, we first derive several identities for the generalized Apostol–Bernoulli and Apostol–Euler polynomials. Based on these identities, we further investigate the sums of products of the Apostol–Bernoulli polynomials and of the Apostol–Euler polynomials.

Before stating the results, we introduce a lemma ([15, p. 147, Eqs. (85) and (86)]; see also [16, pp. 96 and 99]).

Lemma 3.1. The generalized Bernoulli polynomials satisfy

$$B_m^{(m+1)}(x) = (x-1)(x-2)\cdots(x-m) = (x-1)_m,$$

$$\frac{m!}{(m-k)!} B_{m-k}^{(m+1)}(x) = D_x^k [(x-1)(x-2)\cdots(x-m)] = D_x^k (x-1)_m,$$

where D_x is the differential operator defined by $D_x f(x) := \frac{d}{dx} f(x)$.

Then the following theorem holds.

Theorem 3.2. For $n \geq m + 1$, we have

$$\mathfrak{B}_n^{(m+1)}(x; \lambda) = (m + 1) \binom{n}{m + 1} \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{\mathfrak{B}_{n-m+k}(x; \lambda)}{n - m + k} D_x^k(x - 1)_m \tag{3.1}$$

$$= (m + 1) \binom{n}{m + 1} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \frac{\mathfrak{B}_{n-k}(x; \lambda)}{n - k} B_k^{(m+1)}(x). \tag{3.2}$$

Proof. We prove (3.1) by induction. It is clearly true for $m = 1$ in view of the recurrence relation (1.15). Suppose that it is true for $m - 1$; then by making use of (1.15) again, we have

$$\begin{aligned} \mathfrak{B}_n^{(m+1)}(x; \lambda) &= \left(1 - \frac{n}{m}\right) \mathfrak{B}_n^{(m)}(x; \lambda) + (x - m) \frac{n}{m} \mathfrak{B}_{n-1}^{(m)}(x; \lambda) \\ &= (m + 1) \binom{n}{m + 1} \sum_{k=1}^m \frac{(-1)^k}{k!} \frac{\mathfrak{B}_{n-m+k}(x; \lambda)}{n - m + k} k D_x^{k-1}(x - 1)_{m-1} \\ &\quad + (m + 1) \binom{n}{m + 1} \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{\mathfrak{B}_{n-m+k}(x; \lambda)}{n - m + k} (x - m) D_x^k(x - 1)_{m-1}. \end{aligned}$$

According to the Leibniz’s rule,

$$\begin{aligned} D_x^k(x - 1)_m &= \sum_{j=0}^k \binom{k}{j} D_x^j(x - 1)_{m-1} D_x^{k-j}(x - m) \\ &= (x - m) D_x^k(x - 1)_{m-1} + k D_x^{k-1}(x - 1)_{m-1}. \end{aligned}$$

Therefore, Eq. (3.1) holds. Moreover, based on Lemma 3.1, (3.2) also holds. This completes the proof. \square

By appealing to the recurrence relation (1.17), we can establish the corresponding results for the generalized Apostol–Euler polynomials.

Theorem 3.3. For $n \geq m + 1$, we have

$$\mathfrak{E}_n^{(m+1)}(x; \lambda) = \frac{2^m}{m!} \sum_{k=0}^m \frac{(-1)^{m-k}}{k!} \mathfrak{E}_{n+k}(x; \lambda) D_x^k(x - 1)_m \tag{3.3}$$

$$= \frac{2^m}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \mathfrak{E}_{n+k}(x; \lambda) B_{m-k}^{(m+1)}(x). \tag{3.4}$$

When $\lambda = 1$, Theorems 3.2 and 3.3 will reduce to the corresponding identities for the generalized Bernoulli and Euler polynomials (see [15, p. 148, Eqs. (87) and (88)]).

Now, we consider the sums of products of the Apostol–Bernoulli polynomials and of the Apostol–Euler polynomials.

In analogy with [13], we denote for $m \geq 2$,

$$S_m(n; x_1, \dots, x_m) = \sum \binom{n}{j_1, \dots, j_m} \mathfrak{B}_{j_1}(x_1; \lambda) \cdots \mathfrak{B}_{j_m}(x_m; \lambda),$$

where the sum is taken over all nonnegative integers j_1, \dots, j_m such that $j_1 + \cdots + j_m = n$. Then the next theorem holds.

Theorem 3.4. Let $y := x_1 + \cdots + x_m$. Then for $n \geq m$, we have

$$\begin{aligned} S_m(n; x_1, \dots, x_m) &= (-1)^{m-1} m \binom{n}{m} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} B_k^{(m)}(y) \frac{\mathfrak{B}_{n-k}(y; \lambda)}{n - k} \end{aligned} \tag{3.5}$$

$$= (-1)^{m-1} m \binom{n}{m} \sum_{k=0}^{m-1} (-1)^k \left\{ \sum_{j=0}^k \binom{m-k-1+j}{j} s(m, m-k+j) y^j \right\} \frac{\mathfrak{B}_{n-k}(y; \lambda)}{n-k}, \tag{3.6}$$

where $s(n, k)$ are the Stirling numbers of the first kind (see, e.g., [12, p. 212]).

Proof. This is in fact the same as the proofs due to Dilcher [13, Section 3, Lemma 4 and Theorem 3]. From the generating function of the Apostol–Bernoulli polynomials, we have

$$S_m(n; x_1, \dots, x_m) = \mathfrak{B}_n^{(m)}(x_1 + \dots + x_m; \lambda) = \mathfrak{B}_n^{(m)}(y; \lambda).$$

Thus, by (3.2) and the identity

$$\binom{m-1}{k} B_k^{(m)}(y) = \sum_{j=0}^k \binom{m-k-1+j}{j} s(m, m-k+j) y^j$$

(see [13, p. 32, Eq. (3.7)]), the desired results can be obtained. \square

For example, when $m = 2$, (3.6) gives

$$\sum_{k=0}^n \binom{n}{k} \mathfrak{B}_k(x; \lambda) \mathfrak{B}_{n-k}(y; \lambda) = n(x+y-1) \mathfrak{B}_{n-1}(x+y; \lambda) - (n-1) \mathfrak{B}_n(x+y; \lambda).$$

Corollary 3.5. *If $x_1 + \dots + x_m = 0$, then for $n \geq m$, we have*

$$S_m(n; x_1, \dots, x_m) = (-1)^{m-1} m \binom{n}{m} \sum_{k=0}^{m-1} (-1)^k s(m, m-k) \frac{\mathfrak{B}_{n-k}(\lambda)}{n-k}. \tag{3.7}$$

This follows from (3.6) with the substitution $y = 0$ and the fact that $\mathfrak{B}_n(0; \lambda) = \mathfrak{B}_n(\lambda)$. In particular, by setting $x_1 = \dots = x_m = 0$, the right side of (3.7) leads us at once to an expression for

$$\sum \binom{n}{j_1, \dots, j_m} \mathfrak{B}_{j_1}(\lambda) \dots \mathfrak{B}_{j_m}(\lambda).$$

We can deal with the sums of the products of the Apostol–Euler polynomials in an analogous way. Let us denote

$$T_m(n; x_1, \dots, x_m) = \sum \binom{n}{j_1, \dots, j_m} \mathfrak{E}_{j_1}(x_1; \lambda) \dots \mathfrak{E}_{j_m}(x_m; \lambda),$$

where the sum is again taken over all nonnegative integers j_1, \dots, j_m such that $j_1 + \dots + j_m = n$. In light of the assertion (3.4) of Theorem 3.3, the next theorem can be obtained.

Theorem 3.6. *Let $y := x_1 + \dots + x_m$. Then for $n \geq m$, we have*

$$\begin{aligned} T_m(n; x_1, \dots, x_m) &= \frac{2^{m-1}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} B_k^{(m)}(y) \mathfrak{E}_{n+m-1-k}(y; \lambda) \end{aligned} \tag{3.8}$$

$$= \frac{2^{m-1}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k \left\{ \sum_{j=0}^k \binom{m-k-1+j}{j} s(m, m-k+j) y^j \right\} \mathfrak{E}_{n+m-1-k}(y; \lambda). \tag{3.9}$$

For example, when $m = 2$, (3.9) gives

$$\sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k(x; \lambda) \mathfrak{E}_{n-k}(y; \lambda) = 2(1-x-y) \mathfrak{E}_n(x+y; \lambda) + 2 \mathfrak{E}_{n+1}(x+y; \lambda).$$

Moreover, Theorem 3.6 has the following special case.

Corollary 3.7. If $x_1 + \dots + x_m = 0$, then for $n \geq m$, we have

$$T_m(n; x_1, \dots, x_m) = \frac{(-2)^{m-1}}{(m-1)!} \sum_{k=0}^{m-1} (-1)^k s(m, k+1) \mathfrak{E}_{n+k}(0; \lambda).$$

In addition to these, since $\mathfrak{E}_n(\lambda) = 2^n \mathfrak{E}_n\left(\frac{1}{2}; \lambda\right)$, we have an explicit expression for the sums of products of the Apostol–Euler numbers.

Corollary 3.8. For $n \geq m$, we have

$$\begin{aligned} & \sum \binom{n}{j_1, \dots, j_m} \mathfrak{E}_{j_1}(\lambda) \cdots \mathfrak{E}_{j_m}(\lambda) \\ &= \frac{1}{(m-1)!} \sum_{k=0}^{m-1} (-2)^k \left\{ \sum_{j=0}^k \binom{m-k-1+j}{j} s(m, m-k+j) \left(\frac{m}{2}\right)^j \right\} \mathfrak{E}_{n+m-1-k}(\lambda). \end{aligned}$$

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