

SOME NEW IDENTITIES ON THE APOSTOL-BERNOULLI POLYNOMIALS OF HIGHER ORDER DERIVED FROM BERNOULLI BASIS

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ABSTRACT. In the present paper, we aim to obtain some new interesting relations and identities of the Apostol-Bernoulli polynomials of higher order. These identities are derived using a suitable polynomial basis, for which we employ a Bernoulli basis. Finally, by utilizing our method, we also derive formulas for the convolutions of Bernoulli and Euler polynomials, expressed via Apostol-Bernoulli polynomials of higher order.

1. INTRODUCTION

For $t \in \mathbb{C}$, the Euler polynomials have the following Taylor expansion at $t = 0$:

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = e^{tE(x)} = \frac{2}{e^t + 1} e^{xt}, \quad (|t| < \pi) \quad (1)$$

with the usual convention about replacing of $(E(x))^n := E_n(x)$ (see [1], [2], [9], [8], [13], [14], [15], [16], [20], [21], [22]).

There are also explicit formulas for the Euler polynomials, e.g.

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}$$

where E_k means the Euler numbers. Conversely, the Euler numbers are expressed with the Euler polynomials through $E_k = 2^k E_k(1/2)$. These numbers can be computed by:

$$(E+1)^n + (E-1)^n = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

(see [7], [17]).

For $|t| < 2\pi$ with $t \in \mathbb{C}$, the Bernoulli polynomials are defined by

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = e^{tB(x)} = \frac{t}{e^t - 1} e^{xt}$$

where we have used $(B(x))^n := B_n(x)$, symbolically. In the case $x = 0$, we have $B_n(0) := B_n$ that stands for n -th Bernoulli number. This number can be computed

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via

$$(B + 1)^n - B_n = \delta_{n,1}$$

where $\delta_{n,1}$ stands for Kronecker delta (see [4], [5], [12], [13]).

The Euler polynomials of order k are defined by the exponential generating function as follows:

$$\left(\frac{2}{e^t + 1}\right)^k e^{xt} = e^{tE^{(k)}(x)} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}), \quad (2)$$

with the usual convention about replacing $(E^{(k)}(x))^n$ by $E_n^{(k)}(x)$. In the special case, $x = 0$, $E_n^{(k)}(0) := E_n^{(k)}$ are called Apostol-Euler numbers of order k (see [15], [16], [18]).

In the complex plane, Apostol-Euler polynomials $E_n(x | \lambda)$ and Apostol-Bernoulli polynomials $B_n(x | \lambda)$ are given by [14]

$$\frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x | \lambda) \frac{t^n}{n!}, \quad (|t| < \log(-\lambda)) \quad (3)$$

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x | \lambda) \frac{t^n}{n!}, \quad (|t| < \log \lambda). \quad (4)$$

In [16], Apostol-Euler polynomials of higher order $E_n^{(k)}(x | \lambda)$ and Apostol-Bernoulli polynomials of higher order $B_n^{(k)}(x | \lambda)$ are given by the following generating functions:

$$\left(\frac{2}{\lambda e^t + 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x | \lambda) \frac{t^n}{n!}, \quad (|t| < \log(-\lambda)) \quad (5)$$

$$\frac{t^k}{(\lambda e^t - 1)^k} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x | \lambda) \frac{t^n}{n!}, \quad (|t| < \log \lambda). \quad (6)$$

In the above expressions, we take the principal value of the logarithm $\log \lambda$, i.e., $\log \lambda = \log |\lambda| + i \arg \lambda$ ($-\pi < \arg \lambda \leq \pi$) when $\lambda \neq 1$; set $\log 1 = 0$ when $\lambda = 1$. Additionally, in the special case, $x = 0$ in (5), we have $E_n^{(k)}(0 | \lambda) := E_n^{(k)}(\lambda)$ and $B_n^{(k)}(0 | \lambda) := B_n^{(k)}(\lambda)$ that stand for Apostol-Euler numbers and Apostol-Bernoulli numbers, respectively. Apostol-Euler polynomials of higher order and Apostol-Bernoulli polynomials of higher order can be expressed in terms of their numbers as follows:

$$E_n^{(k)}(x | \lambda) = \sum_{l=0}^n \binom{n}{l} x^l E_{n-l}^{(k)}(\lambda) \quad (7)$$

and

$$B_n^{(k)}(x | \lambda) = \sum_{l=0}^n \binom{n}{l} x^l B_{n-l}^{(k)}(\lambda) \quad (8)$$

From (1), (2), (3) and (5), we have

$$\begin{aligned} E_n^{(1)}(x | \lambda) & : = E_n(x | \lambda) \text{ and } E_n^{(1)}(x | 1) := E_n(x | 1) := E_n(x) \\ B_n^{(1)}(x | \lambda) & : = B_n(x | \lambda) \text{ and } B_n^{(1)}(x | 1) := B_n(x | 1) := B_n(x) \end{aligned}$$

By (1), we easily get

$$E_n^{(0)}(x | \lambda) = B_n^{(0)}(x | \lambda) = x^n. \quad (9)$$

Applying derivative operator in the both sides of (8), we have

$$\frac{d}{dx} B_n^{(k)}(x | \lambda) = n B_{n-1}^{(k)}(x | \lambda) \tag{10}$$

By (6), we arrive to

$$\frac{\lambda B_{n+1}^{(k)}(x+1 | \lambda) - B_{n+1}^{(k)}(x | \lambda)}{n+1} = B_n^{(k-1)}(x | \lambda), \quad (\text{see [16]}). \tag{11}$$

The linear operators Λ and D on the space of real-valued differentiable functions are considered as: For $n \in \mathbb{N}$

$$\Lambda f(x) = \lambda f(x+1) - f(x) \text{ and } Df(x) = \frac{df(x)}{dx}. \tag{12}$$

Notice that $\Lambda D = D\Lambda$. By (12), we have

$$\begin{aligned} \Lambda^2 f(x) &= \Lambda(\Lambda f(x)) = \lambda^2 f(x+2) - 2\lambda f(x+1) + f(x) \\ &= \sum_{l=0}^2 \binom{2}{l} (-1)^l \lambda^l f(x+l). \end{aligned}$$

By continuing this way, we procure

$$\Lambda^k f(x) = \sum_{l=0}^k (-1)^l \binom{k}{l} \lambda^l f(x+l).$$

Consequently, we give the following Lemma.

Lemma 1.1. *Let f be real valued function and $k \in \mathbb{N}$, we have*

$$\Lambda^k f(x) = \sum_{l=0}^k (-1)^l \binom{k}{l} \lambda^l f(x+l).$$

Furthermore,

$$\Lambda^k f(0) = \sum_{l=0}^k (-1)^l \binom{k}{l} \lambda^l f(l). \tag{13}$$

Let $P_n = \{q(x) \in \mathbb{Q}[x] \mid \deg q(x) \leq n\}$ be the $(n+1)$ -dimensional vector space over \mathbb{Q} . Likely, $\{1, x, \dots, x^n\}$ is the most natural basis for P_n . Additionally, $\{B_0^{(k)}(x | \lambda), B_1^{(k)}(x | \lambda), \dots, B_n^{(k)}(x | \lambda)\}$ is also a good basis for the space P_n for our objective of arithmetical applications of Apostol-Bernoulli polynomials of higher order.

If $q(x) \in P_n$, then $q(x)$ can be written as

$$q(x) = \sum_{j=0}^n b_j B_j^{(k)}(x | \lambda). \tag{14}$$

Recently, many mathematicians have studied on the applications of *polynomials* and *q-polynomials* for their finite evaluation schemes, closure under addition, multiplication, differentiation, integration, and composition, they are also richly utilized in construction of their generating functions for finding many identities and formulas (see [1-21]).

In the paper, we discover methods for determining b_j from the expression of $q(x)$ in (14), and apply those results to arithmetically and combinatorially interesting identities involving $B_0^{(k)}(x|\lambda), B_1^{(k)}(x|\lambda), \dots, B_n^{(k)}(x|\lambda)$.

2. INTERESTING IDENTITIES OF THE APOSTOL-BERNOULLI POLYNOMIALS OF HIGHER ORDER

By (11) and (12), we see that

$$\Lambda B_n^{(k)}(x|\lambda) = \lambda B_n^{(k)}(x+1|\lambda) - B_n^{(k)}(x|\lambda) = n B_{n-1}^{(k-1)}(x|\lambda), \quad (15)$$

and

$$D B_n^{(k)}(x|\lambda) = n B_{n-1}^{(k)}(x|\lambda). \quad (16)$$

Let us assume that $q(x) \in P_n$. Then $q(x)$ can be generated by means of $B_0^{(k)}(x|\lambda), B_1^{(k)}(x|\lambda), \dots, B_n^{(k)}(x|\lambda)$ as follows:

$$q(x) = \sum_{j=0}^n b_j B_j^{(k)}(x|\lambda). \quad (17)$$

Thus, by (17), we get

$$\Lambda q(x) = \sum_{j=0}^n b_j \Lambda B_j^{(k)}(x|\lambda) = \sum_{j=1}^n b_j j B_{j-1}^{(k-1)}(x|\lambda),$$

and

$$\Lambda^2 q(x) = \Lambda [\Lambda q(x)] = \sum_{j=2}^n b_j j(j-1) B_{j-2}^{(k-2)}(x|\lambda).$$

By continuing this way, we have

$$\Lambda^k q(x) = \sum_{j=k}^n b_j j(j-1)\cdots(j-k+1) B_{j-k}^{(0)}(x|\lambda). \quad (18)$$

By (9) and (18), we see that

$$D^s \Lambda^k q(x) = \sum_{j=k+s}^n b_j \frac{j!}{(j-k-s)!} x^{j-k-s} \quad (19)$$

Let us take $x = 0$ in (19). Then we derive the following:

$$\frac{1}{(k+s)!} D^s \Lambda^k q(0) = b_{k+s}. \quad (20)$$

From (14) and (20), we have

$$\begin{aligned} b_{k+s} &= \frac{1}{(k+s)!} D^s \Lambda^k q(0) = \frac{1}{(k+s)!} \Lambda^k D^s q(0) \\ &= \frac{1}{(k+s)!} \sum_{a=0}^k (-1)^a \binom{k}{a} \lambda^a D^s q(a). \end{aligned} \quad (21)$$

Therefore, by (17) and (21), we have the following theorem.

Theorem 2.1. For $k \in \mathbb{Z}_+$ and $q(x) \in P_n$, we have

$$q(x) = \sum_{j=k}^n \left(\frac{1}{j!} \sum_{a=0}^k (-1)^a \binom{k}{a} \lambda^a D^{j-k} q(a) \right) B_j^{(k)}(x | \lambda).$$

Let us take $q(x) = x^n \in P_n$. Then we derive that $D^{j-k} x^n = \frac{n!}{(n-j+k)!} x^{n-j+k}$.

Thus, by Theorem 2.1, we get

$$x^n = \sum_{j=k}^n \left(\frac{1}{j!} \sum_{a=0}^k (-1)^a \binom{k}{a} \lambda^a \frac{n!}{(n-j+k)!} a^{n-j+k} \right) B_j^{(k)}(x | \lambda). \quad (22)$$

Therefore, by (22), we arrive at the following corollary.

Corollary 2.2. For $k, n \in \mathbb{Z}_+$, we have

$$x^n = \sum_{j=k}^n \left(\frac{1}{j!} \sum_{a=0}^k (-1)^a \binom{k}{a} \lambda^a \frac{n!}{(n-j+k)!} a^{n-j+k} \right) B_j^{(k)}(x | \lambda).$$

In [13], [5] and [8], Euler polynomials of higher order are defined by the rule:

$$\sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{2}{e^t + 1} \right)^k e^{xt}, \quad (|t| < \pi).$$

Let $q(x) = E_n^{(k)}(x) \in P_n$. Also, it is well known in [13] that

$$D^{j-k} E_n^{(k)}(x) = \frac{n!}{(n-j+k)!} E_{n-j+k}^{(k)}(x). \quad (23)$$

By Theorem 2.1 and (23), we get the following theorem.

Theorem 2.3. For $k, n \in \mathbb{Z}_+$, we have

$$E_n^{(k)}(x) = \sum_{j=k}^n \sum_{a=0}^k \sum_{l=0}^{n-j+k} \frac{\binom{k}{a} \binom{n-j+k}{l} a^l (-\lambda)^a n!}{j! (n-j+k)!} E_{n-j+k-l}^{(k)} B_j^{(k)}(x | \lambda).$$

In [15], [21] and [22], the Bernoulli polynomials of higher order, $B_n^{(k)}(x)$, is known to be:

$$\left(\frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi)$$

Let us consider $q(x) = B_n^{(k)}(x) \in P_n$. Then we see that

$$D^{j-k} B_n^{(k)}(x) = \frac{n!}{(n-j+k)!} B_{n-j+k}^{(k)}(x). \quad (24)$$

Thanks to Theorem 2.1 and (24), we acquire the following theorem.

Theorem 2.4. For $k, n \in \mathbb{Z}_+$, we have

$$B_n^{(k)}(x) = \sum_{j=k}^n \sum_{a=0}^k \sum_{l=0}^{n-j+k} \frac{(-\lambda)^a \binom{k}{a} \binom{n-j+k}{l} a^l (-\lambda)^a n!}{j! (n-j+k)!} B_{n-j+k-l}^{(k)} B_j^{(k)}(x | \lambda).$$

Hansen [7] derived the following convolution formula:

$$\sum_{k=0}^m \binom{m}{k} B_k(x) B_{m-k}(y) = (1-m) B_m(x+y) + (x+y-1) m B_{m-1}(x+y). \quad (25)$$

We note that the special case $x = y = 0$ of the last identity

$$B_m = -\frac{\sum_{k=2}^{m-2} \binom{m}{k} B_k B_{m-k}}{m+1}$$

is originally constructed by Euler and Ramanujan (cf. [5]).

Let us now write the following

$$q(x) = \sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(y) \in P_n. \quad (26)$$

By using derivative operator D^s in the both sides of (25), we derive

$$\begin{aligned} D^{j-k} q(x) &= (1-n) \frac{n!}{(n-j+k)!} B_{n-j+k}(x+y) \\ &+ (x+y-1) \frac{n!}{(n-j+k-1)!} B_{n-j+k-1}(x+y) \\ &+ (j-k) \frac{n!}{(n-j+k)!} B_{n-j+k}(x+y) \end{aligned} \quad (27)$$

By Theorem 2.1, (26) and (27), we arrive at the following theorem.

Theorem 2.5. *For $k, n \in \mathbb{Z}_+$, we have*

$$\begin{aligned} &\sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(y) \\ &= \sum_{j=k}^n \frac{1}{j!} \sum_{a=0}^k (-1)^a \binom{k}{a} \lambda^a \left\{ (1-n) \frac{n!}{(n-j+k)!} B_{n-j+k}(a+y) \right. \\ &\quad \left. + (a+y-1) \frac{n!}{(n-j+k-1)!} B_{n-j+k-1}(a+y) \right. \\ &\quad \left. + (j-k) \frac{n!}{(n-j+k)!} B_{n-j+k}(a+y) \right\} B_j^{(k)}(x | \lambda). \end{aligned}$$

Dilcher [5] introduced the following interesting identity:

$$\sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) = 2(1-x-y) E_n(x+y) + 2E_{n+1}(x+y).$$

Let $\sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) \in P_n$, then we write that

$$q(x) = \sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y). \quad (28)$$

By (28), we have

$$\begin{aligned}
 D^{j-k}q(x) &= 2\left\{\frac{n!}{(n-j+k)!}(1-x-y)E_{n-j+k}(x+y)\right. \\
 &\quad - (j-k)\frac{n!}{(n-j+k+1)!}E_{n-j+k+1}(x+y) \\
 &\quad \left. + \frac{(n+1)!}{(n+1-j+k)!}E_{n+1-j+k}(x+y)\right\}.
 \end{aligned}$$

As a result of the last identity and Theorem 2.1, we derive the following.

Theorem 2.6. *The following equality holds:*

$$\begin{aligned}
 &\sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) \\
 &= 2 \sum_{j=k}^n \frac{1}{j!} \sum_{a=0}^k (-1)^a \binom{k}{a} \lambda^a \left\{ \frac{n!}{(n-j+k)!} (1-x-y) E_{n-j+k}(x+y) \right. \\
 &\quad - (j-k) \frac{n!}{(n-j+k+1)!} E_{n-j+k+1}(x+y) \\
 &\quad \left. + \frac{(n+1)!}{(n+1-j+k)!} E_{n+1-j+k}(x+y) \right\} B_j^{(k)}(x|\lambda).
 \end{aligned}$$

Remark 2.7. Throughout this paper when we take $\lambda = 1$, our results can easily be related to Bernoulli polynomials of higher order.

Remark 2.8. Theorem 2.1 seems to be plenty large enough for obtaining interesting identities related to special functions in connection with Apostol-Bernoulli polynomials of higher order.

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REFERENCES

- [1] S. Araci, Novel identities for q -Genocchi numbers and polynomials, *Journal of Function Spaces and Applications*, Volume **2012**, Article ID 214961, 13 pages.
- [2] S. Araci and M. Acikgoz, A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, *Advanced Studies in Contemporary Mathematics* 22 (2012), no. 3, 399-406.
- [3] S. Araci, E. Şen, M. Acikgoz, Theorems on Genocchi polynomials of higher order arising from Genocchi basis, *Taiwanese Journal of Math.* (In press).
- [4] A. Bagdasaryan, An elementary and real approach to values of the Riemann zeta function, *Phys. Atom. Nucl.* 73, 251-254 (2010)
- [5] K. Dilcher, Sums of products of Bernoulli numbers, *J. Number Theory*, 60 (1996), 23-41.
- [6] V. Gupta, T. Kim, J. Choi, Y.-H. Kim, Generating functions for q -Bernstein, q -Meyer-König-Zeller and q -Beta basis, *Autom. Comput. Math.* 19(1), 7-11 (2010).

- [7] E. R. Hansen, A table of series and products, *Prentice-Hall Inc.*, Englewood Cliffs, New Jersey, **1975**.
- [8] Y. He and C. Wang Some formulae of products of the Apostol-Bernoulli and Apostol-Euler Polynomials, *Discrete Dynamics in Nature and Society*, vol. **2012**, Article ID 927953, 11 pages.
- [9] H. Jolany and H. Sharifi, Some results for Apostol-Genocchi Polynomials of higher order, *Bull. Malays. Math. Sci. Soc.* 36 (2), 465-479
- [10] T. Kim, Identities involving Frobenius-Euler polynomials arising from non-linear differential equations, *J. Number Theory* 132 (**2012**), no. 12, 2854-865.
- [11] T. Kim, An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p -adic invariant q -integrals on \mathbb{Z}_p , *Rocky Mountain J. Math.* 41 (**2011**), No. 1, 239-247.
- [12] T. Kim, S. H. Rim, D.V. Dolgy, and S. H. Lee, Some identities of Genocchi polynomials arising from Genocchi basis, *Journal of Inequalities and Applications* **2013**, 2013:43
- [13] T. Kim, Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, *Adv. Stud. Contemp. Math.* 20 (**2010**), no. 1, 23-28.
- [14] T. Kim, On the analogs of Euler numbers and polynomials associated with p -adic q -integral on \mathbb{Z}_p at $q = -1$, *J. Math. Anal. Appl.* 331 (**2007**) 779-792.
- [15] D. S. Kim, T. Kim, Some identities of higher-order Euler polynomials arising from Euler basis, *Integral Transforms and Special Functions* 24:9, **2013**, 734-738.
- [16] Q. M. Luo, The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order, *Integral Transforms and Special Functions*, 20:5, **2009**, 377-391.
- [17] N. E. Norlund, Memoire sur les polynomes de Bernoulli, *Acta Math.* 43 (1920), 121-196.
- [18] E. Şen, Theorems on Apostol-Euler polynomials of higher order arising from Euler basis, *Adv. Stud. Contemp. Math.* 23 (**2013**), No. 2, pp. 337-345
- [19] F. Qi, Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind, *Filomat* 28 (**2014**), in press.
- [20] K. Shiratani, On the Euler numbers, *Mem. Fac. Sci., Kyushu Univ.*, Ser. A 27 (**1973**), 1-5.
- [21] J. Choi, P. J. Anderson, and H. M. Srivastava, Some q -extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n , and the multiple Hurwitz zeta function, *Appl. Math. Comput.* 199 (2008), no. 2, 723-737.
- [22] H. M. Srivastava, Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inf. Sci.* 5, 390-444 (2011).

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