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Some properties of the generalized Apostol-type polynomials

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Abstract

In this paper, we study some properties of the generalized Apostol-type polynomials (see (Luo and Srivastava in Appl. Math. Comput. 217:5702-5728, 2011)), including the recurrence relations, the differential equations and some other connected problems, which extend some known results. We also deduce some properties of the generalized Apostol-Euler polynomials, the generalized Apostol-Bernoulli polynomials, and Apostol-Genocchi polynomials of high order.

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1 Introduction, definitions and motivation

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [1], pp.532-533 and [2], p.61 *et seq.*; see also [3] and the references cited therein):

$$\left(\frac{z}{e^z-1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^\infty B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \bigl(|z|<2\pi\bigr), \tag{1.1}$$

$$\left(\frac{2}{e^z+1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^\infty E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \bigl(|z|<\pi\bigr) \tag{1.2}$$

and

$$\left(\frac{2z}{e^z+1}\right)^\alpha \cdot e^{xz} = \sum_{n=0}^\infty G_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \bigl(|z|<\pi\bigr). \tag{1.3}$$

So that, obviously, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given, respectively, by

$$B_n(x) := B_n^{(1)}(x), \qquad E_n(x) := E_n^{(1)}(x) \quad \text{and} \quad G_n(x) := G_n^{(1)}(x) \quad (n \in \mathbb{N}_0). \tag{1.4}$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n of order n, we have

$$B_n := B_n(0) = B_n^{(1)}(0), \qquad E_n := E_n(0) = E_n^{(1)}(0) \quad \text{and} \quad G_n := G_n(0) = G_n^{(1)}(0), \tag{1.5}$$

respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol (see [4], p.165, Eq. (3.1)) and (more recently) by Srivastava (see [5], pp.83-84). We begin by recalling here Apostol's definitions as follows.

Definition 1.1 (Apostol [4]; see also Srivastava [5]) The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ ($\lambda \in \mathbb{C}$) are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{z^n}{n!}$$
$$\bigl(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log\lambda| \text{ when } \lambda \neq 1\bigr) \tag{1.6}$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \quad \text{and} \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \tag{1.7}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers.

Recently, Luo and Srivastava [6] further extended the Apostol-Bernoulli polynomials as the so-called Apostol-Bernoulli polynomials of order α.

Definition 1.2 (Luo and Srivastava [6]) The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ ($\lambda \in \mathbb{C}$) of order $\alpha \in \mathbb{N}_0$ are defined by means of the following generating function:

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}$$
$$\bigl(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log\lambda| \text{ when } \lambda \neq 1\bigr) \tag{1.8}$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda), \tag{1.9}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Bernoulli numbers of order α.

On the other hand, Luo [7], gave an analogous extension of the generalized Euler polynomials as the so-called Apostol-Euler polynomials of order α.

Definition 1.3 (Luo [7]) The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ ($\lambda \in \mathbb{C}$) of order $\alpha \in \mathbb{N}_0$ are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr) \tag{1.10}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{1.11}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Euler numbers of order α.

On the subject of the Genocchi polynomials $G_n(x)$ and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [8–14]). Moreover, Luo (see [12–14]) introduced and investigated the Apostol-Genocchi polynomials of (real or complex) order α, which are defined as follows:

Definition 1.4 The Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ $(\lambda \in \mathbb{C})$ of order $\alpha \in \mathbb{N}_0$ are defined by means of the following generating function:

$$\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr) \tag{1.12}$$

with, of course,

$$\begin{aligned} &G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1), \qquad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0;\lambda), \\ &\mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \end{aligned} \tag{1.13}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol-Genocchi numbers, the Apostol-Genocchi numbers of order α and the Apostol-Genocchi polynomials, respectively.

Recently, Luo and Srivastava [15] introduced a unification (and generalization) of the above-mentioned three families of the generalized Apostol type polynomials.

Definition 1.5 (Luo and Srivastava [15]) The generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$ $(\alpha \in \mathbb{N}_0, \lambda, u, v \in \mathbb{C})$ of order α are defined by means of the following generating function:

$$\left(\frac{2^u z^v}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr), \tag{1.14}$$

where

$$\mathcal{F}_n^{(\alpha)}(\lambda;u,v) := \mathcal{F}_n^{(\alpha)}(0;\lambda;u,v) \tag{1.15}$$

denote the so-called Apostol type numbers of order α.

So that, by comparing Definition 1.5 with Definitions 1.2, 1.3 and 1.4, we have

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^{\alpha}\mathcal{F}_n^{(\alpha)}(x;-\lambda;0,1), \tag{1.16}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1,0), \tag{1.17}$$

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1,1). \tag{1.18}$$

A polynomial $p_n(x)$ ($n \in \mathbb{N}$, $x \in \mathbb{C}$) is said to be a quasi-monomial [16], whenever two operators $\hat{M}$, $\hat{P}$, called multiplicative and derivative (or lowering) operators respectively, can be defined in such a way that

$$\hat{P}p_n(x) = np_{n-1}(x), \tag{1.19}$$

$$\hat{M}p_n(x) = p_{n+1}(x), \tag{1.20}$$

which can be combined to get the identity

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{1.21}$$

The Appell polynomials [17] can be defined by considering the following generating function:

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{1.22}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{R_k}{k!} t^k \quad \bigl(A(0) \neq 0\bigr) \tag{1.23}$$

is analytic function at $t = 0$.

From [18], we know that the multiplicative and derivative operators of $R_n(x)$ are

$$\hat{M} = (x + \alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}, \tag{1.24}$$

$$\hat{P} = D_x, \tag{1.25}$$

where

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{1.26}$$

By using (1.21), we have the following lemma.

Lemma 1.6 ([18]) *The Appell polynomials $R_n(x)$ defined by* (1.22) *satisfy the differential equation*:

$$\frac{\alpha_{n-1}}{(n-1)!} y^{(n)} + \frac{\alpha_{n-2}}{(n-2)!} y^{(n-1)} + \cdots + \frac{\alpha_1}{1!} y'' + (x + \alpha_0) y' - ny = 0, \tag{1.27}$$

where the numerical coefficients α_k, $k = 1, 2, \ldots, n-1$ are defined in (1.26), *and are linked to the values R_k by the following relations*:

$$R_{k+1} = \sum_{h=0}^{k} \binom{k}{h} R_h \alpha_{k-h}.$$

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ be a polynomial set. $\{P_n\}_{n\geq 0}$ is called a σ-Appell polynomial set of transfer power series A is generated by

$$G(x,t) = A(t)G_0(x,t) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \tag{1.28}$$

where $G_0(x,t)$ is a solution of the system:

$$\sigma G_0(x,t) = tG_0(x,t),$$
$$G_0(x,0) = 1.$$

In [19], the authors investigated the connection coefficients between two polynomials. And there is a result about connection coefficients between two σ-Appell polynomial sets.

Lemma 1.7 ([19]) *Let $\sigma \in \Lambda^{(-1)}$. Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two σ-Appell polynomial sets of transfer power series, respectively, A_1 and A_2. Then*

$$Q_n(x) = \sum_{m=0}^{n} \frac{n!}{m!} \alpha_{n-m} P_m(x), \tag{1.29}$$

where

$$\frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty} \alpha_k t^k.$$

In recent years, several authors obtained many interesting results involving the related Bernoulli polynomials and Euler polynomials [5, 20–40]. And in [29], the authors studied some series identities involving the generalized Apostol type and related polynomials.

In this paper, we study some other properties of the generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$, including the recurrence relations, the differential equations and some connection problems, which extend some known results. As special, we obtain some properties of the generalized Apostol-Euler polynomials, the generalized Apostol-Bernoulli polynomials and Apostol-Genocchi polynomials of high order.

2 Recursion formulas and differential equations

From the generating function (1.14), we have

$$\frac{\partial}{\partial x} \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = n\mathcal{F}_{n-1}^{(\alpha)}(x;\lambda;u,v). \tag{2.1}$$

A recurrence relation for the generalized Apostol type polynomials is given by the following theorem.

Theorem 2.1 *For any integral $n \geq 1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, the following recurrence relation for the generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$ holds true*:

$$\left(\frac{\alpha v}{n+1} - 1\right) \mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;u,v) = \frac{\alpha\lambda}{2^u} \cdot \frac{n!}{(n+v)!} \mathcal{F}_{n+v}^{(\alpha+1)}(x+1;\lambda;u,v) - x\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v). \tag{2.2}$$

Proof Differentiating both sides of (1.14) with respect to t, and using some elementary algebra and the identity principle of power series, recursion (2.2) easily follows. □

By setting $\lambda := -\lambda$, $u = 0$ and $v = 1$ in Theorem 2.1, and then multiplying $(-1)^\alpha$ on both sides of the result, we have:

Corollary 2.2 *For any integral $n \geq 1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, the following recurrence relation for the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ holds true*:

$$\left[\alpha - (n+1)\right]\mathcal{B}_{n+1}^{(\alpha)}(x;\lambda) = \alpha\lambda\mathcal{B}_{n+1}^{(\alpha+1)}(x+1;\lambda) - x\mathcal{B}_n^{(\alpha)}(x;\lambda). \tag{2.3}$$

By setting $u = 1$ and $v = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.3 *For any integral $n \geq 1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, the following recurrence relation for the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ holds true*:

$$\mathcal{E}_{n+1}^{(\alpha)}(x;\lambda) = x\mathcal{E}_n^{(\alpha)}(x;\lambda) - \frac{\alpha\lambda}{2}\mathcal{E}_n^{(\alpha+1)}(x+1;\lambda). \tag{2.4}$$

By setting $u = 1$ and $v = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.4 *For any integral $n \geq 1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, the following recurrence relation for the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ holds true*:

$$2\left[\alpha - (n+1)\right]\mathcal{G}_{n+1}^{(\alpha)}(x;\lambda) = \alpha\lambda\mathcal{G}_{n+1}^{(\alpha+1)}(x+1;\lambda) - 2(n+1)x\mathcal{G}_n^{(\alpha)}(x;\lambda). \tag{2.5}$$

From (1.14) and (1.22), we know that the generalized Appostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$ is Appell polynomials with

$$A(t) = \left(\frac{2^u t^v}{\lambda e^t + 1}\right)^\alpha. \tag{2.6}$$

From the Eq. (23) of [15], we know that $\mathcal{G}_0(1;\lambda) = 0$. So from (2.6) and (1.12), we can obtain that if $v = 0$, we have

$$\frac{A'(t)}{A(t)} = \frac{\lambda\alpha}{2}\sum_{n=0}^{\infty}\frac{\mathcal{G}_{n+1}(1;\lambda)}{n+1}\cdot\frac{t^n}{n!}. \tag{2.7}$$

By using (1.24) and (1.26), we can obtain the multiplicative and derivative operators of the generalized Appostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$

$$\hat{M} = \left(x + \frac{\lambda\alpha}{2}\mathcal{G}_1(1;\lambda)\right) + \frac{\lambda\alpha}{2}\sum_{k=0}^{n-1}\frac{\mathcal{G}_{n-k+1}(1;\lambda)}{(n-k+1)!}D_x^{n-k}, \tag{2.8}$$

$$\hat{P} = D_x. \tag{2.9}$$

From (2.1), we can obtain

$$\frac{\partial^p}{\partial x^p}\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \frac{n!}{(n-p)!}\mathcal{F}_{n-p}^{(\alpha)}(x;\lambda;u,v). \tag{2.10}$$

Then by using (1.20), (2.8) and (2.10), we obtain the following result.

Theorem 2.5 *For any integral $n \geq 1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, the following recurrence relation for the generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,0)$ holds true*:

$$\begin{aligned}\mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;u,0) &= \left(x + \frac{\lambda\alpha}{2}\mathcal{G}_1(1;\lambda)\right)\mathcal{F}_n^{(\alpha)}(x;\lambda;u,0) \\ &\quad + \frac{\lambda\alpha}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{\mathcal{G}_{n-k+1}(1;\lambda)}{n-k+1}\mathcal{F}_{n-k}^{(\alpha)}(x;\lambda;u,0). \qquad (2.11)\end{aligned}$$

By setting $u = 1$ in Theorem 2.5, we have the following corollary.

Corollary 2.6 *For any integral $n \geq 1$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, the following recurrence relation for the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ holds true*:

$$\mathcal{E}_{n+1}^{(\alpha)}(x;\lambda) = \left(x + \frac{\lambda\alpha}{2}\mathcal{G}_1(1;\lambda)\right)\mathcal{E}_n^{(\alpha)}(x;\lambda) + \frac{\lambda\alpha}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{\mathcal{G}_{n-k+1}(1;\lambda)}{n-k+1}\mathcal{E}_{n-k}^{(\alpha)}(x;\lambda). \qquad (2.12)$$

Furthermore, applying Lemma 1.7 to $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,0)$, we have the following theorem.

Theorem 2.7 *The generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,0)$ satisfy the differential equation*:

$$\begin{aligned}&\frac{\lambda\alpha}{2}\frac{\mathcal{G}_n(1;\lambda)}{n!}y^{(n)} + \frac{\lambda\alpha}{2}\frac{\mathcal{G}_{n-1}(1;\lambda)}{(n-1)!}y^{(n-1)} + \cdots \\ &\quad + \frac{\lambda\alpha}{2}\frac{\mathcal{G}_2(1;\lambda)}{2}y'' + \left(x + \frac{\lambda\alpha}{2}\mathcal{G}_1(1;\lambda)\right)y' - ny = 0. \qquad (2.13)\end{aligned}$$

Specially, by setting $u = 1$ in Theorem 2.7, then we have the following corollary.

Corollary 2.8 *The generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ satisfy the differential equation*:

$$\begin{aligned}&\frac{\lambda\alpha}{2}\frac{\mathcal{G}_n(1;\lambda)}{n!}y^{(n)} + \frac{\lambda\alpha}{2}\frac{\mathcal{G}_{n-1}(1;\lambda)}{(n-1)!}y^{(n-1)} + \cdots \\ &\quad + \frac{\lambda\alpha}{2}\frac{\mathcal{G}_2(1;\lambda)}{2}y'' + \left(x + \frac{\lambda\alpha}{2}\mathcal{G}_1(1;\lambda)\right)y' - ny = 0. \qquad (2.14)\end{aligned}$$

3 Connection problems

From (1.14) and (1.28), we know that the generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$ are a D_x-Appell polynomial set, where D_x denotes the derivative operator.

From Table 1 in [19], we know that the derivative operators of monomials x^n and the Gould-Hopper polynomials $g_n^m(x,h)$ [30] are all D_x. And their transfer power series $A(t)$ are 1 and e^{ht^m}, respectively.

Applying Lemma 1.7 to $P_n(x) = x^n$ and $Q_n(x) = \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$, we have the following theorem.

Theorem 3.1

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{F}_{n-m}^{(\alpha)}(\lambda;u,v)x^m, \tag{3.1}$$

where $\mathcal{F}_n^{(\alpha)}(\lambda;u,v)$ *is the so-called Apostol type numbers of order* α *defined by* (1.15).

By setting $\lambda := -\lambda$, $u = 0$ and $v = 1$ in Theorem 3.1, and then multiplying $(-1)^\alpha$ on both sides of the result, we have the following corollary.

Corollary 3.2

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha)}(\lambda)x^m, \tag{3.2}$$

which is just Eq. (3.1) *of* [23].

By setting $u = 0$ and $v = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.3

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{n-m}^{(\alpha)}(\lambda)x^m. \tag{3.3}$$

By setting $u = 1$ and $v = 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.4

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}^{(\alpha)}(\lambda)x^m, \tag{3.4}$$

which is just Eq. (24) *of* [15].

Applying Lemma 1.7 to $P_n(x) = \mathcal{F}_n(x;\lambda;u,v)$ and $Q_n(x) = \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$, we have the following theorem.

Theorem 3.5

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{F}_{n-m}^{(\alpha-1)}(\lambda;u,v)\mathcal{F}_m(x;\lambda;u,v), \tag{3.5}$$

where $\mathcal{F}_n^{(\alpha)}(\lambda;u,v)$ *is the so-called Apostol type numbers of order* α *defined by* (1.15).

By setting $\lambda := -\lambda$, $u = 0$ and $v = 1$ in Theorem 3.5, and then multiplying $(-1)^\alpha$ on both sides of the result, we have the following corollary.

Corollary 3.6

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha-1)}(\lambda)\mathcal{B}_m(x;\lambda), \tag{3.6}$$

which is just Eq. (3.2) *of* [23].

By setting $u = 1$ and $v = 0$ in Theorem 3.5, we have the following corollary.

Corollary 3.7

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{n-m}^{(\alpha-1)}(\lambda)\mathcal{E}_m(x;\lambda). \tag{3.7}$$

By setting $u = 1$ and $v = 1$ in Theorem 3.5, we have the following corollary.

Corollary 3.8

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}^{(\alpha-1)}(\lambda)\mathcal{G}_m(x;\lambda). \tag{3.8}$$

Applying Lemma 1.7 to $P_n(x) = g_n^m(x,h)$ and $Q_n(x) = \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$, we have the following theorem.

Theorem 3.9

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{r=0}^{n} \frac{n!}{r!} \left[\sum_{k=0}^{[(n-r)/m]} (-1)^k \frac{h^k}{k!(n-r-mk)!} \mathcal{F}_{n-r-mk}^{(\alpha)}(\lambda;u,v) \right] g_r^m(x,h). \tag{3.9}$$

By setting $\lambda := -\lambda$, $u = 0$ and $v = 1$ in Theorem 3.9, and then multiplying $(-1)^\alpha$ on both sides of the result, we have the following corollary.

Corollary 3.10

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[\sum_{k=0}^{[(n-r)/m]} (-1)^k \frac{h^k}{k!(n-r-mk)!} \mathcal{B}_{n-r-mk}^{(\alpha)}(\lambda) \right] g_r^m(x,h), \tag{3.10}$$

which is just Eq. (3.3) *of* [23].

By setting $u = 1$ and $v = 0$ in Theorem 3.9, we have the following corollary.

Corollary 3.11

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[\sum_{k=0}^{[(n-r)/m]} (-1)^k \frac{h^k}{k!(n-r-mk)!} \mathcal{E}_{n-r-mk}^{(\alpha)}(\lambda) \right] g_r^m(x,h). \tag{3.11}$$

By setting $u = 1$ and $v = 1$ in Theorem 3.9, we have the following corollary.

Corollary 3.12

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[\sum_{k=0}^{[(n-r)/m]} (-1)^k \frac{h^k}{k!(n-r-mk)!} \mathcal{G}_{n-r-mk}^{(\alpha)}(\lambda) \right] g_r^m(x,h). \tag{3.12}$$

When $v\alpha = 1$, applying Lemma 1.7 to $P_n(x) = \mathcal{E}_n^{(\alpha-1)}(x;\lambda)$ and $Q_n(x) = \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$, we have the following theorem.

Theorem 3.13 *If* $v\alpha = 1$, *then we have*

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n} \binom{n}{m} 2^{(u-1)\alpha} \mathcal{G}_{n-m}(\lambda) \mathcal{E}_m^{(\alpha-1)}(x;\lambda). \tag{3.13}$$

By setting $\lambda := -\lambda$, $u = 0$ and $v = 1$ in Theorem 3.13, and then multiplying $(-1)^\alpha$ on both sides of the result, we have the following corollary.

Corollary 3.14

$$\mathcal{B}_n(x;\lambda) = -\frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(-\lambda) x^m. \tag{3.14}$$

By setting $u = 1$ and $v = 1$ in Theorem 3.13, we have the following corollary.

Corollary 3.15

$$\mathcal{G}_n(x;\lambda) = -\frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(\lambda) x^m, \tag{3.15}$$

which is just the case of $\alpha = 1$ *in* (3.4).

When $v = 1$ or $\alpha = 0$, applying Lemma 1.7 to $P_n(x) = \mathcal{G}_n^{(\alpha-1)}(x;\lambda)$ and $Q_n(x) = \mathcal{F}_n^{(\alpha)}(x;\lambda;u,v)$, we can obtain the following theorem.

Theorem 3.16 *If* $v = 1$ *or* $\alpha = 0$, *we have*

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n} \binom{n}{m} 2^{(u-1)\alpha} \mathcal{G}_{n-m}(\lambda) \mathcal{G}_m^{(\alpha-1)}(x;\lambda). \tag{3.16}$$

By setting $\lambda := -\lambda$, $u = 0$ and $v = 1$ in Theorem 3.13, and then multiplying $(-1)^\alpha$ on both sides of the result, we have the following corollary.

Corollary 3.17

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \left(-\frac{1}{2}\right)^\alpha \mathcal{G}_{n-m}(-\lambda) \mathcal{G}_m^{(\alpha-1)}(x;-\lambda). \tag{3.17}$$

When $\alpha = 1$ in (3.17), it is just (3.15).

By setting $u = 1$ and $v = 1$ in Theorem 3.16, we have the following corollary.

Corollary 3.18

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(\lambda) \mathcal{G}_m^{(\alpha-1)}(x;\lambda), \tag{3.18}$$

which is equal to (3.8).

If $\alpha = 0$ in Theorem 3.16, we have:

Corollary 3.19

$$x^n = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(\lambda)\mathcal{G}_m^{(-1)}(x;\lambda). \tag{3.19}$$

4 Hermite-based generalized Apostol type polynomials

Finally, we give a generation of the generalized Apostol type polynomials.

The two-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ are defined by the series [31]

$$H_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r}y^r}{r!(n-2r)!} \tag{4.1}$$

with the following generating function:

$$\exp\left(xt + yt^2\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y). \tag{4.2}$$

And the 2VHKdFP $H_n(x,y)$ are also defined through the operational identity

$$\exp\left(y\frac{\partial^2}{\partial x^2}\right)\left\{x^n\right\} = H_n(x,y). \tag{4.3}$$

Acting the operator $\exp(y\frac{\partial^2}{\partial x^2})$ on (1.14), and by the identity [32]

$$\exp\left(y\frac{\partial^2}{\partial x^2}\right)\left\{\exp\left(-ax^2 + bx\right)\right\} = \frac{1}{\sqrt{1+4ay}} \exp\left(-\frac{ax^2 - bx - b^2y}{1+4ay}\right), \tag{4.4}$$

we define the Hermite-based generalized Apostol type polynomials ${}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;u,v)$ by the generating function

$$\left(\frac{2^u z^v}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;u,v)\frac{t^n}{n!} \quad \left(|t| < \left|\log(-\lambda)\right|\right). \tag{4.5}$$

Clearly, we have

$${}_H\mathcal{F}_n(x,y;\lambda;u,v) = {}_H\mathcal{F}_n^{(1)}(x,y;\lambda;u,v).$$

From the generating function (4.5), we easily obtain

$$\frac{\partial}{\partial x} {}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;u,v) = n {}_H\mathcal{F}_{n-1}^{(\alpha)}(x,y;\lambda;u,v) \tag{4.6}$$

and

$$\frac{\partial}{\partial y} {}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;u,v) = n(n-1) {}_H\mathcal{F}_{n-2}^{(\alpha)}(x,y;\lambda;u,v), \tag{4.7}$$

which can be combined to get the identity

$$\frac{\partial^2}{\partial x^2}{}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;u,v) = \frac{\partial}{\partial y}{}_H\mathcal{F}_n^{(\alpha)}(x,y;\lambda;u,v). \tag{4.8}$$

Acting with the operator $\exp y\frac{\partial^2}{\partial x^2}$ on both sides of (3.1), (3.5), (3.13), (3.18), and by using (4.3), we obtain

$${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{F}_{n-m}^{(\alpha)}(\lambda;u,v)H_m(x,y), \tag{4.9}$$

$${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{F}_{n-m}^{(\alpha-1)}(\lambda;u,v)\,{}_H\mathcal{F}_m(x;\lambda;u,v), \tag{4.10}$$

$${}_H\mathcal{F}_n^{(\alpha)}(x;\lambda;u,v) = \sum_{m=0}^{n}\binom{n}{m}2^{(u-1)\alpha}\mathcal{G}_{n-m}(\lambda)\,{}_H\mathcal{E}_m^{(\alpha-1)}(x;\lambda), \quad \text{where } v\alpha = 1, \tag{4.11}$$

$${}_H\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m}(\lambda)\,{}_H\mathcal{G}_m^{(\alpha-1)}(x;\lambda), \quad \text{where } v = 1 \text{ or } \alpha = 0, \tag{4.12}$$

where ${}_H\mathcal{E}_n^{(\alpha)}(x;\lambda)$ and ${}_H\mathcal{G}_n^{(\alpha)}(x;\lambda)$ are the Hermite-based generalized Apostol-Euler polynomials and the Hermite-based generalized Apostol-Genocchi polynomials respectively, defined by the following generating functions:

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}\cdot e^{xt+yt^2} = \sum_{n=0}^{\infty}{}_H\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad \bigl(|t| < \bigl|\log(-\lambda)\bigr|\bigr),$$

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}\cdot e^{xt+yt^2} = \sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad \bigl(|t| < \bigl|\log(-\lambda)\bigr|\bigr).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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