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Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials

Qiu-Ming Luo [a], H.M. Srivastava [b,*]

[a] *Department of Mathematics, Jiaozuo University, Jiaozuo City, Henan 454003, People's Republic of China*
[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

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Abstract

The main object of this paper is to give analogous definitions of Apostol type (see [T.M. Apostol, On the Lerch Zeta function, Pacific J. Math. 1 (1951) 161–167] and [H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc. 129 (2000) 77–84]) for the so-called Apostol–Bernoulli numbers and polynomials of higher order. We establish their elementary properties, derive several explicit representations for them in terms of the Gaussian hypergeometric function and the Hurwitz (or generalized) Zeta function, and deduce their special cases and applications which are shown here to lead to the corresponding results for the classical Bernoulli numbers and polynomials of higher order.

Keywords: Bernoulli polynomials; Apostol–Bernoulli polynomials; Apostol–Bernoulli polynomials of higher order; Apostol–Euler polynomials; Apostol–Euler polynomials of higher order; Gaussian hypergeometric function; Stirling numbers of the second kind; Hurwitz (or generalized) Zeta function; Hurwitz–Lerch and Lipschitz–Lerch Zeta functions; Lerch's functional equation

* Corresponding author.
E-mail addresses: luomath@126.com (Q.-M. Luo), hmsri@uvvm.uvic.ca, harimsri@math.uvic.ca (H.M. Srivastava).

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1. Introduction, definitions and preliminaries

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [8] and [10, p. 61 et seq.]):

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{z^n}{n!}\quad \bigl(|z|<2\pi;\ 1^\alpha:=1\bigr) \tag{1}$$

and

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{z^n}{n!}\quad \bigl(|z|<\pi;\ 1^\alpha:=1\bigr), \tag{2}$$

so that, obviously,

$$B_n(x):=B_n^{(1)}(x)\quad \text{and}\quad E_n(x):=E_n^{(1)}(x)\quad (n\in\mathbb{N}_0), \tag{3}$$

where

$$\mathbb{N}_0:=\mathbb{N}\cup\{0\}\quad \bigl(\mathbb{N}:=\{1,2,3,\ldots\}\bigr).$$

For the classical Bernoulli numbers B_n and the classical Euler numbers E_n, we readily find from (3) that

$$B_n:=B_n(0)=B_n^{(1)}(0)\quad \text{and}\quad E_n:=E_n(0)=E_n^{(1)}(0)\quad (n\in\mathbb{N}_0). \tag{4}$$

Some interesting analogues of the classical Bernoulli polynomials and numbers were investigated by Apostol [2, Eq. (3.1), p. 165] and (more recently) by Srivastava [9, pp. 83–84]. We begin by recalling here Apostol's definitions as follows.

Definition 1 (Apostol [2]; see also Srivastava [9]). The Apostol–Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{z^n}{n!}\quad \bigl(|z+\log\lambda|<2\pi\bigr) \tag{5}$$

with, of course,

$$B_n(x)=\mathcal{B}_n(x;1)\quad \text{and}\quad \mathcal{B}_n(\lambda):=\mathcal{B}_n(0;\lambda), \tag{6}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol–Bernoulli numbers.

Apostol [2] not only gave elementary properties of the polynomials $\mathcal{B}_n(x;\lambda)$, but also obtained the following recursion formula of the numbers $\mathcal{B}_n(\lambda)$ (see [2, Eq. (3.7), p. 166]):

$$\mathcal{B}_n(\lambda)=n\sum_{k=0}^{n-1}\frac{k!(-\lambda)^k}{(\lambda-1)^{k+1}}S(n-1,k)\quad \bigl(n\in\mathbb{N}_0;\ \lambda\in\mathbb{C}\backslash\{1\}\bigr), \tag{7}$$

where $S(n,k)$ denotes the Stirling numbers of the second kind defined by means of the following expansion (see [4, Theorem B, p. 207]):

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k), \tag{8}$$

so that

$$S(n,0) = \delta_{n,0}, \qquad S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,n-1) = \binom{n}{2},$$

$\delta_{n,k}$ being the Kronecker symbol.

Motivated by the generalizations in (1) and (2) of the classical Bernoulli polynomials and the classical Euler polynomials involving a real or complex parameter α, we introduce and investigate here the so-called Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α and the Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α, which are defined as follows.

Definition 2. The Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} \quad \bigl(|z + \log\lambda| < 2\pi;\ 1^{\alpha} := 1\bigr) \tag{9}$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda), \tag{10}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol–Bernoulli numbers of order α.

Definition 3 (cf. Luo [7]). The Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} \quad \bigl(|z + \log\lambda| < \pi;\ 1^{\alpha} := 1\bigr) \tag{11}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{12}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol–Euler numbers of order α.

By using Definition 3 in conjunction with (2), it is easily observed that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda) \frac{z^n}{n!} &= e^{-x\log\lambda} \left(\frac{2}{e^{z+\log\lambda} + 1}\right)^{\alpha} e^{x(z+\log\lambda)} \\
&= e^{-x\log\lambda} \sum_{k=0}^{\infty} E_k^{(\alpha)}(x) \frac{(z+\log\lambda)^k}{k!} \\
&= e^{-x\log\lambda} \sum_{k=0}^{\infty} E_k^{(\alpha)}(x) \sum_{n=0}^{k} \frac{z^n (\log\lambda)^{k-n}}{(k-n)!\,n!}
\end{aligned}$$

$$= e^{-x \log \lambda} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{\infty} E_{n+k}^{(\alpha)}(x) \frac{(\log \lambda)^k}{k!},$$

which yields the following representation for the Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α in series of the familiar Euler polynomials $E_n^{(\alpha)}(x)$ of order α.

Lemma 1. *The Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α is represented by*

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = e^{-x \log \lambda} \sum_{k=0}^{\infty} E_{n+k}^{(\alpha)}(x) \frac{(\log \lambda)^k}{k!} \quad (n \in \mathbb{N}_0) \tag{13}$$

in series of the familiar Euler polynomials $E_n^{(\alpha)}(x)$ of order α.

In precisely the same manner, Definition 2 would yield the following result.

Lemma 2. *The Apostol–Bernoulli polynomials $\mathcal{B}_n^{(l)}(x;\lambda)$ of order l are represented by*

$$\mathcal{B}_n^{(l)}(x;\lambda) = e^{-x \log \lambda} \sum_{k=0}^{\infty} \binom{n+k-l}{k} \binom{n+k}{k}^{-1} B_{n+k}^{(l)}(x) \frac{(\log \lambda)^k}{k!} \quad (n, l \in \mathbb{N}_0) \tag{14}$$

in series of the familiar Bernoulli polynomials $B_n^{(l)}(x)$ of order l.

Recently, Luo [7] derived several interesting properties and explicit representations for the Apostol–Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α, including (for example) an explicit series representation for $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ involving the Gaussian hypergeometric function $F(a,b;c;z)$ defined by (cf., e.g., [1, p. 556 et seq.])

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$
$$\big(c \notin \mathbb{Z}_0^-;\ |z| < 1;\ z = 1 \text{ and } \Re(c-a-b) > 0;\ z = -1 \text{ and } \Re(c-a-b) > -1\big), \tag{15}$$

where

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \quad \big(\mathbb{Z}^- := \{-1,-2,-3,\ldots\}\big)$$

and $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad (n \in \mathbb{N}).$$

The main object of the present paper is to investigate the corresponding problems for the Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α. And, by closely following the work of Srivastava [9] dealing with the special case $\alpha = 1$, we also derive an explicit series representation for

$$\mathcal{B}_n^{(\alpha)}\left(\frac{p}{q}; e^{2\pi i \xi}\right) \quad (p \in \mathbb{Z};\ q \in \mathbb{N};\ \xi \in \mathbb{R})$$

involving the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ defined by (cf. [3, p. 249] and [10, p. 88])

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \big(\Re(s) > 1;\ a \notin \mathbb{Z}_0^-\big), \tag{16}$$

so that

$$\zeta(s,1) =: \zeta(s) = \frac{1}{2^s-1}\zeta\left(s, \frac{1}{2}\right) \tag{17}$$

for the Riemann Zeta function $\zeta(s)$.

2. Explicit formulas involving the Gaussian hypergeometric function

We begin by stating our main result in this section as Theorem 1 below.

Theorem 1. *For $n, l \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}\backslash\{1\}$, the following explicit series representation holds true*:

$$\mathcal{B}_n^{(l)}(x;\lambda) = l!\binom{n}{l}\sum_{k=0}^{n-l}\binom{n-l}{k}\binom{l+k-1}{k}\frac{\lambda^k}{(\lambda-1)^{k+l}} \cdot \sum_{j=0}^{k}(-1)^j\binom{k}{j} j^k (x+j)^{n-k-l} F\left(k+l-n, k; k+1; \frac{j}{x+j}\right), \tag{18}$$

where $F(a,b;c;z)$ denotes the Gaussian hypergeometric function defined by (15)*. Furthermore, for $n, l \in \mathbb{N}_0$,*

$$\mathcal{B}_n^{(l)}(x;\lambda) = e^{-x\log\lambda}\sum_{k=0}^{\infty}\binom{n+k-l}{k}\binom{n+k}{k}^{-1}\frac{(\log\lambda)^k}{k!} \cdot \sum_{r=0}^{n+k}\binom{n+k}{r}\binom{l+r-1}{r}\frac{r!}{(2r)!}\sum_{j=0}^{r}(-1)^j\binom{r}{j} \cdot j^{2r}(x+j)^{n+k-r} F\left(r-n-k, r-l; 2r+1; \frac{j}{x+j}\right) \tag{19}$$

in terms of the Gaussian hypergeometric function $F(a,b;c;z)$ defined by (15).

Proof. Making use of Taylor's expansion and Leibniz's rule, we find from (9) with $\alpha = l$ $(l \in \mathbb{N}_0)$ that

$$\mathcal{B}_n^{(l)}(x;\lambda) = D_z^n\left\{\left(\frac{z}{\lambda e^z - 1}\right)^l e^{xz}\right\}\bigg|_{z=0} \quad \left(D_z = \frac{d}{dz}\right)$$
$$= \frac{l!}{(\lambda-1)^l}\sum_{k=l}^{n}\binom{n}{k}\binom{k}{l} x^{n-k} D_z^{k-l}$$

$$\cdot \left\{ \left(1 + \frac{\lambda}{\lambda - 1} (e^z - 1) \right)^{-l} \right\} \bigg|_{z=0}. \tag{20}$$

Now, by setting $\alpha = l$ $(l \in \mathbb{N}_0)$ and

$$w = \frac{\lambda}{\lambda - 1} \big(e^z - 1 \big)$$

in the binomial expansion:

$$(1 + w)^{-\alpha} = \sum_{r=0}^{\infty} \binom{\alpha + r - 1}{r} (-w)^r \quad \big(|w| < 1 \big), \tag{21}$$

and using the following known definition (see [10, Eq. 1.5 (15), p. 58]):

$$\big(e^z - 1 \big)^l = l! \sum_{r=l}^{\infty} S(r, l) \frac{z^r}{r!}, \tag{22}$$

we find from (20) that

$$\mathcal{B}_n^{(l)}(x; \lambda) = l! \sum_{k=l}^{n} \binom{n}{k} \binom{k}{l} x^{n-k} \cdot \sum_{r=0}^{k-l} \binom{l + r - 1}{r} \frac{r!(-\lambda)^r}{(\lambda - 1)^{r+l}} S(k - l, r). \tag{23}$$

Upon interchanging the order of summation in (23), if we apply (see [10, Eq. 1.5 (20), p. 58])

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

and the elementary combinatorial identity:

$$\binom{n}{k} \binom{k}{r} = \binom{n}{r} \binom{n - r}{n - k},$$

we readily obtain

$$\mathcal{B}_n^{(l)}(x; \lambda) = l! \binom{n}{l} \sum_{k=0}^{n-l} \binom{l + k - 1}{k} \binom{n - l}{k} \frac{\lambda^k x^{n-k-l}}{(\lambda - 1)^{k+l}}$$
$$\cdot \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k F\left(k + l - n, 1; k + 1; -\frac{j}{x} \right). \tag{24}$$

Finally, we apply the known Pfaff–Kummer hypergeometric transformation [1, Eq. (15.3.4), p. 559]:

$$F(a, b; c; z) = (1 - z)^{-a} F\left(a, c - b; c; \frac{z}{z - 1} \right)$$
$$\big(c \notin \mathbb{Z}_0^-;\ \big| \arg(1 - z) \big| \leqq \pi - \varepsilon \ (0 < \varepsilon < \pi) \big) \tag{25}$$

in (24). We are thus led immediately to the assertion (18) of Theorem 1.

The assertion (19) of Theorem 1 can be proven similarly (or, alternatively, by applying Lemma 2 in conjunction with the special case $\alpha = l$ ($l \in \mathbb{N}_0$) of a known result given earlier by Srivastava and Todorov [11, Eq. (3), p. 510]; see also Eq. (26) below). □

Remark 1. By setting $\lambda = 1$ in (19), we obtain a *special* case $\alpha = l$ ($l \in \mathbb{N}_0$) of the aforementioned known result due to Srivastava and Todorov (see [11, Eq. (3), p. 510]):

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k-1}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k} \cdot F\left(k-n, k-\alpha; 2k+1; \frac{j}{x+j}\right). \tag{26}$$

Remark 2. For the Apostol–Bernoulli numbers $\mathcal{B}_n^{(\alpha)}(\lambda)$, by setting $x = 0$ in (18), we obtain the following explicit representation:

$$\mathcal{B}_n^{(l)}(\lambda) = l! \binom{n}{l} \sum_{k=0}^{n-l} \binom{l+k-1}{k} \frac{k!(-\lambda)^k}{(\lambda-1)^{k+l}} S(n-l, k), \quad \big(n, l \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\setminus\{1\}\big), \tag{27}$$

where we have made use of the Gauss summation theorem [1, Eq. (15.1.20), p. 556]:

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \big(c \notin \mathbb{Z}_0^-;\ \Re(c-a-b) > 0\big)$$

for

$$a = k+l-n, \qquad b = k, \quad \text{and} \quad c = k+1,$$

so that

$$F(k+l-n, k; k+1; 1) = \binom{n-l}{k}^{-1} \quad (k = 0, 1, \ldots, n-l;\ n, l \in \mathbb{N}_0). \tag{28}$$

Remark 3. Apostol's formula (7) is an obvious special case of our formula (27) when $l = 1$.

Remark 4. The following explicit formula for the Bernoulli numbers $B_n^{(\alpha)}$ of order α was given by Todorov [12, Eq. (3), p. 665]:

$$B_n^{(\alpha)} = \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \binom{n+k}{k}^{-1} S(n+k, k). \tag{29}$$

Obviously, as already observed by Srivastava and Todorov [11, p. 513], Todorov's formula (29) is contained in the relatively more general Srivastava–Todorov result (26) above (and hence also in the assertion (19) of Theorem 1, but only for the special case when $\alpha = l$ ($l \in \mathbb{N}_0$)).

Remark 5. The proof of Theorem 1 can be applied *mutatis mutandis* in order to obtain a new explicit formula for the Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ involving the Stirling numbers of the second kind as follows:

$$\mathcal{B}_n^{(l)}(x;\lambda) = l! \sum_{k=l}^{n} \binom{n}{k} \binom{k}{l} x^{n-k} \sum_{j=0}^{k-l} \binom{l+j-1}{j} \cdot \frac{j!(-\lambda)^j}{(\lambda-1)^{j+l}} S(k-l,j) \quad \bigl(n,l \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\backslash\{1\}\bigr). \tag{30}$$

Corollary. *The following explicit representation holds true for the Apostol–Bernoulli polynomials* $\mathcal{B}_n(x;\lambda)$:

$$\mathcal{B}_n(x;\lambda) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\lambda^k}{(\lambda-1)^{k+1}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k (x+j)^{n-k-1} \cdot F\left(k-n+1, k; k+1; \frac{j}{x+j}\right) \quad \bigl(n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\backslash\{1\}\bigr). \tag{31}$$

3. Explicit representations involving the Hurwitz (or generalized) Zeta function

A general Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ defined by (cf., e.g., [10, p. 121 et seq.])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad \bigl(a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\bigr) \tag{32}$$

contains, as its *special* cases, not only the Riemann and Hurwitz (or generalized) Zeta functions (cf. Eqs. (16) and (17)):

$$\zeta(s) = \Phi(1,s,1) \quad \text{and} \quad \zeta(s,a) = \Phi(1,s,a) \tag{33}$$

and the Lerch Zeta function:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi} \Phi\bigl(e^{2\pi i\xi}, s, 1\bigr) \quad \bigl(\xi \in \mathbb{R};\ \Re(s)>1\bigr), \tag{34}$$

but also such other functions as the polylogarithmic function:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\,\Phi(z,s,1) \quad \bigl(s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\bigr) \tag{35}$$

and the Lipschitz–Lerch Zeta function (cf. [10, Eq. 2.5 (11), p. 122]):

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\bigl(e^{2\pi i\xi},s,a\bigr) =: L(\xi,s,a)$$
$$\bigl(a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \Re(s) > 0 \text{ when } \xi \in \mathbb{R}\backslash\mathbb{Z};\ \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}\bigr), \tag{36}$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

For the general Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ defined by (32), it is easily seen by using the elementary series identity:

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q}\sum_{k=0}^{\infty} f(qk+j) \quad (q \in \mathbb{N}) \tag{37}$$

that

$$\Phi(z,s,a) = q^{-s}\sum_{j=1}^{q} \Phi\left(z^q,s,\frac{a+j-1}{q}\right)z^{j-1}, \tag{38}$$

which, in the special case when

$$z = \exp\left(\frac{2p\pi i}{q}\right) \quad (p \in \mathbb{Z};\ q \in \mathbb{N}),$$

yields the following summation formula for the Lipschitz–Lerch Zeta function $\phi(\xi,a,s)$ defined by (36):

$$\phi\left(\frac{p}{q},a,s\right) = q^{-s}\sum_{j=1}^{q} \zeta\left(s,\frac{a+j-1}{q}\right)\exp\left(\frac{2(j-1)p\pi i}{q}\right) \tag{39}$$

in terms of the Hurwitz (or generalized) Zeta function $\zeta(s,a)$.

For $z = 1$, (38) reduces at once to the following familiar identity:

$$\zeta(s,a) = q^{-s}\sum_{j=1}^{q} \zeta\left(s,\frac{a+j-1}{q}\right), \tag{40}$$

which, for $a = 1$, yields a well-known result for the Riemann Zeta function $\zeta(s)$. On the other hand, by setting $a = \frac{1}{2}$ in (38) and (39), we have

$$\sum_{n=1}^{\infty} \frac{z^n}{(2n-1)^s} = (2q)^{-s}\sum_{j=1}^{q} \Phi\left(z^q,s,\frac{2j-1}{2q}\right)z^{j-1} \tag{41}$$

and

$$\sum_{n=0}^{\infty} \frac{e^{(2n+1)p\pi i/q}}{(2n+1)^s} = (2q)^{-s}\sum_{j=1}^{q} \zeta\left(s,\frac{2j-1}{2q}\right)\exp\left(\frac{(2j-1)p\pi i}{q}\right), \tag{42}$$

respectively. Lastly, in their special cases when $a = 1$, (38) and (39) yield the following companions of the summation formulas (41) and (42), respectively:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^s} =: \mathrm{Li}_s(z) = q^{-s}\sum_{j=1}^{q} \Phi\left(z^q,s,\frac{j}{q}\right)z^j \tag{43}$$

and

$$\sum_{n=1}^{\infty} \frac{e^{2np\pi i/q}}{n^s} =: \ell_s\left(\frac{p}{q}\right) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{j}{q}\right) \exp\left(\frac{2jp\pi i}{q}\right). \tag{44}$$

For the Lipschitz–Lerch Zeta function $\phi(\xi, a, s)$ defined by (36), we *now* recall Lerch's functional equation:

$$\begin{aligned}\phi(\xi, a, 1-s) = \frac{\Gamma(s)}{(2\pi)^s} \Big\{ &\exp\left[\left(\frac{1}{2}s - 2a\xi\right)\pi i\right] \phi(-a, \xi, s) \\ &+ \exp\left[\left(-\frac{1}{2}s + 2a(1-\xi)\right)\pi i\right] \phi(a, 1-\xi, s) \Big\}\end{aligned}$$
$$(s \in \mathbb{C};\ 0 < \xi < 1), \tag{45}$$

which was applied recently by Srivastava [9] in conjunction with Apostol's formula [2, p. 164]:

$$\phi(\xi, a, 1-n) = -\frac{\mathcal{B}_n(a; e^{2\pi i\xi})}{n} \quad (n \in \mathbb{N}) \tag{46}$$

with a view to deriving the following explicit representation for the Apostol–Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ defined by (5) (cf. [9, Eq. (4.6), p. 84]; see also [10, Eq. 6.1 (27), p. 341]):

$$\begin{aligned}\mathcal{B}_n\left(\frac{p}{q}; e^{2\pi i\xi}\right) = -\frac{n!}{(2q\pi)^n} \Bigg\{ &\sum_{j=1}^{q} \zeta\left(n, \frac{\xi + j - 1}{q}\right) \exp\left[\left(\frac{n}{2} - \frac{2(\xi + j - 1)p}{q}\right)\pi i\right] \\ &+ \sum_{j=1}^{q} \zeta\left(n, \frac{j - \xi}{q}\right) \exp\left[\left(-\frac{n}{2} + \frac{2(j - \xi)p}{q}\right)\pi i\right] \Bigg\},\end{aligned}$$
$$(n \in \mathbb{N}\backslash\{1\};\ p \in \mathbb{Z};\ q \in \mathbb{N};\ \xi \in \mathbb{R}), \tag{47}$$

which holds true whenever each side exists. Indeed, in its *special* case when $\xi \in \mathbb{Z}$, the summation formula (47) can easily be shown to reduce to a known result given earlier by Cvijović and Klinowski [5, Theorem A, p. 1529].

Remark 6. Srivastava's formula (47) as well as Srivastava's *detailed* derivation of his series representation (47) were subsequently reproduced *verbatim* by Luo [6, p. 513 et seq.] without rightfully attributing the series representation (47) to Srivastava [9] (see also [10, Eq. 6.1 (27), p. 341]). Moreover, *both* [9] *and* [10] were actually included in the list of citations at the end of Luo's paper [6, p. 515].

With a view to applying Srivastava's formula (47) for our present objective, we turn once again to Definition 2, which yields

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!} &= \left(\frac{z}{\lambda e^z - 1}\right)^{\alpha - 1} \cdot \left(\frac{z}{\lambda e^z - 1}\right) e^{xz} \\ &= \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha-1)}(\lambda) \frac{z^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{z^n}{n!}\right).\end{aligned} \tag{48}$$

Upon recognizing the last member in (48) as the Cauchy product of two series, we immediately arrive at Lemma 3 below.

Lemma 3. *The following relationship*:

$$\mathcal{B}_n^{(\alpha)}(x;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}^{(\alpha-1)}(\lambda)\mathcal{B}_k(x;\lambda)\quad (n\in\mathbb{N}_0) \tag{49}$$

holds true between the Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ *of order* α *and the Apostol–Bernoulli numbers* $\mathcal{B}_n^{(\alpha-1)}(\lambda)$ *of order* $\alpha-1$.

Theorem 2. *For the Apostol–Bernoulli polynomials* $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ *of order* α,

$$\begin{aligned}\mathcal{B}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i\xi}\right)&=n\left(e^{2\pi i\xi}-1\right)^{-1}\mathcal{B}_{n-1}^{(\alpha-1)}\left(e^{2\pi i\xi}\right)-\sum_{k=2}^{n}\frac{k!}{(2q\pi)^k}\binom{n}{k}\mathcal{B}_{n-k}^{(\alpha-1)}\left(e^{2\pi i\xi}\right)\\&\quad\cdot\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{k}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.\\&\quad\left.+\sum_{j=1}^{q}\zeta\left(k,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{k}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}\\&\quad\big(n\in\mathbb{N}\backslash\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\backslash\mathbb{Z}\big).\end{aligned}\tag{50}$$

Proof. The proof of Theorem 2 is fairly straightforward. Indeed, by making use of Srivastava's formula (47) in Lemma 3 above, the assertion (50) of Theorem 2 follows immediately upon noting that [10, Eq. 2.5 (46), p. 126]

$$\mathcal{B}_0(x;\lambda)=0\quad\text{and}\quad\mathcal{B}_1(x;\lambda)=\frac{1}{\lambda-1}\quad(\lambda\neq 1). \tag{51}$$

Since

$$\mathcal{B}_n^{(0)}(\lambda):=\mathcal{B}_n^{(0)}(0;\lambda)=\delta_{n,0}\quad(n\in\mathbb{N}_0), \tag{52}$$

Srivastava's formula (47) can be recovered at once from Theorem 2 by setting $\alpha=1$ in (50). More importantly, in the *exceptional* case of the representation formula (50) when $\xi\in\mathbb{Z}$, we can apply the assertion (49) of Lemma 3 (*with* $\lambda=1$) in conjunction with Srivastava's formula [9, Eq. (2.3), p. 79]:

$$\begin{aligned}B_n\left(\frac{p}{q}\right)&=-\frac{2\cdot n!}{(2q\pi)^n}\sum_{j=1}^{q}\zeta\left(n,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right)\\&\quad\big(n\in\mathbb{N}\backslash\{1\};\ p\in\mathbb{N}_0;\ q\in\mathbb{N};\ 0\leqq p\leqq q\big)\end{aligned}\tag{53}$$

with a view to deriving the following *complement* of (50) for $\xi\in\mathbb{Z}$:

$$B_n^{(\alpha)}\left(\frac{p}{q}\right) = B_n^{(\alpha-1)} + n\left(\frac{p}{q} - \frac{1}{2}\right)B_{n-1}^{(\alpha-1)} - \sum_{k=2}^{n} \frac{2 \cdot k!}{(2q\pi)^k} \binom{n}{k} B_{n-k}^{(\alpha-1)} \sum_{j=1}^{q} \zeta\left(k, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q} - \frac{k\pi}{2}\right)$$
$$\big(n \in \mathbb{N}\backslash\{1\};\ p \in \mathbb{N}_0;\ q \in \mathbb{N};\ 0 \leqq p \leqq q\big). \qquad \square \tag{54}$$

4. Miscellaneous results

The following *further* properties of the Apostol–Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α are readily derived from Definition 2. We, therefore, choose to omit the details involved.

Theorem 3. *Let $n \in \mathbb{N}_0$. Suppose also that α and λ are suitable (real or complex) parameters. Then*

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k^{(\alpha)}(\lambda) x^{n-k} \quad \textit{and} \quad \mathcal{B}_n^{(0)}(x;\lambda) = x^n, \tag{55}$$

$$\lambda \mathcal{B}_n^{(\alpha)}(x+1;\lambda) - \mathcal{B}_n^{(\alpha)}(x;\lambda) = n\mathcal{B}_{n-1}^{(\alpha-1)}(x;\lambda), \tag{56}$$

$$\frac{\partial}{\partial x}\mathcal{B}_n^{(\alpha)}(x;\lambda) = n\mathcal{B}_{n-1}^{(\alpha)}(x;\lambda), \tag{57}$$

$$\int_a^b \mathcal{B}_n^{(\alpha)}(x;\lambda)\,dx = \frac{\mathcal{B}_{n+1}^{(\alpha)}(b;\lambda) - \mathcal{B}_{n+1}^{(\alpha)}(a;\lambda)}{n+1}, \tag{58}$$

$$\mathcal{B}_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k^{(\alpha)}(x;\lambda)\mathcal{B}_{n-k}^{(\beta)}(y;\lambda), \tag{59}$$

$$\mathcal{B}_n^{(\alpha)}(\alpha - x;\lambda) = \frac{(-1)^n}{\lambda^\alpha}\mathcal{B}_n^{(\alpha)}\big(x;\lambda^{-1}\big), \tag{60}$$

$$\mathcal{B}_n^{(\alpha)}(\alpha + x;\lambda) = \frac{(-1)^n}{\lambda^\alpha}\mathcal{B}_n^{(\alpha)}\big(-x;\lambda^{-1}\big), \tag{61}$$

$$nx\mathcal{B}_{n-1}^{(\alpha)}(x;\lambda) = (n-\alpha)\mathcal{B}_n^{(\alpha)}(x;\lambda) + \alpha\lambda\mathcal{B}_n^{(\alpha+1)}(x+1;\lambda), \quad \textit{and} \tag{62}$$

$$\mathcal{B}_n^{(\alpha+1)}(x;\lambda) = \left(1 - \frac{n}{\alpha}\right)\mathcal{B}_n^{(\alpha)}(x;\lambda) + n\left(\frac{x}{\alpha} - 1\right)\mathcal{B}_{n-1}^{(\alpha)}(x;\lambda). \tag{63}$$

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