

Research Article

Some New Classes of Generalized Apostol-Euler and Apostol-Genocchi Polynomials

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The main object of this paper is to introduce and investigate two new classes of generalized Apostol-Euler and Apostol-Genocchi polynomials. In particular, we obtain a new addition formula for the new class of the generalized Apostol-Euler polynomials. We also give an extension and some analogues of the Srivastava-Pintér addition theorem obtained in the works by Srivastava and Pintér (2004) and R. Tremblay, S. Gaboury, B.-J. Fugère, and Tremblay et al. (2011). for both classes.

1. Introduction

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, each of degree n as well as in α , are defined, respectively, by the following generating functions (see, [1, volume 3, page 253 et seq.], [2, Section 2.8], and [3]):

$$\begin{aligned} \left(\frac{t}{e^t - 1}\right)^\alpha \cdot e^{xt} &= \sum_{k=0}^{\infty} B_k^{(\alpha)}(x) \frac{t^k}{k!} \quad (|t| < 2\pi; 1^\alpha := 1), \\ \left(\frac{2}{e^t + 1}\right)^\alpha \cdot e^{xt} &= \sum_{k=0}^{\infty} E_k^{(\alpha)}(x) \frac{t^k}{k!} \quad (|t| < \pi; 1^\alpha := 1), \\ \left(\frac{2t}{e^t + 1}\right)^\alpha \cdot e^{xt} &= \sum_{k=0}^{\infty} G_k^{(\alpha)}(x) \frac{t^k}{k!} \quad (|t| < \pi; 1^\alpha := 1). \end{aligned}$$

(1.1)

The literature contains a large number of interesting properties and relationships involving these polynomials [1, 4–7]. These appear in many applications in combinatorics, number theory, and numerical analysis.

Many interesting extensions to these polynomials have been given. In particular, Luo and Srivastava [8, 9] introduced the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$; Luo [10] invented the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ and the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ in [3]. These polynomials are defined, respectively, as follows.

Definition 1.1. The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!} \quad (1.2)$$

($|t| < 2\pi$, if $\lambda = 1$; $|t| < |\log \lambda|$, if $\lambda \neq 1$; $1^\alpha := 1$)

with

$$B_n^{(\alpha)}(x) = \mathfrak{B}_n^{(\alpha)}(x; 1). \quad (1.3)$$

Definition 1.2. The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!} \quad (|t| < |\log(-\lambda)|; 1^\alpha := 1) \quad (1.4)$$

with

$$E_n^{(\alpha)}(x) = \mathfrak{E}_n^{(\alpha)}(x; 1). \quad (1.5)$$

Definition 1.3. The generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x; \lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_k^{(\alpha)}(x; \lambda) \frac{t^k}{k!} \quad (|t| < |\log(-\lambda)|; 1^\alpha := 1) \quad (1.6)$$

with

$$G_n^{(\alpha)}(x) = \mathfrak{G}_n^{(\alpha)}(x; 1). \quad (1.7)$$

Many authors have investigated these polynomials and numerous very interesting papers can be found in the literature. The reader should read [11–20].

Recently, the authors [21] studied a new family of generalized Apostol-Bernoulli polynomials of order α in the following form.

Definition 1.4. For arbitrary real or complex parameter α and for $b, c \in \mathbb{R}^+$, the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1, \alpha]}(x, b, c; \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t = 0$, with $|t \log b| < 2\pi$ if $\lambda = 1$ or with $|t \log b| < |\log \lambda|$ if $\lambda \neq 1$, by means of the generating function:

$$\left(\frac{t^m}{\lambda b^t - \sum_{l=0}^{m-1} \left((t \log b)^l / l! \right)} \right)^\alpha \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_n^{[m-1, \alpha]}(x, b, c; \lambda) \frac{t^k}{k!}. \tag{1.8}$$

It easy to see that if we set $m = 1$, $b = c = e$ in (1.8), we arrive at the following:

$$\left(\frac{t}{\lambda e^t - 1} \right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_n^{[0, \alpha]}(x, e, e; \lambda) \frac{t^k}{k!}. \tag{1.9}$$

This is the generating function for the generalized Apostol-Bernoulli polynomials of order α . Thus, we have

$$\mathfrak{B}_n^{[0, \alpha]}(x, e, e; \lambda) = \mathfrak{B}_n^{(\alpha)}(x; \lambda). \tag{1.10}$$

Obviously, when we set $\lambda = 1$ and $\alpha = 1$ in (1.10) we obtain

$$\mathfrak{B}_n^{[0, 1]}(x, e, e; 1) = B_n(x), \tag{1.11}$$

where $B_n(x)$ are the classical Bernoulli polynomials.

Moreover, Srivastava et al. [22] introduced two new families of generalized Euler and Genocchi polynomials. They investigated the following forms.

Definition 1.5. Let $a, b, c \in \mathbb{R}^+$ ($a \neq b$) and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{2}{\lambda b^t + a^t} \right)^\alpha \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!} \tag{1.12}$$

$$\left(\left| t \log \left(\frac{b}{a} \right) \right| < |\log(-\lambda)|; 1^\alpha := 1; x \in \mathbb{R} \right).$$

Definition 1.6. Let $a, b, c \in \mathbb{R}^+$ ($a \neq b$) and $n \in \mathbb{N}_0$. Then the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^\alpha \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!} \quad (1.13)$$

$$\left(\left|t \log\left(\frac{b}{a}\right)\right| < |\log(-\lambda)|; 1^\alpha := 1; x \in \mathbb{R}\right).$$

It is easy to see that setting $a = 1$ and $b = c = e$ in (1.12) and (1.13) yields the classical results for the Apostol-Euler and Apostol-Genocchi polynomials.

Lately, Kurt [23] presented a new interesting class of generalized Euler polynomials. Explicitly, he introduced the next definition.

Definition 1.7. For arbitrary real or complex parameter α , the generalized Euler polynomials $E_n^{[m-1, \alpha]}(x)$, $m \in \mathbb{N}$, are defined, in a suitable neighborhood of $t = 0$ by means of the generating function:

$$\left(\frac{2^m}{e^t + \sum_{l=0}^{m-1} (t^l/l!)}\right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} E_n^{[m-1, \alpha]}(x) \frac{t^k}{k!}. \quad (1.14)$$

It is easy to see that if we set $m = 1$ in (1.14), we arrive at the following:

$$\left(\frac{2}{e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(x) \frac{t^k}{k!}, \quad (1.15)$$

which is the generating function for the generalized Euler polynomials of order α . Thus, we have

$$E_n^{[0, \alpha]}(x) = E_n^{(\alpha)}(x). \quad (1.16)$$

In this paper, we propose a further generalization of Apostol-Euler polynomials and the Apostol-Genocchi polynomials and we give some properties involving them. For the new class of Apostol-Euler polynomials, we establish a new addition theorem with the help of a result given by Srivastava et al. [24]. We also give an extension of the Srivastava-Pintér theorem [25, 26]. Finally, we exhibit some relationships between the generalized Apostol-Euler polynomials and other polynomials or special functions with the help of the new addition formula.

2. New Classes of Generalized Apostol-Euler and Apostol-Genocchi Polynomials

The following definitions provide a natural generalization of the Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x; \lambda)$, $m \in \mathbb{N}$, of order $\alpha \in \mathbb{C}$ and the Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1, \alpha]}(x; \lambda)$, $m \in \mathbb{N}$, of order $\alpha \in \mathbb{C}$.

Definition 2.1. For arbitrary real or complex parameter α and for $b, c \in \mathbb{R}^+$, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x, b, c; \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t = 0$, with $|t \log b| < |\log(-\lambda)|$ by means of the generating function:

$$\left(\frac{2^m}{\lambda b^t + \sum_{l=0}^{m-1} ((t \log b)^l / l!)} \right)^\alpha \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_n^{[m-1, \alpha]}(x, b, c; \lambda) \frac{t^k}{k!}. \quad (2.1)$$

It is easy to see that if we set $m = 1$, $b = c = e$ in (2.1), we arrive at the following:

$$\left(\frac{2}{\lambda e^t + 1} \right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_n^{[0, \alpha]}(x, e, e; \lambda) \frac{t^k}{k!}. \quad (2.2)$$

This is the generating function for the generalized Apostol-Euler polynomials of order α . Thus, we have

$$\mathfrak{E}_n^{[0, \alpha]}(x, e, e; \lambda) = \mathfrak{E}_n^{(\alpha)}(x; \lambda). \quad (2.3)$$

Definition 2.2. For arbitrary real or complex parameter α and for $b, c \in \mathbb{R}^+$, the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1, \alpha]}(x, b, c; \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t = 0$, with $|t \log b| < |\log(-\lambda)|$ by means of the generating function:

$$\left(\frac{2^m t^m}{\lambda b^t + \sum_{l=0}^{m-1} ((t \log b)^l / l!)} \right)^\alpha \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_n^{[m-1, \alpha]}(x, b, c; \lambda) \frac{t^k}{k!}. \quad (2.4)$$

Obviously, if we set $m = 1$, $b = c = e$ in (2.4), we obtain

$$\left(\frac{2t}{\lambda e^t + 1} \right)^\alpha \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_n^{[0, \alpha]}(x, e, e; \lambda) \frac{t^k}{k!}. \quad (2.5)$$

This is the generating function for the generalized Apostol-Genocchi polynomials of order α . Thus, we have

$$\mathfrak{G}_n^{[0, \alpha]}(x, e, e; \lambda) = \mathfrak{G}_n^{(\alpha)}(x; \lambda). \quad (2.6)$$

The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x, b, c; \lambda)$ defined by (2.1) possess the following interesting properties. These are stated as Theorems 2.3, 2.4, and 2.5 below.

Theorem 2.3. *The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1, l]}(x, b, c; \lambda)$ and the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1, l]}(x, b, c; \lambda)$, $l \in \mathbb{N}_0$, are related by*

$$\mathfrak{B}_n^{[m-1, l]}(x, b, c; \lambda) = \frac{(-1)^l n!}{2^{ml} (n - ml)!} \mathfrak{E}_{n-ml}^{[m-1, l]}(x, b, c; -\lambda) \quad (n, l, m \in \mathbb{N}_0, n \geq ml) \quad (2.7)$$

or, equivalently, by

$$\mathfrak{E}_n^{[m-1,l]}(x, b, c; \lambda) = \frac{(-2^m)^l n!}{(n - ml)!} \mathfrak{B}_{n+ml}^{[m-1,l]}(x, b, c; -\lambda) \quad (n, l, m \in \mathbb{N}_0). \quad (2.8)$$

Proof. Considering the generating function (2.1), the relations (2.7) and (2.8) follow easily. \square

Theorem 2.4. Let $b, c \in \mathbb{R}^+$, α an arbitrary complex number, and $m \in \mathbb{N}$. Then, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ satisfy the following relations:

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x + y, b, c; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x, b, c; \lambda) \mathfrak{E}_{n-k}^{[m-1,\beta]}(y, b, c; \lambda), \quad (2.9)$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x + y, b, c; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x, b, c; \lambda) (y \log c)^{n-k}. \quad (2.10)$$

Proof. Considering the generating function (2.1) and comparing the coefficients of $t^n/n!$ in the both sides of the above equation, we arrive at (2.9). Proof of (2.10) is similar. \square

Theorem 2.5. The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ satisfy the following recurrence relation:

$$\begin{aligned} & \lambda \mathfrak{E}_n^{[m-1,\alpha]}(x + 1, b, c; \lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x, b, c; \lambda) \\ &= 2 \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x, b, c; \lambda) \mathfrak{E}_{n-k}^{(-1)}(0; \lambda; 1, c, a), \end{aligned} \quad (2.11)$$

where $\mathfrak{E}_{n-1-k}^{(-1)}(0; \lambda; 1, c, a)$ are the generalized Apostol-Euler polynomials defined by (1.12).

Proof. Considering the expression $\lambda \mathfrak{E}_n^{[m-1,\alpha]}(x + 1, b, c; \lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ and using the generating functions (2.1) and (1.12), (2.11) follows easily. \square

Remark 2.6. Setting $m = 1$ and $b = c = e$ in (2.11) and with the help of (2.3), we find

$$\lambda \mathfrak{E}_n^{(\alpha)}(x + 1; \lambda) + \mathfrak{E}_n^{(\alpha)}(x; \lambda) = 2 \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(x; \lambda) \mathfrak{E}_{n-k}^{(-1)}(0; \lambda). \quad (2.12)$$

Using the well-known result (see [8])

$$\mathfrak{E}_n^{(\alpha+\beta)}(x + y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{(\alpha)}(x; \lambda) \mathfrak{E}_{n-k}^{(\beta)}(y; \lambda), \quad (2.13)$$

(2.12) becomes the familiar relation for the generalized Apostol-Euler polynomials (see [8]):

$$\lambda \mathfrak{E}_n^{(\alpha)}(x + 1; \lambda) + \mathfrak{E}_n^{(\alpha)}(x; \lambda) = 2 \mathfrak{E}_n^{(\alpha-1)}(x; \lambda). \quad (2.14)$$

Now, let us shift our focus on some interesting properties for the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ defined by (2.4). These are stated as Theorems 2.7, 2.8, and 2.9 below.

Theorem 2.7. *The generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda)$, the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x, b, c; \lambda)$, and the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ are related by*

$$\begin{aligned} \mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda) &= (-2^m)^\alpha \mathfrak{B}_n^{[m-1,\alpha]}(x, b, c; -\lambda) \quad (\alpha \in \mathbb{C}), \\ \mathfrak{G}_n^{[m-1,l]}(x, b, c; \lambda) &= \frac{n!}{(n-ml)!} \mathfrak{E}_{n-ml}^{[m-1,l]}(x, b, c; \lambda) \quad (n, l, m \in \mathbb{N}_0, n \geq ml). \end{aligned} \tag{2.15}$$

Proof. Considering the generating function (2.4), the relations (2.15) follow easily. □

Theorem 2.8. *Let $b, c \in \mathbb{R}^+$, α an arbitrary complex number, and $m \in \mathbb{N}$. Then, the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ satisfy the following relations:*

$$\mathfrak{G}_n^{[m-1,\alpha+\beta]}(x+y, b, c; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{G}_k^{[m-1,\alpha]}(x, b, c; \lambda) \mathfrak{G}_{n-k}^{[m-1,\beta]}(y, b, c; \lambda), \tag{2.16}$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x+y, b, c; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{G}_k^{[m-1,\alpha]}(x, b, c; \lambda) (y \log c)^{n-k}. \tag{2.17}$$

Proof. Considering the generating function (2.4) and comparing the coefficients of $t^n/n!$ in the both sides of the above equation, we arrive at (2.17). Proof of (2.18) is similar. □

Theorem 2.9. *The generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ satisfy the following recurrence relation:*

$$\begin{aligned} \lambda \mathfrak{G}_n^{[m-1,\alpha]}(x+1, b, c; \lambda) + \mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda) \\ = 2n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{G}_k^{[m-1,\alpha]}(x, b, c; \lambda) \mathfrak{G}_{n-1-k}^{(-1)}(0; \lambda; 1, c, a), \end{aligned} \tag{2.18}$$

where $\mathfrak{G}_{n-1-k}^{(-1)}(0; \lambda; 1, c, a)$ are the generalized Apostol-Genocchi polynomials defined by (1.13).

Proof. Considering the expression $\lambda \mathfrak{G}_n^{[m-1,\alpha]}(x+1, b, c; \lambda) + \mathfrak{G}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ and using the generating functions (2.4) and (1.13), (2.19) follows easily. □

Remark 2.10. Putting $m = 1$ and $b = c = e$ in (2.19) and with the help of (2.6), we find

$$\lambda \mathfrak{G}_n^{(\alpha)}(x+1; \lambda) + \mathfrak{G}_n^{(\alpha)}(x; \lambda) = 2n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{G}_k^{(\alpha)}(x; \lambda) \mathfrak{G}_{n-1-k}^{(-1)}(0; \lambda). \tag{2.19}$$

Using the well-known result (see [11])

$$\mathfrak{G}_n^{(\alpha+\beta)}(x+y; \lambda) = \sum_{k=0}^n \binom{n}{k} \mathfrak{G}_k^{(\alpha)}(x; \lambda) \mathfrak{G}_{n-k}^{(\beta)}(y; \lambda), \quad (2.20)$$

(2.20) becomes the familiar relation for the generalized Apostol-Genocchi polynomials (see [11]):

$$\lambda \mathfrak{G}_n^{(\alpha)}(x+1; \lambda) + \mathfrak{G}_n^{(\alpha)}(x; \lambda) = 2n \mathfrak{G}_{n-1}^{(\alpha-1)}(x; \lambda). \quad (2.21)$$

3. An Addition Theorem for the New Class of Generalized Apostol-Euler Polynomials

In this section, we establish a new addition theorem for the generalized Apostol-Euler polynomials. This new formula is based on a result due to Srivastava et al. [24].

The next theorem has been invented by Srivastava et al. [24]. However, the theorem is given without proof (see [24, pages 438–440]).

Theorem 3.1. *Let $B(z)$ and $z^{-1}C(z)$ be arbitrary functions which are analytic in the neighborhood of the origin, and assume (for sake of simplicity) that*

$$B(0) = C'(0) = 1. \quad (3.1)$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ by means of

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{z^n}{n!} = [B(z)]^\alpha \exp(xC(z)), \quad (3.2)$$

where α and x are arbitrary complex numbers independent of z . Then, for arbitrary parameters σ and γ ,

$$f_n^{(\alpha+\sigma\gamma)}(x+\gamma y) = \sum_{k=0}^n \frac{\gamma+n}{\gamma+k} \binom{n}{k} f_k^{(\alpha-\sigma k)}(x-ky) f_{n-k}^{(\sigma k+\sigma\gamma)}(ky+\gamma y), \quad (3.3)$$

provided that $\operatorname{Re}(\gamma) > 0$.

Remark 3.2. The choice of 1 in the conditions of (3.3) is merely a convenient one. In fact, any nonzero constant values may be assumed for $B(0)$ and $C'(0)$.

Now, applying the last theorem with special choices of functions and parameters furnishes the next very interesting addition formula. This formula is contained in the following corollary.

Corollary 3.3. Let $b, c \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then, for arbitrary complex parameters α, σ, x and y , the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1, \alpha]}(x, b, c; \lambda)$ satisfy the addition formula:

$$\begin{aligned} &\mathfrak{E}_n^{[m-1, \alpha+\sigma\gamma]}(x + \gamma y, b, c; \lambda) \\ &= \sum_{k=0}^n \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha-\sigma k]}(x - ky, b, c; \lambda) \mathfrak{E}_{n-k}^{[m-1, \sigma k + \sigma\gamma]}(ky + \gamma y, b, c; \lambda) \end{aligned} \tag{3.4}$$

provided that $\text{Re}(\gamma) > 0$.

Proof. Setting $B(z) = 2^m / (b^t + \sum_{l=0}^{m-1} ((t \log b)^l / l!))$ and $C(z) = t \log c$ in Theorem 3.1, the result follows. \square

Moreover, if we set $\sigma = 0$ in (3.4), we obtain

$$\begin{aligned} &\mathfrak{E}_n^{[m-1, \alpha]}(x + \gamma y, b, c; \lambda) \\ &= \sum_{k=0}^n \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(x - ky, b, c; \lambda) \mathfrak{E}_{n-k}^{[m-1, 0]}(ky + \gamma y, b, c; \lambda) \\ &= \sum_{k=0}^n \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(x - ky, b, c; \lambda) (\gamma + k)^{n-k} (y \log c)^{n-k}. \end{aligned} \tag{3.5}$$

This result (3.5) will be very useful in the next section.

4. Some Analogues of the Srivastava-Pintér Addition Theorem

In this section, we give a generalization of the Srivastava-Pintér addition theorem and an analogue. We end this section by giving two interesting relationships involving the new addition formula (3.5).

Theorem 4.1. The following relationship,

$$\begin{aligned} &\mathfrak{E}_n^{[m-1, \alpha]}(x + y, b, c; \lambda) \\ &= \sum_{k=0}^n \binom{n}{k} \left[2 \sum_{j=0}^k \binom{k}{j} \mathfrak{E}_j^{[m-1, \alpha]}(y, b, c; \lambda) \mathfrak{E}_{k-j}^{(-1)}(0; \lambda; 1, c, a) \right] \mathfrak{E}_{n-k}(x; \lambda) (\log c)^{n-k} \end{aligned} \tag{4.1}$$

$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0),$

holds between the new class of generalized Apostol-Euler polynomials, the classical Apostol-Euler polynomials defined by (1.4), and the generalized Apostol-Euler polynomials defined by (1.12).

Table 1: x^n expressed in terms of sums of special polynomials or numbers.

No.	Special polynomials or numbers	Series representation for x^n
(1)	Hermite polynomials [27, page 194, (4)]	$x^n = \frac{n!}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{H_{n-2k}(x)}{k!(n-2k)!}$
(2)	Legendre polynomials [27, page 181, Theorem 65]	$x^n = \frac{n!}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2n-4k+1)P_{n-2k}(x)}{k!(3/2)_{n-k}}$
(3)	Generalized Laguerre polynomials [27, page 207, (2)]	$x^n = n!(1+\alpha)_n \sum_{k=0}^n \frac{(-1)^k L_k^{(\alpha)}(x)}{(1+\alpha)_k (n-k)!}$
(4)	Gegenbauer polynomials [27, page 283, (36)]	$x^n = \frac{n!}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(\nu+n-2k)C_{n-2k}^\nu(x)}{k!(\nu)_{n+1-k}}$
(5)	Stirling numbers of the second kind [5, page 207, Theorem B]	$x^n = \sum_{k=0}^n \binom{x}{k} k! S(n, k)$
(6)	Bernoulli polynomials [7, page 26]	$x^n = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_k(x)$
(7)	Euler polynomials [7, page 30]	$x^n = \frac{1}{2} \left[E_n(x) + \sum_{k=0}^n \binom{n}{k} E_k(x) \right]$
(8)	Apostol-Bernoulli polynomials [8, page 634, (29)]	$x^n = \frac{1}{n+1} \left[\lambda \sum_{k=0}^{n+1} \binom{n+1}{k} \mathfrak{B}_k(x; \lambda) - \mathfrak{B}_{n+1}(x; \lambda) \right]$
(9)	Apostol-Euler polynomials [8, page 635, (32)]	$x^n = \frac{1}{2} \left[\lambda \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k(x; \lambda) + \mathfrak{E}_n(x; \lambda) \right]$
(10)	Generalized Apostol-Euler polynomials [28, page 1325, (2.4)]	$x^n = \frac{1}{2^\beta} \sum_{k=0}^\infty \binom{\beta}{k} \lambda^k \mathfrak{E}_n^{(\beta)}(x+k; \lambda), \beta \in \mathbb{C}$
(11)	Generalized Bernoulli polynomials and Stirling numbers [28, page 1329, (2.16)]	$x^n = \sum_{l=0}^n \binom{n}{l} \binom{l+j}{j}^{-1} S(l+j, j) B_{n-l}^{(j)}(x), j \in \mathbb{N}_0$
(12)	Generalized Apostol-Bernoulli polynomials and generalized Stirling numbers [28, page 1329, (2.15)]	$x^n = n! \sum_{l=j}^n \frac{j!}{(l+j)!(n-l)!} S(l+j, j; \lambda) \mathfrak{B}_{n-l}^{(j)}(x; \lambda), j \in \mathbb{N}_0$
(13)	Generalized Bernoulli polynomials [29, page 158, (2.6)]	$x^n = \sum_{k=0}^n \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x), m \in \mathbb{N}$

Proof. First of all, if we substitute the entry (9) for x^n from Table 1 into the right-hand side of (2.10), we get

$$\begin{aligned}
& \mathfrak{E}_n^{[m-1, \alpha]}(x+y, b, c; \lambda) \\
&= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) (\log c)^{n-k} \left[\mathfrak{E}_{n-k}(x; \lambda) + \lambda \sum_{j=0}^{n-k} \binom{n-k}{j} \mathfrak{E}_j(x; \lambda) \right] \\
&= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) (\log c)^{n-k} \mathfrak{E}_{n-k}(x; \lambda) \\
&\quad + \frac{\lambda}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) (\log c)^{n-k} \sum_{j=0}^{n-k} \binom{n-k}{j} \mathfrak{E}_j(x; \lambda),
\end{aligned} \tag{4.2}$$

which, upon inverting the order of summation and using the following elementary combinatorial identity:

$$\binom{m}{l} \binom{l}{n} = \binom{m}{n} \binom{m-n}{m-l} \quad (m \geq l \geq n; l, m, n \in \mathbb{N}_0), \tag{4.3}$$

yields

$$\begin{aligned} & \mathfrak{E}_n^{[m-1, \alpha]}(x+y, b, c; \lambda) \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) \mathfrak{E}_{n-k}(x; \lambda) (\log c)^{n-k} \\ &+ \frac{\lambda}{2} \sum_{j=0}^n \binom{n}{j} \mathfrak{E}_j(x; \lambda) (\log c)^j \sum_{k=0}^{n-j} \binom{n-j}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) (\log c)^{n-j-k}. \end{aligned} \tag{4.4}$$

The innermost sum in (4.4) can be calculated with the help of (2.10) with, of course,

$$x = 1 \quad n \mapsto n - j \quad (0 \leq j \leq n; n, j \in \mathbb{N}_0). \tag{4.5}$$

We thus find from (4.4) that

$$\begin{aligned} & \mathfrak{E}_n^{[m-1, \alpha]}(x+y, b, c; \lambda) \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) \mathfrak{E}_{n-k}(x; \lambda) (\log c)^{n-k} \\ &+ \frac{\lambda}{2} \sum_{j=0}^n \binom{n}{n-j} \mathfrak{E}_{n-j}^{[m-1, \alpha]}(y+1, b, c; \lambda) \mathfrak{E}_j(x; \lambda) (\log c)^j \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \left[\mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) + \lambda \mathfrak{E}_k^{[m-1, \alpha]}(y+1, b, c; \lambda) \right] \mathfrak{E}_{n-k}(x; \lambda) (\log c)^{n-k} \end{aligned} \tag{4.6}$$

which, with the relation (2.11), leads us to the relationship (4.7) asserted by Theorem 4.1. \square

Theorem 4.2. *The following relationship,*

$$\begin{aligned} \mathfrak{G}_n^{[m-1, \alpha]}(x+y, b, c; \lambda) &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_{n-k}(x; \lambda) (\log c)^{n-k} \\ &\cdot \left[2k \sum_{j=0}^{k-1} \binom{k-1}{j} \mathfrak{G}_j^{[m-1, \alpha]}(y, b, c; \lambda) \mathfrak{G}_{k-1-j}^{(-1)}(0; \lambda; 1, c, a) \right] \end{aligned} \tag{4.7}$$

$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0),$

holds between the new class of generalized Apostol-Genocchi polynomials, the classical Apostol-Euler polynomials defined by (1.4), and the generalized Apostol-Genocchi polynomials defined by (1.13).

Making use of Table 1 that contains a list of series representation for x^n in terms of special polynomials or numbers, we can find some analogues of the Srivastava-Pintér addition theorem. Let us give an example of such formula.

Theorem 4.3. *The following relationship,*

$$\begin{aligned} & \mathfrak{E}_n^{[m-1, \alpha]}(x+y, b, c; \lambda) \\ &= \sum_{k=0}^n \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(y, b, c; \lambda) (\log c)^{n-k} \frac{(n-k)!}{2^{(n-k)}} \sum_{j=0}^{\lfloor (n-k)/2 \rfloor} \frac{H_{n-k-2j}(x)}{j!(n-k-2j)!}, \end{aligned} \quad (4.8)$$

holds between the new class of generalized Apostol-Euler polynomials and the Hermite polynomials defined by

$$e^{(2xt-t^2)} = \sum_{n=0}^{\infty} H_n(x) t^n. \quad (4.9)$$

Proof. We derived the Proof from the addition theorem (2.10) and entry 1. \square

We end this paper by giving two special cases of the addition theorem (3.4) involving the new class of generalized Apostol-Euler polynomials. These are contained in the two next theorems.

Theorem 4.4. *The following relationship,*

$$\begin{aligned} & \mathfrak{E}_n^{[m-1, \alpha]}(x + \gamma y, b, c; \lambda) \\ &= \sum_{k=0}^n \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(x - ky, b, c; \lambda) (\gamma + k)^{n-k} (\log c)^{n-k} \sum_{j=0}^{n-k} \binom{y}{j} j! S(n-k, j), \end{aligned} \quad (4.10)$$

holds between the new class of generalized Apostol-Euler polynomials and the Stirling numbers of the second kind that could be computed by the formula [30, page 58, (1.5)]

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n. \quad (4.11)$$

Proof. We derived the Proof from the addition theorem (3.5) and entry 5. \square

Theorem 4.5. *The following relationship,*

$$\begin{aligned} & \mathfrak{E}_n^{[m-1, \alpha]}(x + \gamma y, b, c; \lambda) \\ &= \sum_{k=0}^n \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1, \alpha]}(x - ky, b, c; \lambda) (\gamma + k)^{n-k} (\log c)^{n-k} \\ & \cdot (n-k)! \sum_{l=j}^{n-k} \frac{j!}{(l+j)!(n-k-l)!} S(l+j, j; \lambda) \mathfrak{B}_{n-k-l}^{(j)}(y; \lambda) \quad (j \in \mathbb{N}_0), \end{aligned} \quad (4.12)$$

holds between the new class of generalized Apostol-Euler polynomials and the classical Apostol-Bernoulli polynomials and the generalized Stirling numbers.

Proof. We derived the Proof from the addition theorem (3.5) and entry 12. \square

It could be interesting to apply the addition formula (3.3) to other family of polynomials in conjunction with series representation involving some special functions for x^n in order to derive some analogues of the Srivastava-Pintér addition theorem.

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