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Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials

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Abstract

In this paper, we introduce a unified family of Hermite-based Apostol-Bernoulli, Euler and Genocchi polynomials. We obtain some symmetry identities between these polynomials and the generalized sum of integer powers. We give explicit closed-form formulae for this unified family. Furthermore, we prove a finite series relation between this unification and 3*d*-Hermite polynomials.

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1 Introduction

Recently, Khan *et al.* [1] introduced the Hermite-based Appell polynomials via the generating function

$$\mathcal{G}(x,y,z;t) = A(t)\exp(\mathcal{M}t),$$

where

$$\mathcal{M} = x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2}$$

is the multiplicative operator of the 3-variable Hermite polynomials, which are defined by

$$\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} H_n^{(3)}(x,y,z)\frac{t^n}{n!} \tag{1.1}$$

and

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_0 \neq 0.$$

By using the Berry decoupling identity,

$$e^{A+B} = e^{m^2/12}e^{((\frac{-m}{2})A^{1/2}+A)}e^B, \quad [A,B] = mA^{1/2}$$

they obtained the generating function of the Hermite-based Appell polynomials ${}_HA_n(x, y, z)$ as

$$\mathcal{G}(x,y,z;t) = A(t)\exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_HA_n(x,y,z)\frac{t^n}{n!}.$$

Letting $A(t) = \frac{t}{e^t-1}$, they defined Hermite-Bernoulli polynomials ${}_HB_n(x,y,z)$ by

$$\frac{t}{e^t-1}\exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_HB_n(x,y,z)\frac{t^n}{n!}, \quad |t| < 2\pi.$$

For $A(t) = \frac{2}{e^t+1}$, they defined Hermite-Euler polynomials ${}_HE_n(x,y,z)$ by

$$\frac{2}{e^t+1}\exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_HE_n(x,y,z)\frac{t^n}{n!}, \quad |t| < \pi$$

and for $A(t) = \frac{2t}{e^t+1}$, they defined Hermite-Genocchi polynomials ${}_HG_n(x,y,z)$ by

$$\frac{2t}{e^t+1}\exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_HG_n(x,y,z)\frac{t^n}{n!}, \quad |t| < \pi.$$

Recently, the author considered the following unification of the Apostol-Bernoulli, Euler and Genocchi polynomials

$$f_{a,b}^{(\alpha)}(x;t;k,\beta) := \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}$$
$$\bigl(k \in \mathbb{N}_0; a,b \in \mathbb{R}\backslash\{0\}; \alpha,\beta \in \mathbb{C}\bigr)$$

and obtained the explicit representation of this unified family, in terms of Gaussian hypergeometric function. Some symmetry identities and multiplication formula are also given in [2]. Note that the family of polynomials $P_{n,\beta}^{(1)}(x,y,z;k,a,b)$ was investigated in [3].

We organize the paper as follows.

In Section 2, we introduce the unification of the Hermite-based generalized Apostol-Bernoulli, Euler and Genocchi polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)$ and give summation formulas for this unification. In Section 3, we obtain some symmetry identities for these polynomials. In Section 4, we give explicit closed-form formulae for this unified family. Furthermore, we prove a finite series relation between this unification and $3d$-Hermite polynomials.

2 Hermite-based generalized Apostol-Bernoulli, Euler and Genocchi polynomials

In this paper, we consider the following general class of polynomials:

$$f_{a,b}^{(\alpha)}(x,y,z;t;k,\beta) := \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}$$
$$\bigl(k \in \mathbb{N}_0; a,b \in \mathbb{R}\backslash\{0\}; \alpha,\beta \in \mathbb{C}\bigr). \tag{2.1}$$

For the existence of the expansion, we need

(i) $|t| < 2\pi$ when $\alpha \in \mathbb{C}$, $k = 1$ and $(\frac{\beta}{a})^b = 1$; $|t| < 2\pi$ when $\alpha \in \mathbb{N}_0$, $k = 2, 3, \dots$ and $(\frac{\beta}{a})^b = 1$; $|t| < |b \log(\frac{\beta}{a})|$ when $\alpha \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $(\frac{\beta}{a})^b \neq 1$ (or $\neq -1$); $x, y, z \in \mathbb{R}$, $\beta \in \mathbb{C}$, $a, b \in \mathbb{C}/\{0\}$; $1^\alpha := 1$;

(ii) $|t| < \pi$ when $(\frac{\beta}{a})^b = -1$; $|t| < |b \log(\frac{\beta}{a})|$ when $(\frac{\beta}{a})^b \neq -1$; $x, y, z \in \mathbb{R}$, $k = 0$, $\alpha, \beta \in \mathbb{C}$, $a, b \in \mathbb{C}/\{0\}$; $1^\alpha := 1$;

(iii) $|t| < \pi$ when $\alpha \in \mathbb{N}_0$ and $(\frac{\beta}{a})^b = -1$; $x, y, z \in \mathbb{R}$, $k \in \mathbb{N}$, $\beta \in \mathbb{C}$, $a, b \in \mathbb{C}/\{0\}$; $1^\alpha := 1$,

where $w = |w|e^{i\theta}$, $-\pi \leq \theta < \pi$ and $\log(w) = \log(|w|) + i\theta$.

For $k = a = b = 1$ and $\beta = \lambda$ in (2.1), we define the following.

Definition 2.1 Let $\alpha \in \mathbb{N}_0$, λ be an arbitrary (real or complex) parameter and $x, y, z \in \mathbb{R}$. The Hermite-based generalized Apostol-Bernoulli polynomials are defined by

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha \exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\alpha)}(x, y, z; \lambda)\frac{t^n}{n!}$$

$$\bigl(|t| < 2\pi \text{ when } \alpha \in \mathbb{C} \text{ and } \lambda = 1; |t| < \bigl|\log(\lambda)\bigr| \text{ when } \alpha \in \mathbb{N}_0 \text{ and } \lambda \neq 1; x, y, z \in \mathbb{R}; 1^\alpha := 1\bigr).$$

It is clear that

$${}_HP_{n,\lambda}^{(\alpha)}(x, y, z; 1, 1, 1) = {}_H\mathcal{B}_n^{(\alpha)}(x, y, z; \lambda).$$

Some special cases of the Hermite-based generalized Apostol-Bernoulli polynomials (some of which are definition) are listed below:

- ${}_H\mathcal{B}_n^{(1)}(x, y, z; \lambda) := {}_H\mathcal{B}_n(x, y, z; \lambda)$ is called Hermite-based Apostol-Bernoulli polynomials.
- ${}_H\mathcal{B}_n(x, y, z; 1) = {}_HB_n(x, y, z)$ is the Hermite-Bernoulli polynomials.
- ${}_H\mathcal{B}_n(x, 0, 0; \lambda) := \mathcal{B}_n(x; \lambda)$ is the Apostol-Bernoulli polynomials (see [4–7]). When $\lambda = 1$, we have the classical Bernoulli polynomials.
- $\mathcal{B}_n(0; \lambda) := \mathcal{B}_n(\lambda)$ are the Apostol-Bernoulli numbers. $\lambda = 1$ gives the classical Bernoulli numbers.

Setting $k + 1 = -a = b = 1$ and $\beta = \lambda$ in (2.1), we get the following.

Definition 2.2 Let α and λ ($\neq -1$) be an arbitrary (real or complex) parameter and $x, y, z \in \mathbb{R}$. The Hermite-based generalized Apostol-Euler polynomials are defined by

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha \exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_H\mathcal{E}_n^{(\alpha)}(x, y, z; \lambda)\frac{t^n}{n!}$$

$$\bigl(|t| < \pi \text{ when } \lambda = 1; |t| < \bigl|\log(-\lambda)\bigr| \text{ when } \lambda \neq 1; x, y, z \in \mathbb{R}, \alpha \in \mathbb{C}; 1^\alpha := 1\bigr).$$

Obviously, we have

$${}_HP_{n,\lambda}^{(\alpha)}(x, y, z; 0, -1, 1) = {}_H\mathcal{E}_n^{(\alpha)}(x, y, z; \lambda).$$

Some special cases of the Hermite-based generalized Apostol-Euler polynomials (some of which are definition) are listed below:

- ${}_H\mathcal{E}_n^{(1)}(x,y,z;\lambda) := {}_H\mathcal{E}_n(x,y,z;\lambda)$ is called Hermite-based Apostol-Euler polynomials.
- ${}_H\mathcal{E}_n(x,y,z;1) = {}_HE_n(x,y,z)$ is the Hermite-Euler polynomials.
- ${}_H\mathcal{E}_n(x,0,0;\lambda) := \mathcal{E}_n(x;\lambda)$ is the Apostol-Euler polynomials (see [8]). For $\lambda = 1$, we have the classical Euler polynomials.
- $2^n\mathcal{E}_n(\frac{1}{2};\lambda) := \mathcal{E}_n(\lambda)$ are the Apostol-Euler numbers. The case $\lambda = 1$ gives the classical Euler numbers.

Choosing $k = -2a = b = 1$ and $2\beta = \lambda$ in (2.1), we define the following.

Definition 2.3 Let α and λ ($\neq -1$) be an arbitrary (real or complex) parameter and $x,y,z \in \mathbb{R}$. The Hermite-based generalized Apostol-Genocchi polynomials are defined by

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha \exp\bigl(xt + yt^2 + zt^3\bigr) = \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}$$

$$\bigl(|t| < \pi \text{ when } \alpha \in \mathbb{N}_0 \text{ and } \lambda = 1; |t| < \bigl|\log(-\lambda)\bigr|$$
$$\text{when } \alpha \in \mathbb{N}_0 \text{ and } \lambda \neq 1; x,y,z \in \mathbb{R}; 1^\alpha := 1\bigr).$$

It is easily seen that

$${}_HP^{(\alpha)}_{n,\frac{\lambda}{2}}\left(x,y,z;1,\frac{-1}{2},1\right) = {}_H\mathcal{G}_n^{\alpha}(x,y,z;\lambda).$$

Some special cases of the Hermite-based generalized Apostol-Genocchi polynomials (some of which are definition) are listed below:

- ${}_H\mathcal{G}_n^{(1)}(x,y,z;\lambda) := {}_H\mathcal{G}_n(x,y,z;\lambda)$ is called Hermite-based Apostol-Genocchi polynomials.
- ${}_H\mathcal{G}_n(x,y,z;1) = {}_HG_n(x,y,z)$ is the Hermite-Genocchi polynomials.
- ${}_H\mathcal{G}_n(x,0,0;\lambda) := \mathcal{G}_n(x;\lambda)$ is the Apostol-Genocchi polynomials (see [9, 10]). When $\lambda = 1$, we have the classical Genocchi polynomials.
- $\mathcal{G}_n(0;\lambda) := \mathcal{G}_n(\lambda)$ are the Apostol-Genocchi numbers. $\lambda = 1$ gives the classical Genocchi numbers.

Finally we define the unified Hermite-based Apostol polynomials by

$$f_{a,b}^{(1)}(x;t;k,\beta) := \frac{2^{1-k}t^k}{\beta^b e^t - a^b} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HP_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!}$$
$$\bigl(k \in \mathbb{N}_0; a,b \in \mathbb{R}\backslash\{0\}; \beta \in \mathbb{C}\bigr).$$

Thus it is clear that ${}_HP_{n,\beta}(x,y,z;k,a,b) = {}_HP^{(1)}_{n,\beta}(x,y,z;k,a,b)$ and that we have the following observations at once:

- ${}_HP_{n,\lambda}(x,y,z;1,1,1) = {}_H\mathcal{B}_n(x,y,z;\lambda)$ are the Hermite-based Apostol-Bernoulli polynomials.
- ${}_HP_{n,\lambda}(x,y,z;0,-1,1) = {}_H\mathcal{E}(x,y,z;\lambda)$ are the Hermite-based Apostol-Euler polynomials.
- ${}_HP_{n,\frac{\lambda}{2}}(x,y,z;1,\frac{-1}{2},1) = {}_H\mathcal{G}_n(x,y,z;\lambda)$ are the Hermite-based Apostol-Genocchi polynomials.

For the other generalization, we refer [11–25] and [26]. Now we give some relations between the above mentioned Apostol polynomials.

Using (2.1), we get the following identity at once.

Theorem 2.1 *Let* $\alpha, k \in \mathbb{N}_0$; $a, b \in \mathbb{R}\backslash\{0\}$; $\beta \in \mathbb{C}$ *be such that the conditions* (i)-(iii) *are satisfied. Then, the following relation*

$$\sum_{r=0}^{n}\binom{n}{r}{}_HP_{n-r,\beta}^{(\alpha)}(x,y,z;k,a,b){}_HP_{r,\beta}^{(\alpha)}(u,v,w;k,a,b) = {}_HP_{n,\beta}^{(\alpha)}(x+u,y+v,z+w;k,a,b)$$

holds true.

Corollary 2.2 *For each* $n \in \mathbb{N}$, *the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{B}_{n-k}^{(\alpha)}(x,y,z;\lambda){}_H\mathcal{B}_{k}^{(\beta)}(u,v,w;\lambda) = {}_H\mathcal{B}_{n}^{(\alpha+\beta)}(x+u,y+v,z+w;\lambda)$$

holds true for the Hermite-based generalized Apostol-Bernoulli polynomials.

Corollary 2.3 *For each* $n \in \mathbb{N}$, *the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{E}_{n-k}^{(\alpha)}(x,y,z;\lambda){}_H\mathcal{E}_{k}^{(\beta)}(u,v,w;\lambda) = {}_H\mathcal{E}_{n}^{(\alpha+\beta)}(x+u,y+v,z+w;\lambda)$$

holds true for the Hermite-based generalized Apostol-Euler polynomials.

Corollary 2.4 *For each* $n \in \mathbb{N}$, *the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{G}_{n-k}^{(\alpha)}(x,y,z;\lambda){}_H\mathcal{G}_{k}^{(\beta)}(u,v,w;\lambda) = {}_H\mathcal{G}_{n}^{(\alpha+\beta)}(x+u,y+v,z+w;\lambda)$$

holds true for the Hermite-based generalized Apostol-Genocchi polynomials.

Theorem 2.5 *For each* $n \in \mathbb{N}$, *the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}{}_H\mathcal{B}_{n-k}^{(\alpha)}(x,y,z;\lambda){}_H\mathcal{E}_{k}^{(\alpha)}(u,v,w;\lambda) = 2^n_H\mathcal{B}_{n}^{(\alpha)}\left(\frac{x+u}{2},\frac{y+v}{4},\frac{z+w}{8};\lambda^2\right)$$

holds true between the Hermite-based generalized Apostol-Bernoulli and Euler polynomials.

Proof By direct calculations, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}{}_H\mathcal{B}_{n}^{(\alpha)}\left(\frac{x+u}{2},\frac{y+v}{4},\frac{z+w}{8};\lambda^2\right)\frac{(2t)^n}{n!} \\
&\quad= \left(\frac{2t}{\lambda^2e^{2t}-1}\right)^{\alpha}\exp\left[\left(\frac{x+u}{2}\right)2t+\left(\frac{y+v}{4}\right)(2t)^2+\left(\frac{z+w}{8}\right)(2t)^3\right] \\
&\quad= \left(\frac{t}{\lambda e^t-1}\right)^{\alpha}\exp\left(xt+yt^2+zt^3\right)\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}\exp\left(ut+vt^2+wt^3\right)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty} {}_H\mathcal{E}_k^{(\alpha)}(u,v,w;\lambda)\frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} {}_H\mathcal{B}_{n-k}^{(\alpha)}(x,y,z;\lambda) {}_H\mathcal{E}_k^{(\alpha)}(u,v,w;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the result. □

3 Symmetry identities for the unified family

For each $k \in \mathbb{N}_0$, the sum $S_k(n) = \sum_{i=0}^{n} i^k$ is known as the power sum and we have the following generating relation:

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{nt} = \frac{e^{(n+1)t} - 1}{e^t - 1}.$$

For an arbitrary real or complex λ, the generalized sum of integer powers $S_k(n,\lambda)$ is defined, in [27], via the following generating relation:

$$\sum_{k=0}^{\infty} S_k(n,\lambda)\frac{t^k}{k!} = \frac{\lambda e^{(n+1)t} - 1}{\lambda e^t - 1}.$$

It clear that $S_k(n,1) = S_k(n)$.

For each $k \in \mathbb{N}_0$, the sum $M_k(n) = \sum_{i=0}^{n}(-1)^k i^k$ is known as the sum of alternative integer powers. The following generating relation is straightforward:

$$\sum_{k=0}^{\infty} M_k(n)\frac{t^k}{k!} = 1 - e^t + e^{2t} - \cdots + (-1)^n e^{nt} = \frac{1 - (-e^t)^{(n+1)}}{e^t + 1}.$$

For an arbitrary real or complex λ, the generalized sum of alternative integer powers $M_k(n,\lambda)$ is defined, in [27], by

$$\sum_{k=0}^{\infty} M_k(n,\lambda)\frac{t^k}{k!} = \frac{1 - \lambda(-e^t)^{(n+1)}}{\lambda e^t + 1}.$$

Clearly $M_k(n,1) = M_k(n)$. On the other hand, if n is even, then

$$S_k(n,-\lambda) = M_k(n,\lambda). \tag{3.1}$$

We start by obtaining certain symmetry identities, which includes the results given in [28–32] and [27], when $y = z = 0$.

Theorem 3.1 *Let $c,d,m \in \mathbb{N}$, $n \in \mathbb{N}_0$ be such that the conditions* (i)-(iii) *are satisfied with t replaced by ct and dt. Then we have the following symmetry identity*:

$$\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+k} {}_HP_{n-r,\beta}^{(m)}\bigl(dx,d^2y,d^3z;k,a,b\bigr)$$

$$\times \sum_{l=0}^{r}\binom{r}{l}S_l\left(c-1;\left(\frac{\beta}{a}\right)^b\right) {}_HP_{r-l,\beta}^{(m-1)}\bigl(cX,c^2Y,c^3Z;k,a,b\bigr)$$

$$= \sum_{r=0}^{n} \binom{n}{r} d^{n-r} c^{r+k} {}_{H}P_{n-r,\beta}^{(m)}\left(cx, c^2 y, c^3 z; k, a, b\right)$$
$$\times \sum_{l=0}^{r} \binom{r}{l} S_l\left(d-1; \left(\frac{\beta}{a}\right)^b\right) {}_{H}P_{r-l,\beta}^{(m-1)}\left(dX, d^2 Y, d^3 Z; k, a, b\right).$$

Proof Let

$$G(t) := \frac{2^{(1-k)(2m-1)} t^{2km-k} e^{cdxt+y(cdt)^2+z(cdt)^3} (\beta^b e^{cdt} - a^b) e^{cdXt+Y(cdt)^2+Z(cdt)^3}}{(\beta^b e^{ct} - a^b)^m (\beta^b e^{dt} - a^b)^m}.$$

Expanding $G(t)$ into a series, we get

$$G(t) = \frac{1}{c^{km} d^{k(m-1)}} \left(\frac{2^{1-k} c^k t^k}{\beta^b e^{ct} - a^b}\right)^m e^{cdxt+y(cdt)^2+z(cdt)^3} \left(\frac{\beta^b e^{cdt} - a^b}{\beta^b e^{dt} - a^b}\right)$$
$$\times \left(\frac{2^{1-k} d^k t^k}{\beta^b e^{dt} - a^b}\right)^{m-1} e^{cdXt+Y(cdt)^2+Z(cdt)^3}$$
$$= \frac{1}{c^{km} d^{k(m-1)}} \left[\sum_{n=0}^{\infty} {}_{H}P_{n,\beta}^{(m)}\left(dx, d^2 y, d^3 z; k, a, b\right) \frac{(ct)^n}{n!}\right] \left[\sum_{l=0}^{\infty} S_l\left(c-1; \left(\frac{\beta}{a}\right)^b\right) \frac{(dt)^l}{l!}\right]$$
$$\times \left[\sum_{r=0}^{\infty} {}_{H}P_{r,\beta}^{(m-1)}\left(cX, c^2 Y, c^3 Z; k, a, b\right) \frac{(dt)^r}{r!}\right].$$

Now, using Corollary 2 in [33, p.890], we get

$$G(t) = \frac{1}{c^{km} d^{km}} \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} c^{n-r} d^{r+k} {}_{H}P_{n-r,\beta}^{(m)}\left(dx, d^2 y, d^3 z; k, a, b\right)\right.$$
$$\left.\times \sum_{l=0}^{r} \binom{r}{l} S_l\left(c-1; \left(\frac{\beta}{a}\right)^b\right) {}_{H}P_{r-l,\beta}^{(m-1)}\left(cX, c^2 Y, c^3 Z; k, a, b\right)\right] \frac{t^n}{n!}. \tag{3.2}$$

In a similar manner,

$$G(t) = \frac{1}{d^{km} c^{k(m-1)}} \left(\frac{2^{1-k} d^k t^k}{\beta^b e^{ct} - a^b}\right)^m e^{cdxt+y(cdt)^2+z(cdt)^3} \left(\frac{\beta^b e^{cdt} - a^b}{\beta^b e^{dt} - a^b}\right)$$
$$\times \left(\frac{2^{1-k} c^k t^k}{\beta^b e^{dt} - a^b}\right)^{m-1} e^{cdXt+Y(cdt)^2+Z(cdt)^3}$$
$$= \frac{1}{c^{km} d^{km}} \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} \binom{n}{r} d^{n-r} c^{r+k} {}_{H}P_{n-r,\beta}^{(m)}\left(cx, c^2 y, c^3 z; k, a, b\right)\right.$$
$$\left.\times \sum_{l=0}^{r} \binom{r}{l} S_l\left(d-1; \left(\frac{\beta}{a}\right)^b\right) {}_{H}P_{r-l,\beta}^{(m-1)}\left(dX, d^2 Y, d^3 Z; k, a, b\right)\right] \frac{t^n}{n!}. \tag{3.3}$$

From (3.2) and (3.3), we get the result. □

For $k = a = b = 1$ and $\beta = \lambda$ we get the following corollary at once.

Corollary 3.2 *For all $c, d, m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we have the following symmetry identity for the Hermite based generalized Apostol-Bernoulli polynomials*:

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+1}{}_H\mathcal{B}_{n-r}^{(m)}\bigl(dx, d^2y, d^3z, \lambda\bigr)\\&\qquad\times\sum_{l=0}^{r}\binom{r}{l}S_l(c-1;\lambda)_H\mathcal{B}_{r-l}^{(m-1)}\bigl(cX, c^2Y, c^3Z, \lambda\bigr)\\&\quad=\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+1}{}_H\mathcal{B}_{n-r}^{(m)}\bigl(cx, c^2y, c^3z, \lambda\bigr)\\&\qquad\times\sum_{l=0}^{r}\binom{r}{l}S_l(d-1;\lambda)_H\mathcal{B}_{r-l}^{(m-1)}\bigl(dX, d^2Y, d^3Z, \lambda\bigr).\end{aligned}$$

For $k+1=-a=b=1$ and $\beta=\lambda$ we get, by considering (3.1) that

Corollary 3.3 *For all $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we have for each pair of positive even integers c and d, or for each pair of positive odd integers c and d,*

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+1}{}_H\mathcal{E}_{n-r}^{(m)}\bigl(dx, d^2y, d^3z, \lambda\bigr)\\&\qquad\times\sum_{l=0}^{r}\binom{r}{l}M_l(c-1;\lambda)_H\mathcal{E}_{r-l}^{(m-1)}\bigl(cX, c^2Y, c^3Z, \lambda\bigr)\\&\quad=\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+1}{}_H\mathcal{E}_{n-r}^{(m)}\bigl(cx, c^2y, c^3z, \lambda\bigr)\\&\qquad\times\sum_{l=0}^{r}\binom{r}{l}M_l(d-1;\lambda)_H\mathcal{E}_{r-l}^{(m-1)}\bigl(dX, d^2Y, d^3Z, \lambda\bigr).\end{aligned}$$

Letting $k=-2a=b=1$ and $2\beta=\lambda$ and taking into account (3.1) that we have the following.

Corollary 3.4 *For all $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we have for each pair of positive even integers c and d, or for each pair of positive odd integers c and d, that*

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}c^{n-r}d^{r+1}{}_H\mathcal{G}_{n-r}^{(m)}\bigl(dx, d^2y, d^3z, \lambda\bigr)\\&\qquad\times\sum_{l=0}^{r}\binom{r}{l}M_l(c-1;\lambda)_H\mathcal{G}_{r-l}^{(m-1)}\bigl(cX, c^2Y, c^3Z, \lambda\bigr)\\&\quad=\sum_{r=0}^{n}\binom{n}{r}d^{n-r}c^{r+1}{}_H\mathcal{G}_{n-r}^{(m)}\bigl(cx, c^2y, c^3z, \lambda\bigr)\\&\qquad\times\sum_{l=0}^{r}\binom{r}{l}M_l(d-1;\lambda)_H\mathcal{G}_{r-l}^{(m-1)}\bigl(dX, d^2Y, d^3Z, \lambda\bigr).\end{aligned}$$

4 Closed-form formulae for Hermite-based generalized Apostol polynomials

In this section, taking into account the relations

$$f_{a,b}^{(\alpha)}(x,y,z;t;k,\beta) := \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!},$$

$$f_{a,b}^{(1)}(x,y,z;t;k,\beta) := \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right) e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HP_{n,\beta}(x,y,z;k,a,b)\frac{t^n}{n!},$$

we observe the following fact:

$$\left[f_{a,b}^{(1)}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};t;k,\beta\right)\right]^{\alpha} = f_{a,b}^{(\alpha)}(x,y,z;t;k,\beta). \tag{4.1}$$

Using (4.1), we start by proving the following closed form summation formula:

Theorem 4.1 *Let the conditions* (i)-(iii) *be satisfied. The following summation formula*:

$$\sum_{l=0}^{n}\binom{n}{l}\left[{}_HP_{n-l+1,\beta}^{(\alpha)}(x,y,z;k,a,b)\,{}_HP_{l,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\right.$$
$$\left. - \alpha\,{}_HP_{n-l,\beta}^{(\alpha)}(x,y,z;k,a,b)\,{}_HP_{l+1,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\right] = 0$$

holds true.

Proof Taking logarithms on both sides of (4.1) and then differentiating with respect to t, we get

$$\frac{\partial f_{a,b}^{(\alpha)}(x,y,z;t;k,\beta)}{\partial t} f_{a,b}^{(1)}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};t;k,\beta\right)$$
$$= \alpha f_{a,b}^{(\alpha)}(x,y,z;t;k,\beta)\frac{\partial f_{a,b}^{(1)}(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};t;k,\beta)}{\partial t}.$$

Inserting the corresponding generating relations, we obtain

$$\sum_{n=1}^{\infty} n\,{}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^{n-1}}{n!}\sum_{l=0}^{\infty} {}_HP_{l,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\frac{t^l}{l!}$$
$$= \alpha\sum_{n=0}^{\infty} {}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\sum_{l=0}^{\infty} l\,{}_HP_{l,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\frac{t^{l-1}}{l!},$$

and hence

$$\sum_{n=0}^{\infty} {}_HP_{n+1,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\sum_{l=0}^{\infty} {}_HP_{l,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\frac{t^l}{l!}$$
$$= \alpha\sum_{n=0}^{\infty} {}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\sum_{l=0}^{\infty} {}_HP_{l+1,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\frac{t^l}{l!}.$$

Using the fact that (see [34, p.101, Lemma 3])

$$\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}A(n,l)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}A(n-l,l), \tag{4.2}$$

we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}{}_{H}P_{n-l+1,\beta}^{(\alpha)}(x,y,z;k,a,b)_{H}P_{l,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\right]\frac{t^{n}}{n!}\\&\quad=\alpha\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}{}_{H}P_{n-l,\beta}^{(\alpha)}(x,y,z;k,a,b)_{H}P_{l+1,\beta}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};k,a,b\right)\right]\frac{t^{n}}{n!}.\end{aligned}$$

Whence the result. □

Corollary 4.2 *Let $k=a=b=1$ and $\beta=\lambda$. For all $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, we have the following closed form summation formula for the generalized Apostol-Bernoulli polynomials*:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left[{}_{H}\mathcal{B}_{n-k+1}^{(\alpha)}(x,y,z;\lambda)_{H}\mathcal{B}_{k}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};\lambda\right)\right.\\&\quad\left.-\alpha_{H}\mathcal{B}_{n-k}^{(\alpha)}(x,y,z;\lambda)\mathcal{B}_{k+1}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};\lambda\right)\right]=0.\end{aligned}$$

Corollary 4.3 *Let $k+1=-a=b=1$ and $\beta=\lambda$. For all $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, we have the following closed form summation formula for the generalized Apostol-Euler polynomials*:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left[{}_{H}\mathcal{E}_{n-k+1}^{(\alpha)}(x,y,z;\lambda)_{H}\mathcal{E}_{k}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};\lambda\right)\right.\\&\quad\left.-\alpha_{H}\mathcal{E}_{n-k}^{(\alpha)}(x,y,z;\lambda)\mathcal{E}_{k+1}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};\lambda\right)\right]=0.\end{aligned}$$

Corollary 4.4 *Let $k=-2a=b=1$ and $2\beta=\lambda$. For all $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, we have the following closed form summation formula for the generalized Apostol-Genocchi polynomials*:

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left[{}_{H}\mathcal{G}_{n-k+1}^{(\alpha)}(x,y,z;\lambda)_{H}\mathcal{G}_{k}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};\lambda\right)\right.\\&\quad\left.-\alpha_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y,z;\lambda)\mathcal{G}_{k+1}\left(\frac{x}{\alpha},\frac{y}{\alpha},\frac{z}{\alpha};\lambda\right)\right]=0.\end{aligned}$$

Theorem 4.5 *Let the conditions* (i)-(iii) *be satisfied. Then we have the following relation between Hermite based Apostol polynomials and* 3*d-Hermite polynomials*:

$$\begin{aligned}&{}_{H}P_{n+m,\beta}^{(\alpha)}(X,Y,Z;k,a,b)\\&\quad=\sum_{r,l=0}^{n,m}\binom{n}{r}\binom{m}{l}H_{r+l}^{(3)}(X-x,Y-y,Z-z)_{H}P_{n+m-r-l}^{(\alpha)}(x,y,z;k,a,b).\end{aligned}$$

Proof From (2.1), we can write that

$$\left(\frac{2^{1-k}(t+w)^k}{\beta^b e^{t+w}-a^b}\right)^\alpha e^{x(t+w)+y(t+w)^2+z(t+w)^3} = \sum_{n=0}^\infty {}_HP_{n,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{(t+w)^n}{n!} = \sum_{n,m=0}^\infty {}_HP_{n+m,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\frac{w^m}{m!}. \tag{4.3}$$

Therefore, we get

$$\left(\frac{2^{1-k}(t+w)^k}{\beta^b e^{t+w}-a^b}\right)^\alpha = e^{-x(t+w)-y(t+w)^2-z(t+w)^3}\sum_{n,m=0}^\infty {}_HP_{n+m,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\frac{w^m}{m!}.$$

Multiplying both sides by $e^{X(t+w)+Y(t+w)^2+Z(t+w)^3}$, we have

$$\left(\frac{2^{1-k}(t+w)^k}{\beta^b e^{t+w}-a^b}\right)^\alpha e^{X(t+w)+Y(t+w)^2+Z(t+w)^3} = e^{(X-x)(t+w)+(Y-y)(t+w)^2+(Z-z)(t+w)^3}\sum_{n,m=0}^\infty {}_HP_{n+m,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\frac{w^m}{m!}.$$

Taking into account (1.1) and (4.3), then using (4.2), we get

$$\begin{aligned}\sum_{n,m=0}^\infty {}_HP_{n+m,\beta}^{(\alpha)}(X,Y,Z;k,a,b)\frac{t^n}{n!}\frac{w^m}{m!} &= \sum_{n,m=0}^\infty {}_HP_{n+m,\beta}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\frac{w^m}{m!}\sum_{r,l=0}^\infty H_{r+l}^{(3)}(X-x,Y-y,Z-z)\frac{t^r}{r!}\frac{w^l}{l!} \\ &= \sum_{n,m=0}^\infty\sum_{r,l=0}^{n,m}\binom{n}{r}\binom{m}{l}H_{r+l}^{(3)}(X-x,Y-y,Z-z)\,{}_HP_{n+m-r-l}^{(\alpha)}(x,y,z;k,a,b)\frac{t^n}{n!}\frac{w^m}{m!}.\end{aligned}$$

Whence the result. □

Corollary 4.6 *Let $k=a=b=1$ and $\beta=\lambda$. For all $c,d,m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, we have the following summation formula between the Hermite-based generalized Apostol-Bernoulli polynomials and 3d-Hermite polynomials*:

$${}_H\mathcal{B}_{n+m}^{(\alpha)}(X,Y,Z;\lambda) = \sum_{k,l=0}^{n,m}\binom{n}{k}\binom{m}{l}H_{k+l}^{(3)}(X-x,Y-y,Z-z)\,{}_H\mathcal{B}_{n+m-k-l}^{(\alpha)}(x,y,z;\lambda).$$

Corollary 4.7 *Let $k+1=-a=b=1$ and $\beta=\lambda$. For all $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, we have the following summation formula between the Hermite-based generalized Apostol-Euler polynomials and 3d-Hermite polynomials*:

$${}_H\mathcal{E}_{n+m}^{(\alpha)}(X,Y,Z;\lambda) = \sum_{k,l=0}^{n,m}\binom{n}{k}\binom{m}{l}H_{k+l}^{(3)}(X-x,Y-y,Z-z)\,{}_H\mathcal{E}_{n+m-k-l}^{(\alpha)}(x,y,z;\lambda).$$

Corollary 4.8 *Let $k = -2a = b = 1$ and $2\beta = \lambda$. For all $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we have the following summation formula between the Hermite-based generalized Apostol-Genocchi polynomials and 3d-Hermite polynomials*:

$$
\begin{aligned}
&{}_H\mathcal{G}_{n+m}^{(\alpha)}(X,Y,Z;\lambda) \\
&\quad = \sum_{k,l=0}^{n,m} \binom{n}{k}\binom{m}{l} H_{k+l}^{(3)}(X-x,Y-y,Z-z)\, {}_H\mathcal{G}_{n+m-k-l}^{(\alpha)}(x,y,z;\lambda).
\end{aligned}
$$

Competing interests
The author declares that they have no competing interests.

Author's contributions
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