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q-Extensions of Some Results Involving the Luo-Srivastava Generalizations of the Apostol-Bernoulli and Apostol-Euler Polynomials

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Abstract. Carlitz firstly defined the q-Bernoulli and q-Euler polynomials [*Duke Math. J.*, **15** (1948), 987–1000]. Recently, M. Cenkci and M. Can [*Adv. Stud. Contemp. Math.*, **12** (2006), 213–223], J. Choi, P. J. Anderson and H. M. Srivastava [*Appl. Math. Comput.*, **199** (2008), 723–737] further defined the q-Apostol-Bernoulli and q-Apostol-Euler polynomials. In this paper, we show the generating functions and basic properties of the q-Apostol-Bernoulli and q-Apostol-Euler polynomials, and obtain some relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials which are the corresponding q-extensions of some known results. Some formulas in series of q-Stirling numbers of the second kind are also considered.

1. Introduction, definitions and motivation

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{C}$ denotes the set of complex numbers.

The falling and rising factorial are defined by

$$\begin{aligned} &\{n\}_0 = 1, \quad \{n\}_k = n(n-1)\cdots(n-k+1), \\ &(n)_0 = 1, \quad (n)_k = n(n+1)\cdots(n+k-1) \quad (n, k \in \mathbb{N}), \end{aligned}$$

respectively.

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The q-shifted factorial are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k) \quad (n \in \mathbb{N}),$$

$$(a;q)_\infty = (1-a)(1-aq)\cdots(1-aq^n)\cdots = \prod_{k=0}^{\infty}(1-aq^k) \quad (|q|<1;\ a,q \in \mathbb{C}).$$

Clearly,

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

The q-numbers, q-numbers factorial and q-numbers shifted factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \qquad [0]_q! = 1, \quad [n]_q! = [1]_q[2]_q\cdots[n]_q \quad (n \in \mathbb{N});$$

$$([a]_q)_n = [a]_q[a+1]_q\cdots[a+n-1]_q \quad (n \in \mathbb{N},\ a \in \mathbb{C})$$

respectively. Clearly,

$$\lim_{q\to 1}[a]_q = a, \quad \lim_{q\to 1}[n]_q! = n!, \quad \lim_{q\to 1}([a]_q)_n = (a)_n.$$

The q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (z,q \in \mathbb{C};\ |z|<1, \left|q\right|<1). \tag{1.1}$$

When $a = q^\alpha$ ($\alpha \in \mathbb{C}$), then the formula (1.1) becomes the following form:

$$\frac{1}{(z;q)_\alpha} = \frac{(q^\alpha z;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\frac{(q^\alpha;q)_n}{(q;q)_n}z^n := \sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}z^n \tag{1.2}$$

$$(z,q,\alpha \in \mathbb{C};\ |z|<1, \left|q\right|<1).$$

The above q-standard notations can be found in [1] and [14].

Some interesting extensions of the classical Bernoulli polynomials and numbers were first investigated by Apostol [2, p. 165, Eq. (3.1)] and (more recently) by Srivastava [35, p. 83-84]. We begin by recalling here the Apostol's definitions as follows:

Definition 1.1 (Apostol [2]; see also Srivastava [35]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ in x are defined by means of the generating function*

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \tag{1.3}$$

$$(|t|<2\pi \text{ when } \lambda = 1;\ |t| < \left|\log\lambda\right| \text{ when } \lambda \neq 1)$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \qquad \textit{and} \qquad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \tag{1.4}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers (in fact, it is a function in λ).

Recently, Luo and Srivastava further extended the Apostol-Bernoulli polynomials and Apostol-Euler polynomials as follows (for convenience , we also say the *Apostol-type polynomials*):

Definition 1.2 (*cf.* Luo and Srivastava [26]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of higher order in x are defined by means of the generating function:*

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \tag{1.5}$$

$$(|t|<2\pi \text{ when } \lambda=1;\ |t|<\left|\log\lambda\right| \text{ when } \lambda\neq 1),$$

with, of course,

$$\begin{aligned} &B_n^{(\alpha)}(x):=\mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda):=\mathcal{B}_n^{(\alpha)}(0;\lambda),\\ &\mathcal{B}_n(x;\lambda):=\mathcal{B}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{B}_n(\lambda):=\mathcal{B}_n^{(1)}(\lambda), \end{aligned} \tag{1.6}$$

where $\mathcal{B}_n(\lambda)$, $\mathcal{B}_n^{(\alpha)}(\lambda)$, $\mathcal{B}_n(x;\lambda)$ and $B_n^{(\alpha)}(x)$ denote the so-called Apostol-Bernoulli numbers, Apostol-Bernoulli numbers of higher order (in fact, they are the functions in λ), Apostol-Bernoulli polynomials and Bernoulli polynomials of higher order respectively.

Remark 1.3. *When $\lambda\neq 1$ in* (1.5)*, the order α should tacitly be restricted to nonnegative integer values.*

Definition 1.4 (*cf.* Luo [16]). *The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of higher order in x are defined by means of the generating function:*

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \qquad \left(|t|<\left|\log(-\lambda)\right|\right), \tag{1.7}$$

with, of course,

$$\begin{aligned} &E_n^{(\alpha)}(x):=\mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda):=2^n\mathcal{E}_n^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right),\\ &\mathcal{E}_n(x;\lambda):=\mathcal{E}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{E}_n(\lambda):=2^n\mathcal{E}_n\left(\frac{1}{2};\lambda\right), \end{aligned} \tag{1.8}$$

where $\mathcal{E}_n(\lambda)$, $\mathcal{E}_n^{(\alpha)}(\lambda)$, $\mathcal{E}_n(x;\lambda)$ and $E_n^{(\alpha)}(x)$ $(n\in\mathbb{N}_0)$ denote the so-called Apostol-Euler numbers, Apostol-Euler numbers of higher order (in fact, they are the functions in λ), Apostol-Euler polynomials and Euler polynomials of higher order respectively.

1948, Carlitz firstly extended the classical Bernoulli and Euler numbers and polynomials (of higher order) as the q-Bernoulli and q-Euler numbers and polynomials (of higher order)(see, [5–7]).

Recently, Cenkci and Can [9] further defined the q-extensions of Apostol-Bernoulli numbers and polynomials. Subsequently, J. Choi, P. J. Anderson and H. M. Srivastava [11] gave the following q-extensions of Apostol-Bernoulli and Apostol-Euler polynomials of higher order:

Definition 1.5. *For $q,\ \alpha,\ \lambda\in\mathbb{C}$; $|q|<1$, the q-Apostol-Bernoulli numbers and polynomials of higher order in q^x are respectively defined by means of the generating function*

$$U_{\lambda;q}^{(\alpha)}(t)=(-t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}\lambda^n q^n e^{[n]_q t}=\sum_{n=0}^{\infty}\mathcal{B}_{n;q}^{(\alpha)}(\lambda)\frac{t^n}{n!}, \tag{1.9}$$

$$U_{x;\lambda;q}^{(\alpha)}(t)=(-t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}\lambda^n q^{n+x} e^{[n+x]_q t}=\sum_{n=0}^{\infty}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)\frac{t^n}{n!}. \tag{1.10}$$

Obviously, we have

$$\mathcal{B}_{n;q}^{(\alpha)}(\lambda)=\mathcal{B}_{n;q}^{(\alpha)}(0;\lambda),\quad B_{n;q}^{(\alpha)}=B_{n;q}^{(\alpha)}(0),$$

$$\lim_{q\to 1}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)=\mathcal{B}_n^{(\alpha)}(x;\lambda),\qquad \lim_{q\to 1}B_{n;q}^{(\alpha)}(x)=B_n^{(\alpha)}(x),\qquad \lim_{q\to 1}B_{n;q}^{(\alpha)}=B_n^{(\alpha)},$$

where $B_{n;q}^{(\alpha)} := \mathcal{B}_{n;q}^{(\alpha)}(1)$ *and* $B_{n;q}^{(\alpha)}(x) := \mathcal{B}_{n;q}^{(\alpha)}(x;1)$ *denote the q-Bernoulli numbers and polynomials of higher order respectively;* $B_{n;q} := B_{n;q}^{(1)}$ *and* $B_{n;q}(x) := B_{n;q}^{(1)}(x)$ *denote the q-Bernoulli numbers and polynomials respectively.*

Definition 1.6. *For* $q,\ \alpha,\ \lambda \in \mathbb{C};\ |q| < 1$, *the q-Apostol-Euler numbers and polynomials of higher order in* q^x *are respectively defined by means of the generating function*

$$V_{\lambda;q}^{(\alpha)}(t) = 2^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+\frac{\alpha}{2}} e^{2[n+\frac{\alpha}{2}]_q t} = \sum_{n=0}^{\infty} \mathcal{E}_{n;q}^{(\alpha)}(\lambda)\frac{t^n}{n!}, \tag{1.11}$$

$$V_{x;\lambda;q}^{(\alpha)}(t) = 2^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-\lambda)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)\frac{t^n}{n!}. \tag{1.12}$$

Obviously,

$$\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = 2^n \mathcal{E}_{n;q}^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right), \quad E_{n;q}^{(\alpha)} = 2^n E_{n;q}^{(\alpha)}\left(\frac{\alpha}{2}\right),$$
$$\lim_{q\to 1} \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \mathcal{E}_n^{(\alpha)}(x;\lambda), \quad \lim_{q\to 1} E_{n;q}^{(\alpha)}(x) = E_n^{(\alpha)}(x), \quad \lim_{q\to 1} E_{n;q}^{(\alpha)} = E_n^{(\alpha)},$$

where $E_{n;q}^{(\alpha)} := \mathcal{E}_{n;q}^{(\alpha)}(1)$ *and* $E_{n;q}^{(\alpha)}(x) := \mathcal{E}_{n;q}^{(\alpha)}(x;1)$ *denote q-Euler numbers and polynomials of higher order respectively;* $E_{n;q} := \mathcal{E}_{n;q}^{(1)}(1)$ *and* $E_{n;q}(x) := \mathcal{E}_{n;q}^{(1)}(x;1)$ *denote q-Euler numbers and polynomials respectively.*

On the subject of the Apostol type polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature (see [3, 9–12, 15–30, 32–43]; see also the references cited in each of these works).

Remark 1.7. *Throughout this paper, we take the principal value of the logarithm* $\log\lambda$, *i.e.,* $\log\lambda = \log|\lambda| + i\arg\lambda\ (-\pi < \arg\lambda \le \pi)$ *when* $\lambda \ne 1$; *We choose* $\log 1 = 0$ *when* $\lambda = 1$.

The aim of this paper is to derive the generating functions and basic properties of q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order, and to research some relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order. We show some q-extensions of some results of Luo and Srivastava [27] (below Theorem A), Srivastava and Pintér [43] (below Theorem B), Cheon [8] (below (4.5)). Furthermore, other formulas involving the q-Stirling numbers of the second kind are also given.

The paper is organized as follows: In the first section we introduce some notation and rewrite some definitions. In the second section we derive the generating functions of the q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order. In the third section we display the basic properties of q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order. In the fourth section we investigate some relationships between these q-polynomials based on some fairly standard techniques for series rearrangement. In the fifth section we give the formulas in terms of the Carlitz's q-Stirling numbers of the second kind. We provide some new and interesting formulas for the Apostol-Bernoulli and Apostol-Euler polynomials in the sixth section.

2. The generating functions of q-Apostol-Bernoulli and q-Apostol-Euler polynomials

In the present section, by Definition 1.5 and Definition 1.6 we can derive the generating functions and the closed formulas for q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order in order to prove some basic properties in Section 3.

By (1.10) and noting that q-binomial theorem (1.2) yields that

$$\begin{aligned}
U^{(\alpha)}_{x;\lambda;q}(t) &= (-t)^{\alpha}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}\lambda^n q^{n+x}e^{[n+x]_q t}\\
&= (-t)^{\alpha}e^{\frac{t}{1-q}}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}\lambda^n q^{n+x}e^{-\frac{q^{n+x}}{1-q}t}\\
&= (-t)^{\alpha}e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{(1-q)^k}\frac{t^k}{k!}\sum_{n=0}^{\infty}\frac{([\alpha]_q)_n}{[n]_q!}(\lambda q^{k+1})^n\\
&= (-t)^{\alpha}e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{(\lambda q^{k+1};q)_\alpha}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!}.
\end{aligned} \tag{2.1}$$

Therefore, we obtain the generating function of $\mathcal{B}^{(\alpha)}_{n;q}(x;\lambda)$ as follows:

$$U^{(\alpha)}_{x;\lambda;q}(t) = (-t)^{\alpha}e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{(\lambda q^{k+1};q)_\alpha}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}\mathcal{B}^{(\alpha)}_{n;q}(x;\lambda)\frac{t^n}{n!}. \tag{2.2}$$

Clearly, by setting $x = 0$ in (2.2) we have the generating function of $\mathcal{B}^{(\alpha)}_{n;q}(\lambda)$:

$$U^{(\alpha)}_{\lambda;q}(t) = (-t)^{\alpha}e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(\lambda q^{k+1};q)_\alpha}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}\mathcal{B}^{(\alpha)}_{n;q}(\lambda)\frac{t^n}{n!}. \tag{2.3}$$

When $\lambda = 1$ then (2.2) and (2.3) become the generating functions of $B^{(\alpha)}_{n;q}(x)$ and $B^{(\alpha)}_{n;q}$ respectively.

Setting $\alpha = \ell \in \mathbb{N}_0$ in (2.2) and (2.3), via simple calculation, we can get the following closed formulas:

$$\mathcal{B}^{(\ell)}_{n;q}(x;\lambda) = \frac{(-1)^{\ell}\{n\}_\ell}{(1-q)^{n-\ell}}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k}\frac{(-1)^k q^{(k+1)x}}{(\lambda q^{k+1};q)_\ell} \tag{2.4}$$

and

$$\mathcal{B}^{(\ell)}_{n;q}(\lambda) = \frac{(-1)^{\ell}\{n\}_\ell}{(1-q)^{n-\ell}}\sum_{k=0}^{n-\ell}\binom{n-\ell}{k}\frac{(-1)^k}{(\lambda q^{k+1};q)_\ell}. \tag{2.5}$$

Remark 2.1. *The special cases of* (2.4) *and* (2.5) *by putting* $\lambda = 1$, $\ell = 1$ *are just the Carlitz's results (4.7) and (4.11) of [5, p. 992] respectively.*

Similarly, we obtain the following generating functions of q-Apostol-Euler polynomials by using (1.2) and (1.12).

$$V^{(\alpha)}_{x;\lambda;q}(t) = 2^{\alpha}e^{\frac{t}{1-q}}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1};q)_\alpha}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}\mathcal{E}^{(\alpha)}_{n;q}(x;\lambda)\frac{t^n}{n!}. \tag{2.6}$$

Clearly, by setting $x = \frac{\alpha}{2}$ and $t \mapsto 2t$ in (2.6) and noting that $\mathcal{E}^{(\alpha)}_{n;q}(\lambda) = 2^n\mathcal{E}^{(\alpha)}_{n;q}(\frac{\alpha}{2};\lambda)$, we obtain the generating function of q-Apostol-Euler numbers $\mathcal{E}^{(\alpha)}_{n;q}(\lambda)$ as follows:

$$V^{(\alpha)}_{\lambda;q}(t) = 2^{\alpha}e^{\frac{2t}{1-q}}\sum_{k=0}^{\infty}\frac{(-2)^k q^{\frac{(k+1)\alpha}{2}}}{(-\lambda q^{k+1};q)_\alpha}\left(\frac{1}{1-q}\right)^k\frac{t^k}{k!} = \sum_{n=0}^{\infty}\mathcal{E}^{(\alpha)}_{n;q}(\lambda)\frac{t^n}{n!}. \tag{2.7}$$

Putting $\lambda = 1$ in (2.6) and (2.7), we can deduce the generating functions of $E_{n;q}^{(\alpha)}(x)$ and $E_{n;q}^{(\alpha)}$ respectively.

By (2.6) and (2.7), we readily derive the following closed formulas:

$$\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \frac{2^{\alpha}}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{(k+1)x}}{(-\lambda q^{k+1};q)_{\alpha}} \tag{2.8}$$

and

$$\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = \frac{2^{n+\alpha}}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{\frac{(k+1)\alpha}{2}}}{(-\lambda q^{k+1};q)_{\alpha}}. \tag{2.9}$$

Remark 2.2. *The special cases of* (2.8) *and* (2.9) *by setting* $\lambda = 1,\ \alpha = 1$ *are some analogues of Carlitz's numbers* ϵ_m *and polynomials* $\epsilon_m(x)$ *in [5, Eqs. (8.14) and (8.17)] respestively.*

3. The basic properties of q-Apostol-Bernoulli and q-Apostol-Euler polynomials

The following elementary properties of the q-Apostol-Bernoulli and q-Apostol-Euler polynomials are readily derived from Definition 1.5 and Definition 1.6. We, therefore, choose to omit the details involved.

Proposition 3.1. *For* $n,\ell \in \mathbb{N};\ \alpha,\lambda \in \mathbb{C}$,

$$\begin{aligned} &\mathcal{B}_{n;q}^{(\alpha)}(\lambda) = \mathcal{B}_{n;q}^{(\alpha)}(0;\lambda), \qquad \mathcal{B}_{n;q}^{(0)}(x;\lambda) = q^x[x]_q^n,\\ &\mathcal{B}_{0;q}^{(\alpha)}(x;\lambda) = \mathcal{B}_{0;q}^{(\alpha)}(\lambda) = \delta_{\alpha,0}, \qquad \mathcal{B}_{n;q}^{(\ell)}(x;\lambda) = 0 \quad (0 \leqq n \leqq \ell - 1). \end{aligned} \tag{3.1}$$

$\delta_{n,k}$ *being the Kronecker's symbol.*

Proposition 3.2. *A expansion for the q-Apostol-Bernoulli polynomials of higher order*

$$\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k;q}^{(\alpha)}(\lambda)q^{(k-\alpha+1)x}[x]_q^{n-k}. \tag{3.2}$$

Proposition 3.3 (difference equation).

$$\lambda q^{\alpha-1}\mathcal{B}_{n;q}^{(\alpha)}(x+1;\lambda) - \mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = n\mathcal{B}_{n-1;q}^{(\alpha-1)}(x;\lambda) \quad (n \geqq 1). \tag{3.3}$$

Proposition 3.4 (differential relationship).

$$\frac{\partial}{\partial x}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) = \mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)\log q + n\frac{\log q}{q-1}q^x\mathcal{B}_{n-1;q}^{(\alpha)}(x;\lambda q). \tag{3.4}$$

Proposition 3.5 (addition theorem).

$$\mathcal{B}_{n;q}^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k;q}^{(\alpha)}(x;\lambda)q^{(k-\alpha+1)y}[y]_q^{n-k}. \tag{3.5}$$

Proposition 3.6 (theorem of complement).

$$\mathcal{B}_{n;q}^{(\ell)}(\ell-x;\lambda) = \frac{(-1)^n}{\lambda^{\ell}}q^{\ell-n-\binom{\ell}{2}}\mathcal{B}_{n;q^{-1}}^{(\ell)}(x;\lambda^{-1}), \tag{3.6}$$

$$\mathcal{B}_{n;q}^{(\ell)}(\ell+x;\lambda) = \frac{(-1)^n}{\lambda^{\ell}}q^{\ell-n-\binom{\ell}{2}}\mathcal{B}_{n;q^{-1}}^{(\ell)}(-x;\lambda^{-1}). \tag{3.7}$$

Proposition 3.7 (two recursive formulas).

$$(n-\alpha)\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda)=n[x]_q\mathcal{B}_{n-1;q}^{(\alpha)}(x;\lambda)-\lambda[\alpha]_q q^x\mathcal{B}_{n;q}^{(\alpha+1)}(x+1;\lambda), \tag{3.8}$$

$$[\alpha]_q q^{x-\alpha}\mathcal{B}_{n;q}^{(\alpha+1)}(x;\lambda)=n\left([x]_q-[\alpha]_q q^{x-\alpha}\right)\mathcal{B}_{n-1;q}^{(\alpha)}(x;\lambda)+(\alpha-n)\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda). \tag{3.9}$$

Proposition 3.8. *For $n,\in\mathbb{N};\ \alpha,\lambda\in\mathbb{C}$,*

$$\begin{aligned}&\mathcal{E}_{n;q}^{(\alpha)}(\lambda)=2^n\mathcal{E}_{n;q}^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right),\qquad \mathcal{E}_{n;q}^{(0)}(x;\lambda)=q^x[x]_q^n,\\ &\mathcal{E}_{0;q}^{(\alpha)}(\lambda)=\frac{(2\sqrt{q})^\alpha}{(-\lambda q;q)_\alpha},\qquad \mathcal{E}_{0;q}^{(\alpha)}(x;\lambda)=\frac{2^\alpha q^x}{(-\lambda q;q)_\alpha}.\end{aligned} \tag{3.10}$$

Proposition 3.9. *The formula of the q-Apostol-Euler polynomials of higher order*

$$\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)=\sum_{k=0}^{n}\binom{n}{k}2^{-k}\mathcal{E}_{k;q}^{(\alpha)}(\lambda)q^{(k+1)(x-\frac{\alpha}{2})}\left[x-\frac{\alpha}{2}\right]_q^{n-k}. \tag{3.11}$$

Proposition 3.10 (difference equation).

$$\lambda q^{\alpha-1}\mathcal{E}_{n;q}^{(\alpha)}(x+1;\lambda)+\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)=2\mathcal{E}_{n;q}^{(\alpha-1)}(x;\lambda)\quad(n\geqq 0). \tag{3.12}$$

Proposition 3.11 (differential relationship).

$$\frac{\partial}{\partial x}\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)=\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)\log q+n\frac{\log q}{q-1}q^x\mathcal{E}_{n-1;q}^{(\alpha)}(x;\lambda q). \tag{3.13}$$

Proposition 3.12 (integral formula).

$$\int_a^b q^x\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda q)\,\mathrm{d}x=\frac{1-q}{n+1}\int_a^b\mathcal{E}_{n+1;q}^{(\alpha)}(x;\lambda)\,\mathrm{d}x+\frac{q-1}{\log q}\,\frac{\mathcal{E}_{n+1;q}^{(\alpha)}(b;\lambda)-\mathcal{E}_{n+1;q}^{(\alpha)}(a;\lambda)}{n+1}. \tag{3.14}$$

Proposition 3.13 (addition theorem).

$$\mathcal{E}_{n;q}^{(\alpha)}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{k;q}^{(\alpha)}(x;\lambda)q^{(k+1)y}[y]_q^{n-k}. \tag{3.15}$$

Proposition 3.14 (theorem of complement).

$$\mathcal{E}_{n;q}^{(\alpha)}(\alpha-x;\lambda)=\frac{(-1)^n}{\lambda^\alpha q^{\binom{\alpha}{2}+n}}\mathcal{E}_{n;q^{-1}}^{(\alpha)}(x;\lambda^{-1}), \tag{3.16}$$

$$\mathcal{E}_{n;q}^{(\alpha)}(\alpha+x;\lambda)=\frac{(-1)^n}{\lambda^\alpha q^{\binom{\alpha}{2}+n}}\mathcal{E}_{n;q^{-1}}^{(\alpha)}(-x;\lambda^{-1}). \tag{3.17}$$

Proposition 3.15 (two recursive formulas).

$$\mathcal{E}_{n+1;q}^{(\alpha)}(x;\lambda)=[x]_q\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)-\frac{\lambda}{2}[\alpha]_q q^x\mathcal{E}_{n;q}^{(\alpha+1)}(x+1;\lambda), \tag{3.18}$$

$$[\alpha]_q q^{x-\alpha}\mathcal{E}_{n;q}^{(\alpha+1)}(x;\lambda)=2\mathcal{E}_{n+1;q}^{(\alpha)}(x;\lambda)+2\left([\alpha]_q q^{x-\alpha}-[x]_q\right)\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda). \tag{3.19}$$

Remark 3.16. *The Proposition 3.1–Proposition 3.7 are the q-extensions of the basic properties for Apostol-Bernoulli polynomials of higher order (see, [26, p. 301, Eqs. (55)–(63)]).*

When $\lambda=1$, the Proposition 3.1–Proposition 3.7 will become the corresponding properties for the q-Bernoulli numbers and polynomials of higher order.

When $\lambda=1,\ \alpha=1$ or $\ell=1$, the Proposition 3.1–Proposition 3.7 will become the corresponding basic properties of Carlitz's numbers η_m and polynomials $\eta_m(x)$ in [5, p. 991–993].

Remark 3.17. *The Proposition 3.8–Proposition 3.15 are the q-extensions of the basic properties for Apostol-Euler polynomials of higher order (see, [16, p. 918–919, Eqs. (3)–(13)]).*

When $\lambda = 1$, the Proposition 3.8–Proposition 3.15 will become the corresponding proerties for the q-Euler numbers and polynomials of higher order.

When $\lambda = 1$, $\alpha = 1$ or $\ell = 1$, the Proposition 3.8–Proposition 3.15 will become some analogues of the basic properties of Carlitz's numbers ϵ_m and polynomials $\epsilon_m(x)$ in [5, p. 998–1000].

4. Some explicit relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials of higher order

In this section we shall investigate some explicit relationships between the q-Apostol-Bernoulli and q-Apostol-Euler polynomials based on the techniques for series rearrangement.

We now begin by recalling some earlier results of Luo and Srivastava (see, [27]) given by Theorem A below.

Theorem A. For $n \in \mathbb{N}_0;\ \alpha \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \{-1\}$, the following relationships

$$\mathcal{B}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \left[\mathcal{B}_k^{(\alpha)}(y;\lambda) + \frac{k}{2}\mathcal{B}_{k-1}^{(\alpha-1)}(y;\lambda) \right] \mathcal{E}_{n-k}(x;\lambda), \tag{4.1}$$

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(x+y;\lambda) = &\sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[\mathcal{E}_{k+1}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1}^{(\alpha)}(y;\lambda) \right] \mathcal{B}_{n-k}(x;\lambda)\\ &+ \frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^{\alpha} \mathcal{B}_{n+1}(x;\lambda)\end{aligned} \tag{4.2}$$

hold true.

The special cases of Theorem A for $\lambda = 1$ are just the following elegant results of Srivastava and Á. Pintér [43]:

Theorem B. For $n \in \mathbb{N}_0;\ \alpha \in \mathbb{C}$, the following relationships

$$B_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left[B_k^{(\alpha)}(y) + \frac{k}{2}B_{k-1}^{(\alpha-1)}(y) \right] E_{n-k}(x), \tag{4.3}$$

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[E_{k+1}^{(\alpha-1)}(y) - E_{k+1}^{(\alpha)}(y) \right] B_{n-k}(x) \tag{4.4}$$

hold true.

If further putting $\alpha = 1$ in (4.3) of Theorem B and then letting $y \to 0$, in view of that $B_n^{(0)}(x) = x^n$ and $B_1 = -\frac{1}{2}$, gives us the following Cheon's main result [8]:

$$B_n(x) = \sum_{\substack{k=0\\(k\neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) \qquad (n \in \mathbb{N}_0). \tag{4.5}$$

In order to obtain the main results of this paper we need the following facts and lemmas.

Taking $y = 1$ in (3.5), we get

$$\mathcal{B}_{n;q}^{(\alpha)}(x+1;\lambda) = \sum_{k=0}^{n} \binom{n}{k} q^{k-\alpha+1} \mathcal{B}_{k;q}^{(\alpha)}(x;\lambda). \tag{4.6}$$

It follows from (3.3) and (4.6) that

$$\mathcal{B}_{n;q}^{(\alpha-1)}(x;\lambda)=\frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}q^k\mathcal{B}_{k;q}^{(\alpha)}(x;\lambda)-\mathcal{B}_{n+1;q}^{(\alpha)}(x;\lambda)\right]\qquad(n\in\mathbb{N}_0),\tag{4.7}$$

which, in the special case when $\alpha=1$ and noting that $\mathcal{B}_{n;q}^{(0)}(x;\lambda)=q^x[x]_q^n$, we find the following explicit expansion:

$$q^x[x]_q^n=\frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}q^k\mathcal{B}_{k;q}(x;\lambda)-\mathcal{B}_{n+1;q}(x;\lambda)\right],\tag{4.8}$$

which is an q-extension of the known expansion [27, p. 634, Eq. (29)]:

$$x^n=\frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_k(x;\lambda)-\mathcal{B}_{n+1}(x;\lambda)\right].\tag{4.9}$$

Further, setting $\lambda=1$ in (4.8), we easily obtain the following expansion:

$$q^x[x]_q^n=\frac{1}{n+1}\left[\sum_{k=0}^{n}\binom{n+1}{k}q^kB_{k;q}(x)-(1-q^{n+1})B_{n+1;q}(x)\right],\tag{4.10}$$

which is an q-extension of the familiar expansion (*e.g.*, [31, p. 26]):

$$x^n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x).\tag{4.11}$$

It is obvious that

$$[my]_q=[y]_{q^m}[m]_q\tag{4.12}$$

From (3.5) and (4.12) we have

$$\begin{aligned}\mathcal{B}_{n;q^m}^{(\alpha)}(x+y;\lambda)&=\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)[y]_{q^m}^{n-k}\\&=[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)[my]_q^{n-k}.\end{aligned}$$

Upon setting $y=\frac{1}{m}$, we obtain the following formula:

$$\sum_{k=0}^{n}\binom{n}{k}q^{k-\alpha+1}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda)=[m]_q^n\mathcal{B}_{n;q^m}^{(\alpha)}\left(x+\frac{1}{m};\lambda\right).\tag{4.13}$$

We define the following polynomials in q^x:

$$\mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda)=\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda).\tag{4.14}$$

Clearly, we have

$$\lim_{q\to1}\mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda)=m^n\mathcal{B}_n^{(\alpha)}\left(x+\frac{1}{m};\lambda\right),\tag{4.15}$$

$$\mathcal{B}_{n;q^m;\frac{1}{m}}^{(\alpha)}(x+1;\lambda)=[m]_q^n\mathcal{B}_{n;q^m}^{(\alpha)}\left(x+\frac{1}{m};\lambda\right),\tag{4.16}$$

$$\mathcal{B}_{n;q}^{(\alpha)}(x+1;\lambda)=\mathcal{B}_{n;q;1}^{(\alpha)}(x+1;\lambda),\qquad\mathcal{B}_{n;q^m;y}(x+1;\lambda)=\mathcal{B}_{n;q^m;y}^{(1)}(x+1;\lambda),$$

$$B_{n;q^m;y}^{(\alpha)}(x+1)=\mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;1),\qquad B_{n;q^m;y}(x+1)=\mathcal{B}_{n;q^m;y}(x+1;1).$$

It is easy to see that the equations (4.13) and (4.14) are q-extensions of the equation (see, [27, p. 634, Eq.(26)] for $x \longleftrightarrow y, y = \frac{1}{m}$)

$$\sum_{k=0}^{n} \binom{n}{k} m^k \mathcal{B}_k^{(\alpha)}(x;\lambda) = m^n \mathcal{B}_n^{(\alpha)}\left(x + \frac{1}{m};\lambda\right). \tag{4.17}$$

The following special values of $\mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda)$ are easily obtained from (4.14) by simple computation.

$$\mathcal{B}_{n;q^m;y}^{(0)}(x;\lambda) = q^{m(x+y-1)}(1 + q^{my}[mx - m]_q)^n, \tag{4.18}$$

$$\mathcal{B}_{0;q^m;y}^{(\alpha)}(x;\lambda) = q^{m(x+y-1)}\delta_{\alpha,0}, \ \mathcal{B}_{n;q^m;y}^{(\ell)}(x;\lambda) = 0 \quad (0 \leqq n \leqq \ell - 1), \tag{4.19}$$

where $\delta_{n,k}$ denotes the Kronecker's symbol.

The polynomials $\mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda)$ satisfies the following difference relationship.

Lemma 4.1. *For $n \geqq 1$,*

$$\lambda q^{m(\alpha-1)} \mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda) - \mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda) = n[m]_q \mathcal{B}_{n-1;q^m;y}^{(\alpha-1)}(x;\lambda). \tag{4.20}$$

Proof. By (4.14) and applying (3.3), we obtain

$$\begin{aligned}
&\lambda q^{m(\alpha-1)} \mathcal{B}_{n;q^m;y}^{(\alpha)}(x+1;\lambda) - \mathcal{B}_{n;q^m;y}^{(\alpha)}(x;\lambda) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} q^{m(k-\alpha+1)y} [m]_q^k \Big[\lambda q^{m(\alpha-1)} \mathcal{B}_{k;q^m}^{(\alpha)}(x;\lambda) - \mathcal{B}_{k;q^m}^{(\alpha)}(x-1;\lambda)\Big] \\
&\quad = \sum_{k=0}^{n} k \binom{n}{k} q^{m(k-\alpha+1)y} [m]_q^k \mathcal{B}_{k-1;q^m}^{(\alpha-1)}(x-1;\lambda) \\
&\quad = n \sum_{k=0}^{n-1} \binom{n-1}{k} q^{m(k-\alpha+2)y} [m]_q^{k+1} \mathcal{B}_{k;q^m}^{(\alpha-1)}(x-1;\lambda) \\
&\quad = n[m]_q \mathcal{B}_{n-1;q^m;y}^{(\alpha-1)}(x-1;\lambda).
\end{aligned}$$

Hence, the formula (4.20) follows. □

On the other hand, if setting $y = 1$ in (3.15), we have

$$\mathcal{E}_{n;q}^{(\alpha)}(x+1;\lambda) = \sum_{k=0}^{n} \binom{n}{k} q^{k+1} \mathcal{E}_{k;q}^{(\alpha)}(x;\lambda). \tag{4.21}$$

It follows from (3.12) and (4.21) that

$$\mathcal{E}_{n;q}^{(\alpha-1)}(x;\lambda) = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} q^{k+\alpha} \mathcal{E}_{k;q}^{(\alpha)}(x;\lambda) + \mathcal{E}_{n;q}^{(\alpha)}(x;\lambda)\right] \qquad (n \in \mathbb{N}_0) \tag{4.22}$$

which, in the special case when $\alpha = 1$ and noting that $\mathcal{E}_{n;q}^{(0)}(x;\lambda) = q^x[x]_q^n$, we arrive at the following explicit expansion:

$$q^x[x]_q^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} q^{k+1} \mathcal{E}_{k;q}(x;\lambda) + \mathcal{E}_{n;q}(x;\lambda)\right], \tag{4.23}$$

which is an q-extension of the known expansion (see, [27, p. 635, Eq. (32)])

$$x^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k(x;\lambda) + \mathcal{E}_n(x;\lambda)\right]. \tag{4.24}$$

Further, setting $\lambda = 1$ in (4.23), we easily obtain the following expansion:

$$q^x [x]_q^n = \frac{1}{2}\left[\sum_{k=0}^{n} \binom{n}{k} q^{k+1} E_{k;q}(x) + E_{n;q}(x)\right], \tag{4.25}$$

which is an q-extension of the well-known expansion (*e.g.*, [43, p. 378, Eq. (29)]):

$$x^n = \frac{1}{2}\left[\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x)\right]. \tag{4.26}$$

From (3.15) and (4.12) we have

$$\begin{aligned}\mathcal{E}^{(\alpha)}_{n;q^m}(x+y;\lambda) &= \sum_{k=0}^{n} \binom{n}{k} q^{m(k+1)y} \mathcal{E}^{(\alpha)}_{k;q^m}(x;\lambda)[y]_{q^m}^{n-k} \\ &= [m]_q^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{m(k+1)y} [m]_q^k \mathcal{E}^{(\alpha)}_{k;q^m}(x;\lambda)[my]_q^{n-k}.\end{aligned}$$

If we set $y = \frac{1}{m}$, we obtain the following formula:

$$\sum_{k=0}^{n} \binom{n}{k} q^{k+1} [m]_q^k \mathcal{E}^{(\alpha)}_{k;q^m}(x;\lambda) = [m]_q^n \mathcal{E}^{(\alpha)}_{n;q^m}\left(x + \frac{1}{m};\lambda\right). \tag{4.27}$$

We define the following polynomials in q^x:

$$\mathcal{E}^{(\alpha)}_{n;q^m;y}(x+1;\lambda) = \sum_{k=0}^{n} \binom{n}{k} q^{m(k+1)y} [m]_q^k \mathcal{E}^{(\alpha)}_{k;q^m}(x;\lambda). \tag{4.28}$$

Clearly, we have

$$\lim_{q\to 1} \mathcal{E}^{(\alpha)}_{n;q^m;y}(x+1;\lambda) = m^n \mathcal{E}^{(\alpha)}_n\left(x + \frac{1}{m};\lambda\right), \tag{4.29}$$

$$\mathcal{E}^{(\alpha)}_{n;q^m;\frac{1}{m}}(x+1;\lambda) = [m]_q^n \mathcal{E}^{(\alpha)}_{n;q^m}\left(x + \frac{1}{m};\lambda\right), \tag{4.30}$$

$$\mathcal{E}^{(\alpha)}_{n;q}(x+1;\lambda) = \mathcal{E}^{(\alpha)}_{n;q;1}(x+1;\lambda), \qquad \mathcal{E}_{n;q^m;y}(x+1;\lambda) = \mathcal{E}^{(1)}_{n;q^m;y}(x+1;\lambda),$$

$$E^{(\alpha)}_{n;q^m;y}(x+1) = \mathcal{E}^{(\alpha)}_{n;q^m;y}(x+1;1), \qquad E_{n;q^m;y}(x+1) = \mathcal{E}_{n;q^m;y}(x+1;1).$$

It is easy to observe that the equations (4.27) and (4.28) are q-extensions of the equation (see, [27, p. 634, Eq.(27)] for $x \longleftrightarrow y, y = \frac{1}{m}$)

$$\sum_{k=0}^{n} \binom{n}{k} m^k \mathcal{E}^{(\alpha)}_k(x;\lambda) = m^n \mathcal{E}^{(\alpha)}_n\left(x + \frac{1}{m};\lambda\right). \tag{4.31}$$

The following special values of $\mathcal{E}^{(\alpha)}_{n;q^m;y}(x;\lambda)$ are easily obtained from (4.28).

$$\mathcal{E}^{(0)}_{n;q^m;y}(x;\lambda) = q^{m(x+y-1)}(1 + q^{my}[mx-m]_q)^n, \tag{4.32}$$

$$\mathcal{E}^{(\alpha)}_{0;q^m;y}(x;\lambda) = \frac{2^\alpha q^{m(x+y-1)}}{(-\lambda q^m;q^m)_\alpha}. \tag{4.33}$$

Similarly, by (3.12) and (4.28) the polynomials $\mathcal{E}^{(\alpha)}_{n;q^m;y}(x;\lambda)$ in q^x also satisfy the following difference relationship:

Lemma 4.2. *For $n \geqq 0$,*

$$\lambda q^{m(\alpha-1)}\mathcal{E}^{(\alpha)}_{n;q^m;y}(x+1;\lambda)+\mathcal{E}^{(\alpha)}_{n;q^m;y}(x;\lambda)=2\mathcal{E}^{(\alpha-1)}_{n;q^m;y}(x;\lambda). \tag{4.34}$$

Next, by making use of the above formulas and results, we now prove the following formulas of the q-Apostol-Bernoulli polynomials of higher order.

Theorem 4.3. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; α, $\lambda \in \mathbb{C}$, the following relationship:*

$$\begin{aligned}\mathcal{B}^{(\alpha)}_{n;q^m}(x+y;\lambda)=&\frac{1}{2[m]_q^n}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\\&+q^{n-k-m(y+\alpha-1)+1}\Big[\mathcal{B}^{(\alpha)}_{k;q^m;y}(x;\lambda)+k[m]_q\mathcal{B}^{(\alpha-1)}_{k-1;q^m;y}(x;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(my;\lambda)\end{aligned} \tag{4.35}$$

holds true between the q-Apostol-Bernoulli polynomials of higher order and q-Apostol-Euler polynomials.

Proof. First replacing q by q^m in (3.5), and then applying the relation (4.12) and making the suitable substitution in (4.23), we obtain

$$\begin{aligned}\mathcal{B}^{(\alpha)}_{n;q^m}(x+y;\lambda)=&[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha+1)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)[my]_q^{n-k}\\=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\left[\lambda\sum_{j=0}^{n-k}\binom{n-k}{j}q^{j+1}\mathcal{E}_{j;q}(my;\lambda)+\mathcal{E}_{n-k;q}(my;\lambda)\right]\\=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\mathcal{E}_{n-k;q}(my;\lambda)\\&+\frac{1}{2}\lambda[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\sum_{j=0}^{n-k}\binom{n-k}{j}q^{j+1}\mathcal{E}_{j;q}(my;\lambda)\\=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\mathcal{E}_{n-k;q}(my;\lambda)\\&+\frac{1}{2}\lambda[m]_q^{-n}\sum_{j=0}^{n}\binom{n}{j}q^{j+1}\mathcal{E}_{j;q}(my;\lambda)\sum_{k=0}^{n-j}\binom{n-j}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\\=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)\mathcal{E}_{n-k;q}(my;\lambda)\\&+\frac{1}{2}\lambda[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k+1}\mathcal{E}_{n-k;q}(my;\lambda)\sum_{j=0}^{k}\binom{k}{j}q^{m(j-\alpha)y}[m]_q^j\mathcal{B}^{(\alpha)}_{j;q^m}(x;\lambda)\\=&\frac{1}{2}[m]_q^{-n}\sum_{k=0}^{n}\binom{n}{k}\left[q^{m(k-\alpha)y}[m]_q^k\mathcal{B}^{(\alpha)}_{k;q^m}(x;\lambda)+\lambda q^{n-k-my+1}\mathcal{B}^{(\alpha)}_{k;q^m;y}(x+1;\lambda)\right]\mathcal{E}_{n-k;q}(my;\lambda).\end{aligned}$$

In the above process we have inverted the order of summation and applied the following elementary combinatorial identity:

$$\binom{m}{l}\binom{l}{n}=\binom{m}{n}\binom{m-n}{m-l}. \tag{4.36}$$

Finally, in light of the difference relationship (4.20) of Lemma 4.1, we obtain the assertion (4.35) at once. This proof is complete. □

In view of the symmetry of x, y in Theorem 4.3, the formula (4.35) can be rewritten in the following form:

$$\begin{aligned}\mathcal{B}_{n;q^m}^{(\alpha)}(x+y;\lambda) =& \frac{1}{2[m]_q^n}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{m(k-\alpha)x}[m]_q^k\mathcal{B}_{k;q^m}^{(\alpha)}(y;\lambda) + q^{n-k-m(x+\alpha-1)+1} \\ &\times\Big[\mathcal{B}_{k;q^m;x}^{(\alpha)}(y;\lambda) + k[m]_q\mathcal{B}_{k-1;q^m;x}^{(\alpha-1)}(y;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(mx;\lambda).\end{aligned} \tag{4.37}$$

It follows from (4.37) that we give the following corollaries which are the corresponding q-extensions for some well-known results.

Setting $m = 1$ in (4.37), we obtain

Corollary 4.4. *For $n \in \mathbb{N}_0;\ \alpha,\ \lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned}\mathcal{B}_{n;q}^{(\alpha)}(x+y;\lambda) =& \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{(k-\alpha)x}\mathcal{B}_{k;q}^{(\alpha)}(y;\lambda) \\ &+ q^{n-k-x-\alpha+2}\Big[\mathcal{B}_{k;q;x}^{(\alpha)}(y;\lambda) + k\mathcal{B}_{k-1;q;x}^{(\alpha-1)}(y;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(x;\lambda)\end{aligned} \tag{4.38}$$

holds true.

Obviously, the formula (4.38) when $q \to 1$ reduces to (4.1) of Theorem A. Therefore, the formula (4.38) is just an q-extension of the *main* formula (4.1) of Luo and Srivastava [27, p. 636, Theorem 1].

Further, we set $y = 0$ in (4.38), we deduce that

$$\begin{aligned}\mathcal{B}_{n;q}^{(\alpha)}(x;\lambda) =& \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{(k-\alpha)x}\mathcal{B}_{k;q}^{(\alpha)}(\lambda) \\ &+ q^{n-k-x-\alpha+2}\Big[\mathcal{B}_{k;q;x}^{(\alpha)}(0;\lambda) + k\mathcal{B}_{k-1;q;x}^{(\alpha-1)}(0;\lambda)\Big]\Big]\mathcal{E}_{n-k;q}(x;\lambda),\end{aligned} \tag{4.39}$$

which is just an q-extension of the formula of Luo and Srivastava (see, [27, p. 637, Eq. (49)]):

$$\mathcal{B}_{n}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\Big[\mathcal{B}_{k}^{(\alpha)}(\lambda) + \frac{k}{2}\mathcal{B}_{k-1}^{(\alpha-1)}(\lambda)\Big]\mathcal{E}_{n-k}(x;\lambda). \tag{4.40}$$

Putting $\lambda = 1$ in (4.37), we have

Corollary 4.5. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned}B_{n;q^m}^{(\alpha)}(x+y) =& \frac{1}{2[m]_q^n}\sum_{k=0}^{n}\binom{n}{k}\Big[q^{m(k-\alpha)x}[m]_q^k B_{k;q^m}^{(\alpha)}(y) \\ &+ q^{n-k-m(x+\alpha-1)+1}\Big[B_{k;q^m;x}^{(\alpha)}(y) + k[m]_q B_{k-1;q^m;x}^{(\alpha-1)}(y)\Big]\Big]E_{n-k;q}(mx)\end{aligned} \tag{4.41}$$

holds true between the q-Bernoulli polynomials of higher order and q-Euler polynomials.

In particular, setting $\lambda = 1$ in (4.38) or $m = 1$ in (4.41), we thus arrive at the following corollary.

Corollary 4.6. [28, p. 249, Theorem 1, Eq. (3.1)] *For $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned} B_{n;q}^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-\alpha)x} B_{k;q}^{(\alpha)}(y) + \frac{1}{2} q^{n-k-x-\alpha+2} B_{k;q;x}^{(\alpha)}(y) \\ + q^{n-k-x-\alpha+2} \frac{k}{2} B_{k-1;q;x}^{(\alpha-1)}(y) \Big] E_{n-k;q}(x) \end{aligned} \tag{4.42}$$

holds true.

It is obvious that the formula (4.42) when $q \to 1$ reduces to (4.3) of Theorem B. Hence, the formula (4.42) is indeed an q-extension of the *main* formula (4.3) of Srivastava and Á. Pintér (see, [43, p. 379, Theorem 1]).

Setting $\alpha = 1$ in (4.37) and noting that (4.18), we have

Corollary 4.7. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned} \mathcal{B}_{n;q^m}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{m(k-1)x} [m]_q^{k-n} \mathcal{B}_{k;q^m}(y;\lambda) + \frac{1}{2} q^{n-k-mx+1} [m]_q^{-n} \mathcal{B}_{k;q^m;x}(y;\lambda) \\ + \frac{k}{2} [m]_q^{1-n} q^{n-k-m+my+1} (1 + q^{mx}[my-m]_q)^{k-1} \Big] \mathcal{E}_{n-k;q}(mx;\lambda) \end{aligned} \tag{4.43}$$

holds true between the q-Apostol-Bernoulli polynomials and q-Apostol-Euler polynomials.

Further, putting $m = 1$ in (4.43), we get

$$\begin{aligned} \mathcal{B}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-1)x} \mathcal{B}_{k;q}(y;\lambda) + \frac{1}{2} q^{n-k-x+1} \mathcal{B}_{k;q;x}(y;\lambda) \\ + \frac{k}{2} q^{n-k+y} (1 + q^{x}[y-1]_q)^{k-1} \Big] \mathcal{E}_{n-k;q}(x;\lambda), \end{aligned} \tag{4.44}$$

which is an q-extension of the known formula (see, [27, p. 638, Eq. (54)])

$$\mathcal{B}_n(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \left[\mathcal{B}_k(y;\lambda) + \frac{k}{2} y^{k-1} \right] \mathcal{E}_{n-k}(x;\lambda). \tag{4.45}$$

Letting $y \to 0$ in (4.44), we obtain

Corollary 4.8. *For $n \in \mathbb{N}_0$; $\lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned} \mathcal{B}_{n;q}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-1)x} \mathcal{B}_{k;q}(\lambda) + \frac{1}{2} q^{n-k-x+1} \mathcal{B}_{k;q;x}(0;\lambda) \\ + \frac{k}{2} q^{n-k} (1 - q^{x-1})^{k-1} \Big] \mathcal{E}_{n-k;q}(x;\lambda) \end{aligned} \tag{4.46}$$

holds true.

When $q \to 1$, the formula (4.46) reduces to the following form (see, [27, p. 637, Eq. (51)]):

$$\mathcal{B}_n(x;\lambda) = \sum_{\substack{k=0 \\ (k \neq 1)}}^{n} \binom{n}{k} \mathcal{B}_k(\lambda) \mathcal{E}_{n-k}(x;\lambda) + n \left[\mathcal{B}_1(\lambda) + \frac{1}{2} \right] \mathcal{E}_{n-1}(x;\lambda) \tag{4.47}$$

$(\lambda \in \mathbb{C},\ n \in \mathbb{N}_0)$.

Therefore, the formula (4.46) is an q-extension of (4.47).

When $\lambda = 1$, the formula (4.43) reduces to the following known result:

Corollary 4.9. [24, p. 11, Eq. (3.1)] *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\lambda \in \mathbb{C}$, the following relationship*

$$\begin{aligned} B_{n;q^m}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \Big[& \frac{1}{2} q^{m(k-1)x} [m]_q^{k-n} B_{k;q^m}(y) + \frac{1}{2} q^{n-k-mx+1} [m]_q^{-n} B_{k;q^m;x}(y) \\ & + \frac{k}{2} [m]_q^{1-n} q^{n-k-m+my+1} (1 + q^{mx}[my-m]_q)^{k-1} \Big] E_{n-k;q}(mx) \end{aligned} \tag{4.48}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

If we take $\lambda = 1$ in (4.46), we have

Corollary 4.10. [24, p. 13, Eq. (3.8)] *For $n \in \mathbb{N}_0$, the following relationship*

$$B_{n;q}(x) = \sum_{k=0}^{n} \binom{n}{k} \Big[\frac{1}{2} q^{(k-1)x} B_{k;q} + \frac{1}{2} q^{n-k-x+1} B_{k;q;x}(0) + \frac{k}{2} q^{n-k} (1 - q^{x-1})^{k-1} \Big] E_{n-k;q}(x) \tag{4.49}$$

holds true between the q-Bernoulli polynomials and q-Euler polynomials.

It is easy to verify that the formula (4.49) is an q-extension of the Cheon's *main* result (4.5) (see, [8, p. 368, Theorem 3]).

In the same manner, we can obtain the following theorem.

Theorem 4.11. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{-1\}$, the following relationship*

$$\begin{aligned} \mathcal{E}_{n;q^m}^{(\alpha)}(x+y;\lambda) = & [m]_q^{-n} \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \Big[q^{n-k-m(x+\alpha-1)} \Big[2\mathcal{E}_{k+1;q^m;x}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1;q^m;x}^{(\alpha)}(y;\lambda) \Big] \\ & - [m]_q^{k+1} q^{m(k+1)x} \mathcal{E}_{k+1;q^m}^{(\alpha)}(y;\lambda) \Big] \mathcal{B}_{n-k;q}(mx;\lambda) + \frac{2^\alpha q^{my}(\lambda q^{n+1}-1)}{(n+1)[m]_q^n (-\lambda q^m; q^m)_\alpha} \mathcal{B}_{n+1;q}(mx;\lambda) \end{aligned} \tag{4.50}$$

holds true between the q-Apostol-Euler polynomials of higher order and q-Apostol-Bernoulli polynomials.

In the following we give some interesting special cases of (4.50).

Setting $\lambda = 1$ in (4.50), we have

Corollary 4.12. *For $n \in \mathbb{N}_0$; $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned} E_{n;q^m}^{(\alpha)}(x+y) = & [m]_q^{-n} \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \Big[q^{n-k-m(x+\alpha-1)} \Big[2E_{k+1;q^m;x}^{(\alpha-1)}(y) - E_{k+1;q^m;x}^{(\alpha)}(y) \Big] \\ & - [m]_q^{k+1} q^{m(k+1)x} E_{k+1;q^m}^{(\alpha)}(y) \Big] B_{n-k;q}(mx) + \frac{2^\alpha q^{my}(q^{n+1}-1)}{(n+1)[m]_q^n (-q^m; q^m)_\alpha} B_{n+1;q}(mx) \end{aligned} \tag{4.51}$$

holds true between the q-Apostol-Euler polynomials of higher order and q-Apostol-Bernoulli polynomials.

Taking $m = 1$ in (4.50) we get

Corollary 4.13. *For $n \in \mathbb{N}_0$; $\alpha \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{-1\}$, the following relationship*

$$\begin{aligned} \mathcal{E}_{n;q}^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \Big[& q^{n-k-x-\alpha+1} \Big[2\mathcal{E}_{k+1;q;x}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1;q;x}^{(\alpha)}(y;\lambda) \Big] \\ & - q^{(k+1)x} \mathcal{E}_{k+1;q}^{(\alpha)}(y;\lambda) \Big] \mathcal{B}_{n-k;q}(x;\lambda) + \frac{2^\alpha q^{y}(\lambda q^{n+1}-1)}{(n+1)(-\lambda q; q)_\alpha} \mathcal{B}_{n+1;q}(x;\lambda) \end{aligned} \tag{4.52}$$

holds true between the q-Apostol-Euler polynomials of higher order and q-Apostol-Bernoulli polynomials.

The formula (4.52) when $q \to 1$ reduces to (4.2) of Theorem A. Therefore, the formula (4.52) is an q-extension of the *main* formula (4.2) of Luo and Srivastava (see, [27, p. 638, Theorem 2]).

Further, we put $x = 0$ in (4.52) and then replace y by x, we deduce that

$$\begin{aligned}\mathcal{E}_{n;q}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\Big[q^{n-k-\alpha+1}\Big[2\mathcal{E}_{k+1;q;0}^{(\alpha-1)}(x;\lambda) - \mathcal{E}_{k+1;q;0}^{(\alpha)}(x;\lambda)\Big]\\ - \mathcal{E}_{k+1;q}^{(\alpha)}(x;\lambda)\Big]\mathcal{B}_{n-k;q}(\lambda) + \frac{2^\alpha q^x(\lambda q^{n+1}-1)}{(n+1)(-\lambda q;q)_\alpha}\mathcal{B}_{n+1;q}(\lambda),\end{aligned} \tag{4.53}$$

which is an q-extension of the formula of Luo and Srivastava (see, [27, p. 638, Eq. (63)]):

$$\mathcal{E}_{n}^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\Big[\mathcal{E}_{k+1}^{(\alpha-1)}(x;\lambda) - \mathcal{E}_{k+1}^{(\alpha)}(x;\lambda)\Big]\mathcal{B}_{n-k}(\lambda) + \frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^\alpha \mathcal{B}_{n+1}(\lambda). \tag{4.54}$$

If we put $x = \dfrac{\alpha}{2}$ in (4.53) and note that $\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = 2^n \mathcal{E}_{n;q}^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right)$, we derive that

$$\begin{aligned}\mathcal{E}_{n;q}^{(\alpha)}(\lambda) = \sum_{k=0}^{n} \frac{2^{n-k-1}}{k+1}\binom{n}{k}\Big[q^{n-k-\alpha+1}\Big[2^{k+2}\mathcal{E}_{k+1;q;0}^{(\alpha-1)}\left(\frac{\alpha}{2};\lambda\right) - \mathcal{E}_{k+1;q;0}^{(\alpha)}(\lambda)\Big] - \mathcal{E}_{k+1;q}^{(\alpha)}(\lambda)\Big]\mathcal{B}_{n-k;q}(\lambda)\\ + \frac{2^\alpha q^{\frac{\alpha}{2}}(\lambda q^{n+1}-1)}{(n+1)(-\lambda q;q)_\alpha}\mathcal{B}_{n+1;q}(\lambda),\end{aligned} \tag{4.55}$$

which is an q-extension of the formula of Luo and Srivastava (see, [27, p. 638, Eq. (63)]):

$$\mathcal{E}_{n}^{(\alpha)}(\lambda) = \sum_{k=0}^{n} \frac{2^{n-k}}{k+1}\binom{n}{k}\Big[2^{k+1}\mathcal{E}_{k+1}^{(\alpha-1)}\left(\frac{\alpha}{2};\lambda\right) - \mathcal{E}_{k+1}^{(\alpha)}(\lambda)\Big]\mathcal{B}_{n-k}(\lambda) + \frac{\lambda-1}{n+1}\left(\frac{2}{\lambda+1}\right)^\alpha \mathcal{B}_{n+1}(\lambda). \tag{4.56}$$

If we put $\alpha = 1$ and $\lambda = 1$ in (4.55), we have

$$\begin{aligned}E_{n;q} = \sum_{k=0}^{n} \frac{2^{n-k-1}}{k+1}\binom{n}{k}\Big[q^{n-k-\alpha+1}\Big[2^{k+2}E_{k+1;q;0}^{(0)}\left(\frac{1}{2}\right) - E_{k+1;q;0}\Big] - E_{k+1;q}\Big]B_{n-k;q}\\ + \frac{2q^{\frac{1}{2}}(q^{n+1}-1)}{(n+1)(\lambda+1)}B_{n+1;q},\end{aligned} \tag{4.57}$$

which is an q-extension of the familiar formula (see, [27, p. 638, Eqs. (65)]):

$$E_n = \sum_{k=0}^{n} \frac{2^{n-k}}{k+1}\binom{n}{k}(1 - E_{k+1})B_{n-k}. \tag{4.58}$$

Putting $\lambda = 1$ in (4.52), we arrive at

Corollary 4.14. [28, p. 249, Theorem 1, Eq. (3.2)] *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned}E_{n;q}^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\Big[q^{n-k-x-\alpha+1}\Big[2E_{k+1;q;x}^{(\alpha-1)}(y) - E_{k+1;q;x}^{(\alpha)}(y)\Big]\\ - q^{(k+1)x}E_{k+1;q}^{(\alpha)}(y)\Big]B_{n-k;q}(x) + \frac{2^\alpha q^y(q^{n+1}-1)}{(n+1)(-q;q)_\alpha}B_{n+1;q}(x)\end{aligned} \tag{4.59}$$

holds true between the q-Euler polynomials of higher order and q-Bernoulli polynomials.

The formula (4.59) reduces to (4.4) of Theorem B when $q \to 1$. Hence, the formula (4.59) is an q-extension of the *main* formula (4.4) of Srivastava and Á. Pintér (see, [43, p. 380, Theorem 2]).

Letting $\alpha = 1$ in (4.50) and noting that (4.32), we have

Corollary 4.15. *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\lambda \in \mathbb{C} \setminus \{-1\}$, the following relationship*

$$\begin{aligned}\mathcal{E}_{n;q^m}(x+y;\lambda) =&[m]_q^{-n}\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k+m(y-1)}(1+q^{mx}[my-m]_q)^{k+1}\\&-q^{n-k-mx}\mathcal{E}_{k+1;q^m;x}(y;\lambda)-[m]_q^{k+1}q^{m(k+1)x}\mathcal{E}_{k+1;q^m}(y;\lambda)\Big]\mathcal{B}_{n-k;q}(mx;\lambda)\\&+\frac{2q^{my}(\lambda q^{n+1}-1)}{(n+1)[m]_q^n(\lambda q^m+1)}\mathcal{B}_{n+1;q}(mx;\lambda)\end{aligned} \tag{4.60}$$

holds true between the q-Apostol-Euler polynomials and q-Apostol-Bernoulli polynomials.

Setting $m = 1$, the formula (4.60) becomes that

$$\begin{aligned}\mathcal{E}_{n;q}(x+y;\lambda) =&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k+y-1}(1+q^{x}[y-1]_q)^{k+1}\\&-q^{n-k-x}\mathcal{E}_{k+1;q;x}(y;\lambda)-q^{(k+1)x}\mathcal{E}_{k+1;q}(y;\lambda)\Big]\mathcal{B}_{n-k;q}(x;\lambda)\\&+\frac{2q^{y}(\lambda q^{n+1}-1)}{(n+1)(\lambda q+1)}\mathcal{B}_{n+1;q}(x;\lambda),\end{aligned} \tag{4.61}$$

which is an q-extension of the formula of Luo and Srivastava [27, p. 638, Eq. (56)]:

$$\mathcal{E}_n(x+y;\lambda)=\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[y^{k+1}-\mathcal{E}_{k+1}(y;\lambda)\right]\mathcal{B}_{n-k}(x;\lambda)+\frac{2(\lambda-1)}{(n+1)(\lambda+1)}\mathcal{B}_{n+1}(x;\lambda).$$

Setting $y = 0$ in (4.61), we have

$$\begin{aligned}\mathcal{E}_{n;q}(x;\lambda) =&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k-1}(1-q^{x-1})^{k+1}-q^{n-k-x}\mathcal{E}_{k+1;q;x}(0;\lambda)-q^{(k+1)x}\mathcal{E}_{k+1;q}(0;\lambda)\Big]\mathcal{B}_{n-k;q}(x;\lambda)\\&+\frac{2(\lambda q^{n+1}-1)}{(n+1)(\lambda q+1)}\mathcal{B}_{n+1;q}(x;\lambda),\end{aligned} \tag{4.62}$$

is an q-extension of the formula of Luo and Srivastava [27, p. 638, Eq. (57)]:

$$\mathcal{E}_n(x;\lambda)=-\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\mathcal{E}_{k+1}(0;\lambda)\mathcal{B}_{n-k}(x;\lambda)+\frac{2(\lambda-1)}{(n+1)(\lambda+1)}\mathcal{B}_{n+1}(x;\lambda).$$

Taking $\lambda = 1$ in (4.60), we have

Corollary 4.16. [24, p. 13, Eq. (3.10)] *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$; $\alpha \in \mathbb{C}$, the following relationship*

$$\begin{aligned}E_{n;q^m}(x+y) =&[m]_q^{-n}\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k+m(y-1)}(1+q^{mx}[my-m]_q)^{k+1}\\&-q^{n-k-mx}E_{k+1;q^m;x}(y)-[m]_q^{k+1}q^{m(k+1)x}E_{k+1;q^m}(y)\Big]B_{n-k;q}(mx)\\&+\frac{2q^{my}(q^{n+1}-1)}{(n+1)[m]_q^n(q^m+1)}B_{n+1;q}(mx)\end{aligned} \tag{4.63}$$

holds true between the q-Euler polynomials and q-Bernoulli polynomials.

Setting $y = 0$ in (4.63), we deduce that (see, [24, p. 13, Eq. (3.13)]):

$$\begin{aligned} E_{n;q^m}(x) =&[m]_q^{-n}\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k-m}(1-q^{m(x-1)}[m]_q)^{k+1}-q^{n-k-mx}E_{k+1;q^m;x}(0)\\ &-[m]_q^{k+1}q^{m(k+1)x}E_{k+1;q^m}(0)\Big]B_{n-k;q}(mx)+\frac{2(q^{n+1}-1)}{(n+1)[m]_q^n(q^m+1)}B_{n+1;q}(mx). \end{aligned} \tag{4.64}$$

By setting $m = 1$ in (4.64) we deduce that (see, [24, p. 13, Eq. (3.16)]):

$$\begin{aligned} E_{n;q}(x) =&\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\Big[2q^{n-k-1}(1-q^{x-1})^{k+1}-q^{n-k-x}E_{k+1;q;x}(0)-q^{(k+1)x}E_{k+1;q}(0)\Big]B_{n-k;q}(x)\\ &+\frac{2(q^{n+1}-1)}{(n+1)(q+1)}B_{n+1;q}(x). \end{aligned} \tag{4.65}$$

5. Some formulas involving the q-Stirling numbers of the second kind

In this section we provide some formulas for the q-Apostol-Bernoulli and q-Apostol-Euler polynomials in series of the q-Stirling numbers of the second kind. Some interesting special cases are also considered. We know that the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k},$$

which satisfies the following relationships:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \quad (0 \le k \le n), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = 0 \quad (n < k),$$
$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_q + q^k\begin{bmatrix} x-1 \\ k \end{bmatrix}_q \qquad (n,k \in \mathbb{N};\ x \in \mathbb{C}).$$

We recall that the Stirling numbers of the second kind $S(n,k)$ are defined by means of the following expansion (see, [13, p. 207, Theorem B])

$$x^n = \sum_{k=0}^{n}\binom{x}{k}k!S(n,k). \tag{5.1}$$

So that

$$S(n,0) = \delta_{n,0} \qquad S(n,1) = S(n,n) = 1 \qquad S(n,n-1) = \binom{n}{2},$$

where $\delta_{m,n}$ denotes the Kronecker's symbol.

In 1948, Carlitz firstly gave an q-extension of the Stirling numbers of the second kind, i.e., the so-called q-Stirling numbers of the second kind $S_q(n,k)$ are defined by (see, [5, p. 989, Eq. (3.1)])

$$[x]_q^n = \sum_{k=0}^{n}S_q(n,k)[k]_q!\begin{bmatrix} x \\ k \end{bmatrix}_q q^{\binom{k}{2}}. \tag{5.2}$$

Carlitz also showed that the q-Stirling numbers of the second kind $S_q(n,k)$ satisfy the following relationships (see, [5, p. 990, Eq. (3.2) and (3.5)]):

$$S_q(n+1,k) = S_q(n,k-1) + [k]_q S_q(n,k),$$
$$S_q(n,k) = (q-1)^{k-n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\begin{bmatrix} j \\ k \end{bmatrix}_q.$$

Obviously,

$$S_q(n,0) = \delta_{n,0} \qquad S_q(n,1) = S_q(n,n) = 1 \qquad S_q(n,n-1) = \frac{n-[n]_q}{1-q}.$$

Noting that (4.12) and making the appropriate substitution in (5.2) into the right side of the formulas (3.5) and (3.15) respectively, we obtain Theorem 5.1 below.

Theorem 5.1. *For $\alpha,\ \lambda \in \mathbb{C}$; $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the following relationships*

$$\mathcal{B}^{(\alpha)}_{n;q^m}(x+y;\lambda) = \sum_{k=0}^{n}[k]_q!\begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k}\binom{n}{j} q^{m(j-\alpha+1)x+\binom{k}{2}}[m]_q^{j-n}\mathcal{B}^{(\alpha)}_{j;q^m}(y;\lambda)S_q(n-j,k), \tag{5.3}$$

$$\mathcal{E}^{(\alpha)}_{n;q^m}(x+y;\lambda) = \sum_{k=0}^{n}[k]_q!\begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k}\binom{n}{j} q^{m(j+1)x+\binom{k}{2}}[m]_q^{j-n}\mathcal{E}^{(\alpha)}_{j;q^m}(y;\lambda)S_q(n-j,k) \tag{5.4}$$

hold true between the q-Apostol-type polynomials of higher order and q-Stirling numbers of the second kind.

Setting $m = 1$ in (5.3) and (5.4) of Theorem 5.1, we then obtain the following corollary:

Corollary 5.2. *For $\alpha,\ \lambda \in \mathbb{C}$; $n \in \mathbb{N}_0$, the following relationships*

$$\mathcal{B}^{(\alpha)}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n}[k]_q!\begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k}\binom{n}{j} q^{(j-\alpha+1)x+\binom{k}{2}}\mathcal{B}^{(\alpha)}_{j;q}(y;\lambda)S_q(n-j,k), \tag{5.5}$$

$$\mathcal{E}^{(\alpha)}_{n;q}(x+y;\lambda) = \sum_{k=0}^{n}[k]_q!\begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k}\binom{n}{j} q^{(j+1)x+\binom{k}{2}}\mathcal{E}^{(\alpha)}_{j;q}(y;\lambda)S_q(n-j,k) \tag{5.6}$$

hold true.

It is easy to verify that the formulas (5.5) and (5.6) are respectively the q-extensions of the corresponding formulas (75) and (76) of [27, p. 641].

By setting $\lambda = 1$ and $m = 1$ in (5.3), and taking $\lambda = 1$ and $m = 1$ in (5.4), we have

Corollary 5.3. [28, p. 253, Theorem 3, Eq. (4.11) and (4.12)] *For $\alpha \in \mathbb{C}$; $n \in \mathbb{N}_0$, the following relationships*

$$B^{(\alpha)}_{n;q}(x+y) = \sum_{k=0}^{n}[k]_q!\begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k}\binom{n}{j} q^{(j-\alpha+1)x+\binom{k}{2}}B^{(\alpha)}_{j;q}(y)S_q(n-j,k), \tag{5.7}$$

$$E^{(\alpha)}_{n;q}(x+y) = \sum_{k=0}^{n}[k]_q!\begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k}\binom{n}{j} q^{(j+1)x+\binom{k}{2}}E^{(\alpha)}_{j;q}(y)S_q(n-j,k) \tag{5.8}$$

hold true.

Letting $q \to 1$ in (5.7) and (5.8), we obtain the corresponding formulas of Bernoulli and Euler polynomials of higher order.

Setting $\lambda = 1$ and $\alpha = 1$ in (5.3) and (5.4) of Theorem 5.1, then we obtain the following corollary:

Corollary 5.4. [24, p. 14, Eq. (4.5) and (4.6)] *For $m \in \mathbb{N},\ n \in \mathbb{N}_0$, the following relationships*

$$B_{n;q^m}(x+y) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{mjx+\binom{k}{2}} [m]_q^{j-n} B_{j;q^m}(y) S_q(n-j,k), \tag{5.9}$$

$$E_{n;q^m}(x+y) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} mx \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{m(j+1)x+\binom{k}{2}} [m]_q^{j-n} E_{j;q^m}(y) S_q(n-j,k) \tag{5.10}$$

hold true.

6. Further observations and consequences

In this section we give some new and interesting formulas of the Apostol-Bernoulli and Apostol-Euler polynomials of higher order based on the corresponding formulas in Section 4.

Letting $q \to 1$ in (4.37) and (4.50) and noting that (4.15) and (4.29), we obtain the following interesting formulas for Apostol-Bernoulli and Apostol-Euler polynomials of higher order.

Theorem 6.1. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \alpha \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \{-1\}$, the following relationships:*

$$\mathcal{B}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[\mathcal{B}_k^{(\alpha)}(y;\lambda) + \mathcal{B}_k^{(\alpha)}\left(y + \frac{1-m}{m};\lambda\right) + k\mathcal{B}_{k-1}^{(\alpha-1)}\left(y + \frac{1-m}{m};\lambda\right) \right] \mathcal{E}_{n-k}(mx;\lambda), \tag{6.1}$$

$$\begin{aligned} \mathcal{E}_n^{(\alpha)}(x+y;\lambda) = &\sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \left[2\mathcal{E}_{k+1}^{(\alpha-1)}\left(y + \frac{1-m}{m};\lambda\right) - \mathcal{E}_{k+1}^{(\alpha)}\left(y + \frac{1-m}{m};\lambda\right) - \mathcal{E}_{k+1}^{(\alpha)}(y;\lambda) \right] \mathcal{B}_{n-k}(mx;\lambda) \\ &+ \frac{\lambda-1}{m^n(n+1)} \left(\frac{2}{\lambda+1}\right)^{\alpha} \mathcal{B}_{n+1}(mx;\lambda) \end{aligned} \tag{6.2}$$

hold true.

Clearly, the above formulas (6.1) and (6.2) are the corresponding extensions of the formulas (4.1) and (4.2) of Theorem A.

If we set $\lambda = 1$ in (6.1) and (6.2), we obtain the following Corollary.

Corollary 6.2. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \alpha \in \mathbb{C}$, the following relationships*

$$B_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[B_k^{(\alpha)}(y) + B_k^{(\alpha)}\left(y + \frac{1-m}{m}\right) + kB_{k-1}^{(\alpha-1)}\left(y + \frac{1-m}{m}\right) \right] E_{n-k}(mx), \tag{6.3}$$

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \left[2E_{k+1}^{(\alpha-1)}\left(y + \frac{1-m}{m}\right) - E_{k+1}^{(\alpha)}\left(y + \frac{1-m}{m}\right) - E_{k+1}^{(\alpha)}(y) \right] B_{n-k}(mx) \tag{6.4}$$

hold true.

Obviously, the above formulas (6.3) and (6.4) are the corresponding extensions of the formulas (4.3) and (4.4) of Theorem B.

Taking $\alpha = 1$ in (6.1) and (6.2) of Theorem 6.1, we have

Corollary 6.3. *For $n \in \mathbb{N}_0,\ m \in \mathbb{N};\ \lambda \in \mathbb{C} \setminus \{-1\}$, the following relationships*

$$\mathcal{B}_n(x+y;\lambda) = \sum_{k=0}^{n} \frac{m^{k-n}}{2} \binom{n}{k} \left[\mathcal{B}_k(y;\lambda) + \mathcal{B}_k\left(y + \frac{1-m}{m};\lambda\right) + k\left(y + \frac{1-m}{m}\right)^{k-1} \right] \mathcal{E}_{n-k}(mx;\lambda), \tag{6.5}$$

$$\begin{aligned} \mathcal{E}_n(x+y;\lambda) = &\sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1} \binom{n}{k} \left[2\left(y + \frac{1-m}{m}\right)^{k+1} - \mathcal{E}_{k+1}\left(y + \frac{1-m}{m};\lambda\right) \right. \\ &\left. - \mathcal{E}_{k+1}(y;\lambda) \right] \mathcal{B}_{n-k}(mx;\lambda) + \frac{2}{m^n(n+1)} \frac{\lambda-1}{\lambda+1} \mathcal{B}_{n+1}(mx;\lambda) \end{aligned} \tag{6.6}$$

hold true.

Setting $\lambda = 1$ in (6.5) and (6.6), we obtain the following new and interesting formulas respectively.

$$B_n(x+y) = \sum_{k=0}^{n} \frac{m^{k-n}}{2}\binom{n}{k}\left[B_k(y) + B_k\left(y + \frac{1-m}{m}\right) + k\left(y + \frac{1-m}{m}\right)^{k-1}\right]E_{n-k}(mx), \tag{6.7}$$

$$E_n(x+y) = \sum_{k=0}^{n} \frac{m^{k-n+1}}{k+1}\binom{n}{k}\left[2\left(y + \frac{1-m}{m}\right)^{k+1} - E_{k+1}\left(y + \frac{1-m}{m}\right) - E_{k+1}(y)\right]B_{n-k}(mx). \tag{6.8}$$

Obviously, the formulas (6.7) and (6.8) are respectively the extensions of the formulas of Srivastava and Á. Pintér (see, [43]):

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k}\left[B_k(y) + \frac{k}{2}y^{k-1}\right]E_{n-k}(x), \tag{6.9}$$

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left[y^{k+1} - E_{k+1}(y)\right]B_{n-k}(x). \tag{6.10}$$

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