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On sums of Apéry polynomials and related congruences <sup>☆</sup>

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## ABSTRACT

The Apéry polynomials are given by

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \quad (n = 0, 1, 2, \dots).$$

(Those  $A_n = A_n(1)$  are Apéry numbers.) Let  $p$  be an odd prime. We show that

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2},$$

and that

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}$$

for any  $p$ -adic integer  $x \not\equiv 0 \pmod{p}$ . This enables us to determine explicitly  $\sum_{k=0}^{p-1} (\pm 1)^k A_k \pmod{p}$ , and  $\sum_{k=0}^{p-1} (-1)^k A_k \pmod{p^2}$  in the case  $p \equiv 2 \pmod{3}$ . Another consequence states that

$$\begin{aligned} & \sum_{k=0}^{p-1} (-1)^k A_k(-2) \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

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We also prove that for any prime  $p > 3$  we have

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p + \frac{7}{6}p^4B_{p-3} \pmod{p^5}$$

where  $B_0, B_1, B_2, \dots$  are Bernoulli numbers.

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### 1. Introduction

The well-known Apéry numbers given by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, 2, \dots\})$$

play a central role in Apéry's proof of the irrationality of  $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$  (see Apéry [Ap] and van der Poorten [Po]). They also have close connections to modular forms (cf. Ono [O, pp. 198–203]). The Dedekind eta function in the theory of modular forms is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau},$$

where  $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$  and hence  $|q| < 1$ . In 1987 Beukers [B] conjectured that

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2} \quad \text{for any prime } p > 3,$$

where  $a(n)$  ( $n = 1, 2, 3, \dots$ ) are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.$$

This was finally confirmed by Ahlgren and Ono [AO] in 2000.

We define Apéry polynomials by

$$A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 x^k = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 x^k \quad (n \in \mathbb{N}). \tag{1.1}$$

Clearly  $A_n(1) = A_n$ . Motivated by the Apéry polynomials, we also introduce a new kind of polynomials:

$$W_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{n-k}{k}^2 x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 \binom{2k}{k}^2 x^k \quad (n \in \mathbb{N}). \tag{1.2}$$

Recall that Bernoulli numbers  $B_0, B_1, B_2, \dots$  are rational numbers given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^n \binom{n+1}{k} B_k = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

It is well known that  $B_{2n+1} = 0$  for all  $n \in \mathbb{Z}^+$  and

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

Also, Euler numbers  $E_0, E_1, E_2, \dots$  are integers defined by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+.$$

It is well known that  $E_{2n+1} = 0$  for all  $n \in \mathbb{N}$  and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \left( |x| < \frac{\pi}{2} \right).$$

Now we state our first theorem.

**Theorem 1.1.**

(i) Let  $p$  be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} (-1)^k W_k(-x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}. \tag{1.3}$$

Also, for any  $p$ -adic integer  $x \not\equiv 0 \pmod{p}$ , we have

$$\begin{aligned} \sum_{k=0}^{p-1} A_k(x) &\equiv \sum_{k=0}^{p-1} W_k(x) \pmod{p^2} \\ &\equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}, \end{aligned} \tag{1.4}$$

where  $(-)$  denotes the Legendre symbol.

(ii) For any positive integer  $n$  we have

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k. \tag{1.5}$$

If  $p > 3$  is a prime, then

$$\sum_{k=0}^{p-1} (2k + 1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5} \tag{1.6}$$

and

$$\sum_{k=0}^{p-1} (2k + 1)A_k(-1)^k \equiv \left(\frac{-1}{p}\right)p - p^3 E_{p-3} \pmod{p^4}. \tag{1.7}$$

(iii) Given  $\varepsilon \in \{\pm 1\}$  and  $m \in \mathbb{Z}^+$ , for any prime  $p$  we have

$$\sum_{k=0}^{p-1} (2k + 1)\varepsilon^k A_k^m \equiv 0 \pmod{p}.$$

**Remark 1.1.** (i) Let  $p$  be an odd prime. The author [Su1,Su2] had conjectures on  $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / m^k \pmod{p^2}$  with  $m = 1, -8, 16, -64, 256, -512, 4096$ . Motivated by the author’s conjectures on  $\sum_{k=0}^{p-1} A_k(x) \pmod{p^2}$  with  $x = 1, -4, 9$  in an initial version of this paper, Guo and Zeng [GZ, Theorem 1.3] recently showed that

$$\sum_{k=0}^{p-1} A_k(x) \equiv \sum_{k=0}^{(p-1)/2} \binom{p+2k}{4k+1} \binom{2k}{k}^2 x^k \pmod{p^2}.$$

(ii) The values of

$$s_n = \frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)A_k \in \mathbb{Z}$$

with  $n = 1, \dots, 8$  are

$$1, 8, 127, 2624, 61501, 1552760, 41186755, 1131614720$$

respectively. On June 6, 2011 Richard Penner informed the author an interesting application of (1.5): (1.5) with  $x = 1$  implies that  $s_n$  is the trace of the inverse of  $nH_n$  where  $H_n$  refers to the Hilbert matrix  $(\frac{1}{i+j-1})_{1 \leq i, j \leq n}$ .

Can we find integers  $a_0, a_1, a_2, \dots$  such that  $\sum_{k=0}^{p-1} a_k \equiv 4x^2 - 2p \pmod{p^2}$  if  $p = x^2 + y^2$  is a prime with  $x$  odd and  $y$  even? The following corollary provides an affirmative answer!

**Corollary 1.1.** Let  $p$  be any odd prime. Then

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k A_k(-2) &\equiv \sum_{k=0}^{p-1} (-1)^k A_k\left(\frac{1}{4}\right) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \text{ (} 2 \nmid x \text{),} \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned} \tag{1.8}$$

**Proof.** It is known (cf. Ishikawa [I]) that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4} \text{ \& } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

The author conjectured that we can replace  $64^k$  by  $(-8)^k$  in the congruence, and this was recently confirmed by Z.H. Sun [S3]. So, applying (1.3) with  $x = -2, 1/4$  we obtain (1.8).  $\square$

**Corollary 1.2.** Let  $p$  be an odd prime. Then

$$\sum_{k=0}^{p-1} A_k \equiv c(p) \pmod{p} \tag{1.9}$$

where

$$c(p) := \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1, 3 \pmod{8} \text{ \& } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 & \text{if } \left(\frac{-2}{p}\right) = -1, \text{ i.e., } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Also,

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k A_k &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} (-1)^k A_k \left(\frac{1}{16}\right) \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned} \tag{1.10}$$

**Proof.** By [M05] and [Su4], we have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{256^k} \equiv c(p) \pmod{p^2}$$

as conjectured in [RV]. (Here we only need the mod  $p$  version which was proved in [M05].) So (1.9) follows from (1.4). The author [Su2] conjectured that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} &\equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{256^k} \\ &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

This was confirmed by Z.H. Sun [S3] in the case  $p \equiv 2 \pmod{3}$ , and the mod  $p$  version in the case  $p \equiv 1 \pmod{3}$  follows from (4)–(5) in Ahlgren [A, Theorem 5]. So we get (1.10) by applying (1.3) with  $x = 1, 1/16$ .  $\square$

**Remark 1.2.** The author conjectured that (1.9) also holds modulo  $p^2$ , and that (1.10) is also valid modulo  $p^2$  in the case  $p \equiv 1 \pmod{3}$ .

**Corollary 1.3.** For any odd prime  $p$  and integer  $x$ , we have

$$\sum_{k=0}^{p-1} (2k + 1)A_k(x) \equiv p \binom{x}{p} \pmod{p^2}. \tag{1.11}$$

**Proof.** This follows from (1.5) in the case  $n = p$ , for,  $p \mid \binom{p+k}{2k+1}$  for every  $k = 0, \dots, (p - 3)/2$ , and  $p \mid \binom{2k}{k}$  for all  $k = (p + 1)/2, \dots, p - 1$ .  $\square$

We deduce Theorem 1.1(i) from our following result which has its own interest.

**Theorem 1.2.** Let  $p$  be an odd prime and let  $x$  be any  $p$ -adic integer.

(i) If  $x \equiv 2k \pmod{p}$  with  $k \in \{0, \dots, (p - 1)/2\}$ , then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}. \tag{1.12}$$

(ii) If  $x \equiv k \pmod{p}$  with  $k \in \{0, \dots, p - 1\}$ , then

$$\sum_{r=0}^{p-1} \binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}. \tag{1.13}$$

**Remark 1.3.** In contrast with (1.12) and (1.13), we recall the following identities (cf. [G, (3.32) and (3.66)]):

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

**Corollary 1.4.** Let  $p$  be an odd prime.

(i) (Conjectured in [RV] and proved in [M03].) We have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

(ii) (Conjectured by the author [Su1] and confirmed in [S2].) If  $p \equiv 1 \pmod{4}$  and  $p = x^2 + y^2$  with  $x \equiv 1 \pmod{4}$  and  $y \equiv 0 \pmod{2}$ , then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{1.14}$$

**Proof.** Since  $\binom{-1/2}{r} = \binom{2r}{r}/(-4)^r$  for all  $r = 0, 1, \dots$ , applying (1.13) with  $x = -1/2$  and  $k = (p - 1)/2$  we immediately get the congruence in part (i).

When  $p = x^2 + y^2$  with  $x \equiv 1 \pmod{4}$  and  $y \equiv 0 \pmod{2}$ , by (1.12) with  $x = -1/2$  and  $k = (p - 1)/4$  we have

$$\begin{aligned} \sum_{r=0}^{p-1} \frac{\binom{2r}{r}^2}{(-16)^r} &\equiv (-1)^{(p-1)/4} \binom{-1/2}{(p-1)/4} = \frac{\binom{(p-1)/2}{(p-1)/4}}{4^{(p-1)/4}} = \frac{\binom{(p-1)/2}{(p-1)/4}}{2^{(p-1)/2}} \\ &\equiv \frac{2^{p-1} + 1}{2 \times 2^{(p-1)/2}} \left(2x - \frac{p}{2x}\right) \pmod{p^2} \quad (\text{by [CDE] or [BEW, (9.0.2)]}) \\ &\equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2} \end{aligned}$$

since  $((-1)^{(p-1)/4} 2^{(p-1)/2} - 1)^2 \equiv 0 \pmod{p^2}$ . This proves (1.14).  $\square$

**Corollary 1.5.** Let  $a_n := \sum_{k=0}^n \binom{n}{k}^2 C_k$  for  $n = 0, 1, 2, \dots$ , where  $C_k$  denotes the Catalan number  $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$ . Then, for any odd prime  $p$  we have

$$a_1 + \dots + a_{p-1} \equiv 0 \pmod{p^2}. \tag{1.15}$$

**Remark 1.4.** We find no prime  $p \leq 5000$  with  $\sum_{k=1}^{p-1} a_k \equiv 0 \pmod{p^3}$  and no composite number  $n \leq 70\,000$  satisfying  $\sum_{k=1}^{n-1} a_k \equiv 0 \pmod{n^2}$ . We conjecture that (1.15) holds for no composite  $p > 1$ .

The author [Su1, Remark 1.2] conjectured that for any prime  $p > 5$  with  $p \equiv 1 \pmod{4}$  we have

$$\sum_{k=0}^{p^a-1} \frac{k^3 \binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^{2a}} \quad \text{for } a = 1, 2, 3, \dots$$

This was recently confirmed by Z.H. Sun [S3] in the case  $a = 1$ . Note that

$$\frac{k^3 \binom{2k}{k}^3}{64^k} = (-1)^k k^3 \binom{-1/2}{k}^3 = \frac{(-1)^{k-1}}{8} \binom{-3/2}{k-1}^3 \quad \text{for all } k = 1, 2, 3, \dots$$

So, for any prime  $p > 5$  with  $p \equiv 1 \pmod{4}$  we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{-3/2}{r}^3 \equiv 0 \pmod{p^2}.$$

Since  $-3/2 \equiv -2(p+3)/4 \pmod{p}$ , the result just corresponds to the case  $x = -3/2$  of our following general theorem.

**Theorem 1.3.** Let  $p > 3$  be a prime and let  $x$  be a  $p$ -adic integer with  $x \equiv -2k \pmod{p}$  for some  $k \in \{1, \dots, \lfloor (p-1)/3 \rfloor\}$ . Then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^3 \equiv 0 \pmod{p^2}. \tag{1.16}$$

Similar to Apéry numbers, the central Delannoy numbers (see [CHV]) are defined by

$$D_n = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \quad (n \in \mathbb{N}).$$

Such numbers arise naturally in many enumeration problems in combinatorics (cf. Sloane [S]); for example,  $D_n$  is the number of lattice paths from  $(0, 0)$  to  $(n, n)$  with steps  $(1, 0)$ ,  $(0, 1)$  and  $(1, 1)$ .

Now we give our result on central Delannoy numbers.

**Theorem 1.4.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{p-1} D_k \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}. \tag{1.17}$$

We also have

$$\sum_{k=0}^{p-1} (2k + 1)(-1)^k D_k \equiv p - \frac{7}{12} p^4 B_{p-3} \pmod{p^5} \tag{1.18}$$

and

$$\sum_{k=0}^{p-1} (2k + 1) D_k \equiv p + 2p^2 q_p(2) - p^3 q_p(2)^2 \pmod{p^4}, \tag{1.19}$$

where  $q_p(2)$  denotes the Fermat quotient  $(2^{p-1} - 1)/p$ .

**Remark 1.5.** In [Su3] the author determined  $\sum_{k=1}^{p-1} D_k/k$  and  $\sum_{k=1}^{p-1} D_k/k^2$  modulo an odd prime  $p$ .

In the next section we will show Theorems 1.1–1.2 and Corollary 1.5. Section 3 is devoted to our proofs of Theorems 1.3 and 1.4. In Section 4 we are going to raise some related conjectures for further research.

## 2. Proofs of Theorems 1.1–1.2 and Corollary 1.5

We first prove Theorem 1.2.

**Proof of Theorem 1.2.** (i) We now consider the first part of Theorem 1.2. Set

$$f_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{2k + py}{r}^2 \text{ for } k \in \mathbb{N}. \tag{2.1}$$

We want to prove that

$$f_k(y) \equiv (-1)^k \binom{2k + py}{k} \pmod{p^2} \tag{2.2}$$

for any  $p$ -adic integer  $y$  and  $k \in \{0, 1, \dots, (p - 1)/2\}$ .

Applying the Zeilberger algorithm (cf. [PWZ]) via Mathematica 7, we find that

$$\begin{aligned} & (py + 2k + 2)f_{k+1}(y) + 4(py + 2k + 1)f_k(y) \\ &= \frac{(p(y - 1) + 2k + 3)^2 F_k(y)}{(py + 2k + 1)(py + 2k + 2)^2} \binom{py + 2k + 2}{p - 1}^2, \end{aligned} \tag{2.3}$$



where

$$F_k(y) = 14 + 34k + 20k^2 - 10p - 12kp + 2p^2 + 17py + 20kpy - 6p^2y + 5p^2y^2.$$

Now fix a  $p$ -adic integer  $y$ . Observe that

$$\begin{aligned} f_{(p-1)/2}(y) &= \sum_{r=0}^{p-1} (-1)^r \binom{p-1+py}{r}^2 = \sum_{r=0}^{p-1} (-1)^r \prod_{0 < s \leq r} \left(1 - \frac{p(y+1)}{s}\right)^2 \\ &\equiv \sum_{r=0}^{p-1} (-1)^r \left(1 - \sum_{0 < s \leq r} \frac{2p(y+1)}{s}\right) = 1 - \sum_{r=1}^{p-1} (-1)^r \sum_{s=1}^r \frac{2p(y+1)}{s} \\ &= 1 - 2p(y+1) \sum_{s=1}^{p-1} \frac{1}{s} \sum_{r=s}^{p-1} (-1)^r = 1 - p(y+1) \sum_{j=1}^{(p-1)/2} \frac{1}{j} \\ &\equiv (-1)^{(p-1)/2} \binom{p-1+py}{(p-1)/2} \pmod{p^2}. \end{aligned}$$

For each  $k \in \{0, \dots, (p-3)/2\}$ , clearly  $py + 2k + 1, py + 2k + 2 \not\equiv 0 \pmod{p}$ , and also

$$(p(y-1) + 2k + 3) \binom{py + 2k + 2}{p-1}^2 \equiv 0 \pmod{p^2}$$

since  $\binom{py+2k+2}{p-1} = \frac{p}{py+2k+3} \binom{py+2k+3}{p} \equiv 0 \pmod{p}$  if  $0 \leq k < (p-3)/2$ . Thus, by (2.3) we have

$$f_k(y) \equiv -\frac{py + 2k + 2}{4(py + 2k + 1)} f_{k+1}(y) \pmod{p^2} \quad \text{for } k = 0, \dots, \frac{p-3}{2}.$$

If  $0 \leq k < (p-1)/2$  and

$$f_{k+1}(y) \equiv (-1)^{k+1} \binom{2(k+1)+py}{k+1} \pmod{p^2},$$

then

$$\begin{aligned} f_k(y) &\equiv -\frac{py + 2k + 2}{4(py + 2k + 1)} (-1)^{k+1} \binom{2(k+1)+py}{k+1} \\ &= \frac{(-1)^k (py + 2k + 2)^2}{4(k+1)(py + k + 1)} \binom{2k+py}{k} \equiv (-1)^k \binom{2k+py}{k} \pmod{p^2}. \end{aligned}$$

Therefore (2.2) holds for all  $k = 0, 1, \dots, (p-1)/2$ . This proves Theorem 1.2(i).

(ii) The second part of Theorem 1.2 can be proved in a similar way. Here we mention that if we define

$$g_k(y) := \sum_{r=0}^{p-1} \binom{k+py}{r}^2 \quad \text{for } k \in \mathbb{N}, \tag{2.4}$$

then by the Zeilberger algorithm (cf. [PWZ]) we have the recursion

$$\begin{aligned} & (py + k + 1)g_{k+1}(y) - 2(2py + 2k + 1)g_k(y) \\ &= -\frac{(p(y - 1) + k + 2)^2(3py - 2p + 3k + 3)}{(py + k + 1)^2} \binom{py + k + 1}{p - 1}^2. \end{aligned}$$

It follows that if  $k \in \{0, \dots, p - 2\}$  and  $y$  is a  $p$ -adic integer then

$$\begin{aligned} g_{k+1}(y) &\equiv \binom{2(k + 1) + 2py}{k + 1} \pmod{p^2} \\ \implies g_k(y) &\equiv \binom{2k + 2py}{k} \pmod{p^2}. \end{aligned} \tag{2.5}$$

In view of this, we have the second part of Theorem 1.2 by induction. The proof of Theorem 1.2 is now complete.  $\square$

**Proof of Corollary 1.5.** Observe that

$$\sum_{n=0}^{p-1} a_n = \sum_{k=0}^{p-1} C_k \sum_{n=k}^{p-1} \binom{n}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1-k} \binom{k + j}{k}^2.$$

If  $0 \leq k \leq p - 1$  and  $p - k \leq j \leq p - 1$ , then

$$\binom{k + j}{k} = \frac{(k + j)!}{k!j!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1} \binom{k + j}{k}^2 = \sum_{k=0}^{p-1} C_k \sum_{j=0}^{p-1} \binom{x_k}{j}^2,$$

where  $x_k = -k - 1 \equiv p - 1 - k \pmod{p}$ . Applying Theorem 1.2(ii) we get

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \binom{2x_k}{p - 1 - k} = \sum_{k=0}^{p-1} (-1)^k \binom{p + k}{2k + 1} C_k \pmod{p^2}.$$

So it suffices to show that for any  $n \in \mathbb{Z}^+$  we have

$$\sum_{k=0}^{n-1} (-1)^k \binom{n + k}{2k + 1} C_k = 1. \tag{2.6}$$

We prove (2.6) by induction. Clearly, (2.6) holds for  $n = 1$ . Let  $n$  be any positive integer. By the Chu–Vandermonde identity

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n - k} = \binom{x + y}{n}$$

(see, e.g., [GKP, p. 169]), we have

$$\sum_{k=0}^{n-1} \binom{n+1}{k+1} \binom{n+k}{k} (-1)^k = \sum_{k=0}^{n-1} \binom{n+1}{n-k} \binom{-n-1}{k} = -\binom{-n-1}{n}.$$

Thus

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n+1+k}{2k+1} C_k - \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k &= (-1)^n C_n + \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k} C_k \\ &= (-1)^n C_n + \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k+1} \binom{n+k}{k} (-1)^k \\ &= (-1)^n C_n - \frac{1}{n+1} \binom{-n-1}{n} = 0. \end{aligned}$$

This concludes the induction step. We are done.  $\square$

Now we can apply Theorem 1.2 to deduce the first part of Theorem 1.1.

**Proof of Theorem 1.1(i).** Let  $\varepsilon \in \{\pm 1\}$ . Then

$$\begin{aligned} \sum_{m=0}^{p-1} \varepsilon^m A_m(x) &= \sum_{m=0}^{p-1} \varepsilon^m \sum_{k=0}^m \binom{m+k}{2k}^2 \binom{2k}{k}^2 x^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 x^k \sum_{m=k}^{p-1} \varepsilon^m \binom{m+k}{2k}^2 \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 x^k \sum_{r=0}^{p-1-k} \varepsilon^{k+r} \binom{2k+r}{r}^2 \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1-k} \varepsilon^r \binom{p-1-2k-p}{r}^2. \end{aligned}$$

Set  $n = (p - 1)/2$ . Clearly  $\binom{2k}{k} \equiv 0 \pmod{p}$  for  $k = n + 1, \dots, p - 1$ , and

$$\binom{p-1-2k-p}{r} \equiv \binom{p-1-2k}{r} = 0 \pmod{p}$$

if  $0 \leq k \leq n$  and  $p - 1 - 2k < r \leq p - 1$ . Therefore

$$\sum_{m=0}^{p-1} \varepsilon^m A_m(x) \equiv \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1} \varepsilon^r \binom{2(n-k)-p}{r}^2 \pmod{p^2}.$$

Similarly,

$$\begin{aligned}
 \sum_{m=0}^{p-1} \varepsilon^m W_m(\varepsilon x) &= \sum_{m=0}^{p-1} \varepsilon^m \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k}^2 (2k)^2 (\varepsilon x)^k \\
 &= \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{m=2k}^{p-1} \varepsilon^m \binom{m}{2k}^2 \\
 &= \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1-2k} \varepsilon^{2k+r} \binom{2k+r}{r}^2 \\
 &\equiv \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k \sum_{r=0}^{p-1} \varepsilon^r \binom{2(n-k)-p}{r}^2 \pmod{p^2}.
 \end{aligned}$$

So we have

$$\sum_{m=0}^{p-1} \varepsilon^m A_m(x) \equiv \sum_{m=0}^{p-1} \varepsilon^m W_m(\varepsilon x) \equiv \sum_{k=0}^n \binom{2k}{k}^2 \varepsilon^k x^k S_k(\varepsilon) \pmod{p^2}, \tag{2.7}$$

where

$$S_k(\varepsilon) := \sum_{r=0}^{p-1} \varepsilon^r \binom{2(n-k)-p}{r}^2.$$

Applying Theorem 1.2(i) we get

$$\begin{aligned}
 S_k(-1) &\equiv (-1)^{n-k} \binom{2(n-k)-p}{n-k} = (-1)^{n-k} \binom{-2k-1}{n-k} \\
 &= \binom{n+k}{n-k} = \binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.
 \end{aligned}$$

(The last congruence can be easily deduced, see, e.g., [S2, Lemma 2.2].) Combining this with (2.7) in the case  $\varepsilon = -1$  we immediately obtain (1.3).

In view of Theorem 1.2(ii),

$$S_k(1) \equiv \binom{4(n-k)-2p}{2(n-k)} \pmod{p^2}.$$

Recall that  $\binom{n+k}{n-k}(-16)^k \equiv \binom{2k}{k} \pmod{p^2}$ . So, in view of (2.7) with  $\varepsilon = 1$ , we have

$$\begin{aligned}
 \sum_{m=0}^{p-1} A_m(x) &\equiv \sum_{m=0}^{p-1} W_m(x) \equiv \sum_{k=0}^n \binom{n+k}{n-k}^2 (-16)^{2k} x^k \binom{4(n-k)-2p}{2(n-k)} \\
 &= \sum_{j=0}^n \binom{n+(n-j)}{j}^2 256^{n-j} x^{n-j} \binom{4j-2p}{2j} \\
 &= 16^{p-1} \sum_{k=0}^n \frac{\binom{4k-2p}{2k} \binom{2k-p}{k}^2}{256^k} x^{n-k} \pmod{p^2}.
 \end{aligned}$$

If  $x$  is a  $p$ -adic integer with  $x \not\equiv 0 \pmod{p}$ , then

$$16^{p-1} \sum_{k=0}^n \frac{\binom{4k-2p}{2k} \binom{2k-p}{k}^2}{256^k} x^{n-k} \equiv \left(\frac{x}{p}\right) \sum_{k=0}^n \frac{\binom{4k}{2k} \binom{2k}{k}^2}{(256x)^k} \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p},$$

and therefore (1.4) holds.  $\square$

**Lemma 2.1.** *Let  $k \in \mathbb{N}$ . Then, for any  $n \in \mathbb{Z}^+$  we have the identity*

$$\sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k}^2 = \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2. \tag{2.8}$$

**Proof.** Obviously (2.8) holds when  $n = 1$ .

Now assume that  $n > 1$  and (2.8) holds. Then

$$\begin{aligned} \sum_{m=0}^n (2m+1) \binom{m+k}{2k}^2 &= \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2 + (2n+1) \binom{n+k}{2k}^2 \\ &= \frac{(n+k+1)^2}{2k+1} \binom{n+k}{2k}^2 = \frac{(n+1-k)^2}{2k+1} \binom{(n+1)+k}{2k}^2. \end{aligned}$$

Combining the above, we have proved the desired result by induction.  $\square$

**Lemma 2.2.** *Let  $p > 3$  be a prime. Then*

$$\sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{(-1)^k}{2k+1} \equiv -pE_{p-3} \pmod{p^2}. \tag{2.9}$$

**Proof.** Observe that

$$\begin{aligned} \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{(-1)^k}{2k+1} &= \frac{1}{2} \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \left( \frac{(-1)^k}{2k+1} + \frac{(-1)^{p-1-k}}{(2(p-1-k)+1)} \right) \\ &= -p \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{(-1)^k}{(2k+1)(2k+1-2p)} \\ &\equiv -\frac{p}{4} \sum_{k=0}^{p-1} (-1)^k \left(k + \frac{1}{2}\right)^{p-3} \pmod{p^2}. \end{aligned}$$

So we have reduced (2.9) to the following congruence

$$\sum_{k=0}^{p-1} (-1)^k \left(k + \frac{1}{2}\right)^{p-3} \equiv 4E_{p-3} \pmod{p}. \tag{2.10}$$

Recall that the Euler polynomial of degree  $n$  is defined by

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

It is well known that

$$E_n(x) + E_n(x + 1) = 2x^n.$$

Thus

$$\begin{aligned} 2 \sum_{k=0}^{p-1} (-1)^k \left(k + \frac{1}{2}\right)^{p-3} &= \sum_{k=0}^{p-1} \left( (-1)^k E_{p-3} \left(k + \frac{1}{2}\right) - (-1)^{k+1} E_{p-3} \left(k + 1 + \frac{1}{2}\right) \right) \\ &= E_{p-3} \left(\frac{1}{2}\right) - (-1)^p E_{p-3} \left(p + \frac{1}{2}\right) \\ &\equiv 2E_{p-3} \left(\frac{1}{2}\right) = 2 \frac{E_{p-3}}{2^{p-3}} \equiv 8E_{p-3} \pmod{p} \end{aligned}$$

and hence (2.10) follows. We are done.  $\square$

For each  $m = 1, 2, 3, \dots$  those rational numbers

$$H_n^{(m)} := \sum_{0 < k \leq n} \frac{1}{k^m} \quad (n = 0, 1, 2, \dots)$$

are called harmonic numbers of order  $m$ . We simply write  $H_n$  for  $H_n^{(1)}$ . A well-known theorem of Wolstenholme asserts that  $H_{p-1} \equiv 0 \pmod{p^2}$  and  $H_{p-1}^{(2)} \equiv 0 \pmod{p}$  for any prime  $p > 3$ .

**Lemma 2.3.** *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} \equiv -\frac{7}{4} B_{p-3} \pmod{p}. \tag{2.11}$$

**Proof.** Clearly,

$$\sum_{k=1}^{p-1} \frac{1}{k^3} = \sum_{k=1}^{(p-1)/2} \left( \frac{1}{k^3} + \frac{1}{(p-k)^3} \right) \equiv 0 \pmod{p}.$$

By [ST, (5.4)],  $\sum_{k=1}^{p-1} H_k/k^2 \equiv B_{p-3} \pmod{p}$ . Therefore

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} &= \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^k \frac{1}{j^2} = \sum_{j=1}^{p-1} \frac{H_{p-1} - H_{j-1}}{j^2} \\ &\equiv -\sum_{k=1}^{p-1} \frac{H_k}{k^2} + \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv -B_{p-3} \pmod{p}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} &= \sum_{k=1}^{(p-1)/2} \left( \frac{H_k^{(2)}}{k} + \frac{H_{p-k}^{(2)}}{p-k} \right) \\ &\equiv \sum_{k=1}^{(p-1)/2} \left( \frac{H_k^{(2)}}{k} + \frac{H_{p-1}^{(2)} - H_{k-1}^{(2)}}{-k} \right) \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} - H_{(p-1)/2}^{(3)} \pmod{p}. \end{aligned}$$

It is known (see, e.g., [S1, Corollary 5.2]) that

$$H_{(p-1)/2}^{(3)} = \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \equiv -2B_{p-3} \pmod{p}.$$

So we have

$$\sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \equiv \frac{1}{2} \left( \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} + H_{(p-1)/2}^{(3)} \right) \equiv \frac{-B_{p-3} - 2B_{p-3}}{2} = -\frac{3}{2}B_{p-3} \pmod{p}.$$

Clearly

$$H_{(p-1)/2}^{(2)} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left( \frac{1}{k^2} + \frac{1}{(p-k)^2} \right) = \frac{1}{2}H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

Observe that

$$\begin{aligned} \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} &\equiv - \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{p-1-2k} = - \sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}^{(2)}}{2k} \\ &\equiv -\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left( H_{(p-1)/2}^{(2)} - \sum_{j=0}^{k-1} \frac{1}{((p-1)/2-j)^2} \right) \\ &\equiv 2 \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{(2j+1)^2} \equiv 2 \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left( H_{2k}^{(2)} - \sum_{j=1}^k \frac{1}{(2j)^2} \right) \\ &= 4 \sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{2k} - \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \pmod{p} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{2k} &= \sum_{\substack{k=1 \\ 2|k}}^{p-1} \frac{1}{k} \sum_{j=1}^k \frac{1}{j^2} = \sum_{\substack{k=1 \\ 2|k}}^{p-1} \frac{1}{k^3} + \sum_{\substack{1 \leq j < k \leq p-1 \\ 2|k}} \frac{1}{j^2 k} \\ &\equiv \frac{1}{8}H_{(p-1)/2}^{(3)} - \frac{3}{8}B_{p-3} \pmod{p} \text{ (by Pan [P, (2.4)])} \end{aligned}$$

$$\equiv \frac{1}{8}(-2B_{p-3}) - \frac{3}{8}B_{p-3} = -\frac{5}{8}B_{p-3} \pmod{p}.$$

So we finally get

$$\sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} \equiv 4\left(-\frac{5}{8}B_{p-3}\right) - \frac{1}{2}\left(-\frac{3}{2}B_{p-3}\right) = -\frac{7}{4}B_{p-3} \pmod{p}.$$

This concludes the proof of (2.11).  $\square$

**Proof of Theorem 1.1(ii).** (i) Let  $n$  be any positive integer. Then

$$\begin{aligned} \sum_{m=0}^{n-1} (2m+1)A_m(x) &= \sum_{m=0}^{n-1} (2m+1) \sum_{k=0}^m \binom{m+k}{2k}^2 \binom{2k}{k}^2 x^k \\ &= \sum_{k=0}^{n-1} \binom{2k}{k}^2 x^k \sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k}^2 \\ &= \sum_{k=0}^{n-1} \binom{2k}{k}^2 x^k \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2 \quad (\text{by (2.8)}) \\ &= \sum_{k=0}^{n-1} \frac{(n-k)^2}{2k+1} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k. \end{aligned}$$

Since

$$(n-k) \binom{n}{k} = n \binom{n-1}{k} \quad \text{for all } k = 0, \dots, n-1,$$

we have

$$\begin{aligned} \frac{1}{n} \sum_{m=0}^{n-1} (2m+1)A_m(x) &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n-k}{2k+1} \binom{n}{k} \binom{n+k}{k}^2 x^k \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n-k}{2k+1} \binom{n+k}{2k} \binom{2k}{k} \binom{n+k}{k} x^k \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k. \end{aligned}$$

This proves (1.5).

Now we fix a prime  $p > 3$ . By the above,

$$\sum_{m=0}^{p-1} (2m+1)A_m(x) = \sum_{k=0}^{p-1} \frac{p^2}{2k+1} \binom{p-1}{k}^2 \binom{p+k}{k}^2 x^k. \tag{2.12}$$

For  $k \in \{0, \dots, p-1\}$ , clearly



$$\begin{aligned} \binom{p-1}{k}^2 \binom{p+k}{k}^2 &= \prod_{0 < j \leq k} \left( \frac{p-j}{j} \cdot \frac{p+j}{j} \right)^2 = \prod_{0 < j \leq k} \left( 1 - \frac{p^2}{j^2} \right)^2 \\ &\equiv \prod_{0 < j \leq k} \left( 1 - \frac{2p^2}{j^2} \right) \equiv 1 - 2p^2 H_k^{(2)} \pmod{p^4}. \end{aligned}$$

Thus (2.12) implies that

$$\sum_{m=0}^{p-1} (2m+1)A_m(x) \equiv \sum_{k=0}^{p-1} \frac{p^2}{2k+1} (1 - 2p^2 H_k^{(2)}) x^k \pmod{p^5}. \tag{2.13}$$

Since  $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$ , taking  $x = -1$  in (2.13) and applying (2.9) we obtain

$$\sum_{m=0}^{p-1} (2m+1)A_m(-1) \equiv \sum_{k=0}^{p-1} \frac{p^2(-1)^k}{2k+1} \equiv \frac{p^2(-1)^{(p-1)/2}}{2(p-1)/2+1} - p^3 E_{p-3} \pmod{p^4}$$

and hence (1.7) holds.

Now we prove (1.6). In view of (2.13) with  $x = 1$ , we have

$$\begin{aligned} \sum_{m=0}^{p-1} (2m+1)A_m &\equiv \frac{p^2}{2(p-1)/2+1} (1 - 2p^2 H_{(p-1)/2}^{(2)}) \\ &\quad + p^2 \sum_{k=0}^{(p-3)/2} \left( \frac{1 - 2p^2 H_k^{(2)}}{2k+1} + \frac{1 - 2p^2 H_{p-1-k}^{(2)}}{2(p-1-k)+1} \right) \\ &= p - 2p^3 H_{(p-1)/2}^{(2)} + 2p^3 \sum_{k=0}^{(p-3)/2} \frac{2p+2k+1}{(2k+1)(4p^2 - (2k+1)^2)} \\ &\quad - 2p^4 \sum_{k=0}^{(p-3)/2} \left( \frac{H_k^{(2)}}{2k+1} + \frac{H_{p-1}^{(2)} - \sum_{0 < j \leq k} (p-j)^{-2}}{2p - (2k+1)} \right) \\ &\equiv p - 2p^3 H_{(p-1)/2}^{(2)} - 4p^4 \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^3} \\ &\quad - 2p^3 \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^2} - 4p^4 \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} \pmod{p^5}. \end{aligned}$$

By [S1, Corollaries 5.1 and 5.2],

$$\begin{aligned} H_{p-1}^{(2)} &\equiv \frac{2}{3} p B_{p-3} \pmod{p^2}, & H_{(p-1)/2}^{(2)} &\equiv \frac{7}{3} p B_{p-3} \pmod{p^2}, \\ \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^2} &= H_{p-1}^{(2)} - \frac{H_{(p-1)/2}^{(2)}}{4} \equiv \frac{p}{12} B_{p-3} \pmod{p^2}, \end{aligned}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^3} = H_{p-1}^{(3)} - \frac{H_{(p-1)/2}^{(3)}}{8} \equiv 0 - \frac{-2B_{p-3}}{8} = \frac{B_{p-3}}{4} \pmod{p}.$$

Combining these with Lemma 2.3, we finally obtain

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)A_k &\equiv p - 2p^3 \frac{7}{3} p B_{p-3} - 4p^4 \frac{B_{p-3}}{4} - 2p^3 \frac{p}{12} B_{p-3} - 4p^4 \left(-\frac{7}{4} B_{p-3}\right) \\ &= p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5}. \end{aligned}$$

So far we have proved the second part of Theorem 1.1.  $\square$

Part (iii) of Theorem 1.1 is easy.

**Proof of Theorem 1.1(iii).** As  $A_0 = 1$  and  $A_1 = 3$ , the desired congruence with  $p = 2$  holds trivially. Below we assume that  $p$  is an odd prime. If  $k \in \{0, 1, \dots, p-1\}$ , then

$$\begin{aligned} A_{p-1-k} &= \sum_{j=0}^{p-1} \binom{(p-1-k)+j}{2j}^2 \binom{2j}{j}^2 \\ &\equiv \sum_{j=0}^{p-1} \binom{j-k-1}{2j}^2 \binom{2j}{j}^2 = \sum_{j=0}^k \binom{j+k}{2j}^2 \binom{2j}{j}^2 = A_k \pmod{p}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)\varepsilon^k A_k^m &= \sum_{k=0}^{p-1} (2(p-1-k)+1)\varepsilon^{p-1-k} A_{p-1-k}^m \\ &\equiv - \sum_{k=0}^{p-1} (2k+1)\varepsilon^k A_k^m \pmod{p} \end{aligned}$$

and hence we have the desired congruence.  $\square$

### 3. Proofs of Theorems 1.3 and 1.4

**Proof of Theorem 1.3.** Define

$$w_k(y) := \sum_{r=0}^{p-1} (-1)^r \binom{py-2k}{r}^3 \quad \text{for } k \in \mathbb{N}. \tag{3.1}$$

We want to show that  $w_k(y) \equiv 0 \pmod{p^2}$  for any  $p$ -adic integer  $y$  and  $k \in \{1, \dots, \lfloor (p-1)/3 \rfloor\}$ .

By the Zeilberger algorithm (cf. [PWZ]), for  $k = 0, 1, 2, \dots$  we have

$$\begin{aligned} (py-2k)^2 w_k(y) + 3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y) \\ = \frac{P(k, p, y)(p(1-y)+2k-1)^3}{(py-2k)^3(py-2k-1)^3} \binom{py-2k}{p-1}^3 \end{aligned} \tag{3.2}$$

where  $P(k, p, y)$  is a suitable polynomial in  $k, p, y$  with integer coefficients such that  $P(0, p, y) \equiv 0 \pmod{p^2}$ . (Here we omit the explicit expression of  $P(k, p, y)$  since it is complicated.) Note also that

$$w_1(0) = \sum_{r=0}^{p-1} (-1)^r \binom{-2}{r}^3 = \sum_{r=0}^{p-1} (r+1)^3 = \frac{p^2(p+1)^2}{4} \equiv 0 \pmod{p^2}.$$

Fix a  $p$ -adic integer  $y$ . If  $y \neq 0$ , then (3.2) with  $k = 0$  yields

$$\begin{aligned} & 3(3py - 2)(3py - 4)w_1(y) \\ & \equiv \frac{P(0, p, y)(p(1 - y) - 1)^3}{(py)^3(py - 1)^3} \left( \frac{py}{p-1} \binom{p(y-1) + p - 1}{p-2} \right)^3 \equiv 0 \pmod{p^2} \end{aligned}$$

and hence  $w_1(y) \equiv 0 \pmod{p^2}$ . If  $1 < k + 1 \leq \lfloor (p - 1)/3 \rfloor$ , then by (3.2) we have

$$(py - 2k)^2 w_k(y) + 3(3py - 2(3k + 1))(3py - 2(3k + 2))w_{k+1}(y) \equiv 0 \pmod{p^3}$$

since

$$\binom{py - 2k}{p - 1} = \frac{p}{py - 2k + 1} \binom{py - 2k + 1}{p} \equiv 0 \pmod{p}.$$

Thus, when  $1 < k + 1 \leq \lfloor (p - 1)/3 \rfloor$  we have

$$w_k(y) \equiv 0 \pmod{p^2} \implies w_{k+1}(y) \equiv 0 \pmod{p^2}.$$

So, by induction,  $w_k(y) \equiv 0 \pmod{p^2}$  for all  $k = 1, \dots, \lfloor (p - 1)/3 \rfloor$ .

In view of the above, we have completed the proof of Theorem 1.3.  $\square$

**Lemma 3.1.** *Let  $n \in \mathbb{N}$ . Then we have*

$$\sum_{k=0}^n \binom{x+k-1}{k} = \binom{x+n}{n}. \tag{3.3}$$

**Proof.** By the Chu–Vandermonde identity (see, e.g., [GKP, p. 169]),

$$\sum_{k=0}^n \binom{-x}{k} \binom{-1}{n-k} = \binom{-x-1}{n}$$

which is equivalent to (3.3). Of course, it is easy to prove (3.3) by induction.  $\square$

**Proof of Theorem 1.4.** (i) Observe that

$$\begin{aligned} \sum_{n=0}^{p-1} D_n &= \sum_{n=0}^{p-1} \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} = \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{n=k}^{p-1} \binom{n+k}{2k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \sum_{j=0}^{p-1-k} \binom{j+2k}{j} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{2k+1+p-1-k}{p-1-k} \quad (\text{by Lemma 3.1}) \\
 &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{p+k}{2k+1} = \sum_{k=0}^{p-1} \frac{k+1}{2k+1} \binom{2k+1}{k} \binom{p+k}{2k+1}
 \end{aligned}$$

and thus

$$\sum_{n=0}^{p-1} D_n = \sum_{k=0}^{p-1} \frac{k+1}{2k+1} \binom{p+k}{k} \binom{p}{k+1} = p + \sum_{k=1}^{p-1} \frac{p}{2k+1} \binom{p-1}{k} \binom{p+k}{k}.$$

For  $k = 1, \dots, p - 1$  we clearly have

$$\binom{p-1}{k} \binom{p+k}{k} = (-1)^k \prod_{j=1}^k \left(1 - \frac{p^2}{j^2}\right) \equiv (-1)^k (1 - p^2 H_k^{(2)}) \pmod{p^4}; \tag{3.4}$$

in particular,

$$\binom{p-1}{(p-1)/2} \binom{p+(p-1)/2}{(p-1)/2} \equiv (-1)^{(p-1)/2} = \left(\frac{-1}{p}\right) \pmod{p^3}$$

since  $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$ . Therefore

$$\begin{aligned}
 \sum_{n=0}^{p-1} D_n &\equiv \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{p}{2k+1} (-1)^k + \left(\frac{-1}{p}\right) \\
 &\equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3} \quad (\text{by (2.9)}).
 \end{aligned}$$

This proves (1.17).

(ii) Now we prove (1.18) and (1.19).

Let  $n$  be any positive integer. Then

$$\begin{aligned}
 \sum_{m=0}^{n-1} (2m+1)(-1)^m D_m &= \sum_{m=0}^{n-1} (2m+1)(-1)^m \sum_{k=0}^m \binom{m+k}{2k} \binom{2k}{k} \\
 &= \sum_{k=0}^{n-1} \binom{2k}{k} \sum_{m=0}^{n-1} (2m+1)(-1)^m \binom{m+k}{2k}.
 \end{aligned}$$

By induction, we have the identity

$$\sum_{m=0}^{n-1} (2m+1)(-1)^m \binom{m+k}{2k} = (-1)^n (k-n) \binom{n+k}{2k}. \tag{3.5}$$

Thus

$$\begin{aligned} \sum_{m=0}^{n-1} (2m+1)(-1)^m D_m &= (-1)^{n-1} \sum_{k=0}^{n-1} \binom{2k}{k} (n-k) \binom{n+k}{2k} \\ &= (-1)^{n-1} \sum_{k=0}^{n-1} (n-k) \binom{n}{k} \binom{n+k}{k} \\ &= (-1)^{n-1} n \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k}. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{m=0}^{n-1} (2m+1)D_m &= \sum_{m=0}^{n-1} (2m+1) \sum_{k=0}^m \binom{m+k}{2k} \binom{2k}{k} \\ &= \sum_{k=0}^{n-1} \binom{2k}{k} \sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k} \\ &= n \sum_{k=0}^{n-1} C_k (n-k) \binom{n+k}{2k} = \sum_{k=0}^{n-1} \frac{n^2}{k+1} \binom{n-1}{k} \binom{n+k}{k}. \end{aligned}$$

In view of (3.4) and the above,

$$\begin{aligned} \frac{1}{p} \sum_{m=0}^{p-1} (2m+1)(-1)^m D_m &= \sum_{k=0}^{p-1} \binom{p-1}{k} \binom{p+k}{k} \\ &\equiv \sum_{k=0}^{p-1} (-1)^k - p^2 \sum_{k=1}^{p-1} \sum_{0 < j \leq k} \frac{(-1)^k}{j^2} \equiv 1 - p^2 \sum_{j=1}^{p-1} \frac{1}{j^2} \sum_{k=j}^{p-1} (-1)^k \\ &\equiv 1 - p^2 \sum_{i=1}^{(p-1)/2} \frac{1}{(2i)^2} = 1 - \frac{p^2}{4} H_{(p-1)/2}^{(2)} \equiv 1 - \frac{7}{12} p^3 B_{p-3} \pmod{p^4} \end{aligned}$$

and hence (1.18) holds. Similarly,

$$\begin{aligned} \frac{1}{p} \sum_{m=0}^{p-1} (2m+1)D_m &= \sum_{k=0}^{p-1} \frac{p}{k+1} \binom{p-1}{k} \binom{p+k}{k} \\ &\equiv \binom{p+(p-1)}{p-1} + p \sum_{k=0}^{p-2} \frac{(-1)^k}{k+1} (1 - p^2 H_k^{(2)}) \pmod{p^5} \\ &\equiv \binom{2p-1}{p-1} - p \sum_{k=1}^{p-1} \frac{1+(-1)^k}{k} \equiv 1 - p H_{(p-1)/2} \pmod{p^3}. \end{aligned}$$

(We have employed Wolstenholme’s congruences  $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$  and  $H_{p-1} \equiv 0 \pmod{p^2}$ .) To obtain (1.19) it suffices to apply Lehmer’s congruence (cf. [L])

$$H_{(p-1)/2} \equiv -2q_p(2) + pq_p^2(2) \pmod{p^2}.$$

The proof of Theorem 1.4 is now complete.  $\square$

**4. Some related conjectures**

Our following conjecture was motivated by Theorem 1.1(i).

**Conjecture 4.1.** *Let  $p > 3$  be a prime.*

(i) *If  $p \equiv 1 \pmod{3}$ , then*

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}. \tag{4.1}$$

*If  $p \equiv 1, 3 \pmod{8}$ , then*

$$\sum_{k=0}^{p-1} A_k \equiv \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{256^k} \pmod{p^3}. \tag{4.2}$$

(ii) *If  $x$  belongs to the set*

$$\{1, -4, 9, -48, 81, -324, 2401, 9801, -25\,920, -777\,924, 96\,059\,601\} \\ \cup \left\{ \frac{81}{256}, -\frac{9}{16}, \frac{81}{32}, -\frac{3969}{256} \right\}$$

*and  $x \not\equiv 0 \pmod{p}$ , then we must have*

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p^2}.$$

**Remark 4.1.** For those

$$x = -4, 9, -48, 81, -324, 2401, 9801, -25\,920, -777\,924, 96\,059\,601, \frac{81}{256},$$

the author (cf. [Su2]) had conjectures on  $\sum_{k=0}^{p-1} \binom{4k}{k,k,k,k} / (256x)^k \pmod{p^2}$ . Motivated by this, Z.H. Sun [S2] guessed  $\sum_{k=0}^{p-1} \binom{4k}{k,k,k,k} / (256x)^k \pmod{p^2}$  for  $x = -9/16, 81/32, -3969/256$  in a similar way.

We have checked Conjecture 4.1 as well as all the other conjectures in this paper via Mathematica 7. Below we provide numerical evidences for (4.1) and (4.2).

**Example 4.1.** The values of  $A_0, A_1, \dots, A_{10}$  are given by

$$1, 5, 73, 1445, 33\,001, 819\,005, 21\,460\,825, \\ 584\,307\,365, 16\,367\,912\,425, 468\,690\,849\,005, 13\,657\,436\,403\,073$$

respectively. Via computation we find that

$$\sum_{k=0}^6 (-1)^k A_k = 20\,673\,445, \quad \sum_{k=0}^6 \frac{\binom{2k}{k}^3}{16^k} = \frac{18\,825\,543}{262\,144},$$

and also

$$20\,673\,445 \times 262\,144 - 18\,825\,543 = 7^3 \times 15\,800\,002\,159.$$

This verifies (4.1) for  $p = 7$ . By computation we have

$$\sum_{k=0}^{10} A_k = 14\,143\,101\,786\,223, \quad \sum_{k=0}^{10} \frac{\binom{4k}{k,k,k,k}}{256^k} = \frac{22\,821\,835\,381\,970\,859\,184\,405}{18\,889\,465\,931\,478\,580\,854\,784},$$

and also

$$14\,143\,101\,786\,223 \times 18\,889\,465\,931\,478\,580\,854\,784 - 22\,821\,835\,381\,970\,859\,184\,405 \\ = 11^3 \times 200\,717\,985\,992\,690\,007\,194\,123\,778\,899\,817.$$

This verifies (4.2) for  $p = 11$ .

Inspired by parts (ii) and (iii) of Theorem 1.1, we raise the following conjecture.

**Conjecture 4.2.** For any  $\varepsilon \in \{\pm 1\}$ ,  $m, n \in \mathbb{Z}^+$  and  $x \in \mathbb{Z}$ , we have

$$\sum_{k=0}^{n-1} (2k+1)\varepsilon^k A_k(x)^m \equiv 0 \pmod{n}. \tag{4.3}$$

If  $p > 5$  is a prime, then

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6}. \tag{4.4}$$

**Remark 4.2.** After reading an initial version of this paper, Guo and Zeng [GZ] proved the author's following conjectural results:

(a) For any  $n \in \mathbb{Z}^+$  and  $x \in \mathbb{Z}$  we have

$$\sum_{k=0}^{n-1} (2k+1)(-1)^k A_k(x) \equiv 0 \pmod{n}.$$

If  $p$  is an odd prime and  $x$  is an integer, then

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k(x) \equiv p \left( \frac{1-4x}{p} \right) \pmod{p^2}.$$

(b) For any prime  $p > 3$  we have

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k \equiv p \left( \frac{p}{3} \right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k A_k(-2) \equiv p - \frac{4}{3} p^2 q_p(2) \pmod{p^3}.$$

**Example 4.2.** It is easy to check that

$$\sum_{k=0}^9 (2k+1) A_k^2 = 4\,178\,310\,699\,572\,329\,761\,604\,780 \equiv 0 \pmod{10}$$

and

$$\sum_{k=0}^9 (2k+1)(-1)^k A_k^2 = -4\,169\,201\,796\,654\,383\,947\,725\,970 \equiv 0 \pmod{10}.$$

So (4.3) holds when  $m = 2, n = 10, x = 1$  and  $\varepsilon \in \{\pm 1\}$ . Via computation we find that

$$\sum_{k=0}^{10} (2k+1) A_k = 295\,998\,598\,024\,613, \quad H_{10} = \frac{7381}{2520},$$

and

$$295\,998\,598\,024\,613 - 11 + \frac{7}{2} \times 11^2 \times \frac{7381}{2520} = 11^6 \times \frac{120\,300\,114\,181}{720}.$$

This verifies (4.4) for  $p = 11$ .

Recall that for a prime  $p$  and a rational number  $x$ , the  $p$ -adic valuation of  $x$  is given by

$$v_p(x) = \sup\{a \in \mathbb{Z}: \text{the denominator of } p^{-a}x \text{ is not divisible by } p\}.$$

Just like the Apéry polynomial  $A_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^k$  we define

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k.$$

Actually  $D_n((x-1)/2)$  coincides with the Legendre polynomial  $P_n(x)$  of degree  $n$ .



Our following conjecture involves  $p$ -adic valuations.

**Conjecture 4.3.**

(i) For any  $n \in \mathbb{Z}^+$  the numbers

$$s(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k \left(\frac{1}{4}\right)$$

and

$$t(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(-1)^k D_k \left(-\frac{1}{4}\right)^3$$

are rational numbers with denominators  $2^{2\nu_2(n!)}$  and  $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$  respectively. Moreover, the numerators of  $s(1), s(3), s(5), \dots$  are congruent to 1 modulo 12 and the numerators of  $s(2), s(4), s(6), \dots$  are congruent to 7 modulo 12. If  $p$  is an odd prime and  $a \in \mathbb{Z}^+$ , then

$$s(p^a) \equiv t(p^a) \equiv 1 \pmod{p}.$$

For  $p = 3$  and  $a \in \mathbb{Z}^+$  we have

$$s(3^a) \equiv 4 \pmod{3^2} \quad \text{and} \quad t(3^a) \equiv -8 \pmod{3^5}.$$

(ii) Let  $p$  be a prime. For any positive integer  $n$  and  $p$ -adic integer  $x$ , we have

$$\nu_p \left( \frac{1}{n} \sum_{k=0}^{n-1} (2k+1)(-1)^k A_k(x) \right) \geq \min\{\nu_p(n), \nu_p(4x-1)\} \tag{4.5}$$

and

$$\nu_p \left( \frac{1}{n} \sum_{k=0}^{n-1} (2k+1)(-1)^k D_k(x)^3 \right) \geq \min\{\nu_p(n), \nu_p(4x+1)\}. \tag{4.6}$$

**Example 4.3.** We check Conjecture 4.3 with  $n = p = 5$ . For  $n = 5$  we have

$$2^{2\nu_2(n!)} = 2^{2\nu_2(120)} = 2^6 = 64$$

and

$$2^{3(n-1+\nu_2(n!))-\nu_2(n)} = 2^{3(5-1+3)-0} = 2^{21} = 2\,097\,152.$$

Via computation we find that

$$s(5) = \frac{19\,849}{64} \quad \text{and} \quad t(5) = \frac{82\,547}{2\,097\,152}.$$

Note that  $19\,849 \equiv 1 \pmod{12}$  and  $s(5) \equiv t(5) \equiv 1 \pmod{5}$ . Also,

$$\nu_5 \left( \frac{1}{5} \sum_{k=0}^4 (2k+1)(-1)^k A_k(-1) \right) = \nu_5(-13\,095) = 1 = \nu_5(\pm 5)$$

and

$$\nu_5 \left( \frac{1}{5} \sum_{k=0}^4 (2k+1)(-1)^k D_k(1)^3 \right) = \nu_5(59\,189\,205) = 1 = \nu_5(5).$$

Motivated by Theorem 1.3, we pose the following conjecture.

**Conjecture 4.4.** *Let  $p$  be an odd prime and let  $n \geq 2$  be an integer. Suppose that  $x$  is a  $p$ -adic integer with  $x \equiv -2k \pmod{p}$  for some  $k \in \{1, \dots, \lfloor (p+1)/(2n+1) \rfloor\}$ . Then we have*

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^{2n+1} \equiv 0 \pmod{p^2}. \tag{4.7}$$

**Example 4.4.** Clearly  $1/3 \equiv -2 \pmod{7}$  and  $\lfloor (7+1)/5 \rfloor = 1$ . Via computation we find that

$$\sum_{r=0}^6 (-1)^r \binom{1/3}{r}^5 = \frac{12\,107\,415\,300\,799\,972\,328}{12\,157\,665\,459\,056\,928\,801} = 7^2 \times \frac{247\,090\,108\,179\,591\,272}{12\,157\,665\,459\,056\,928\,801}.$$

So (4.7) holds when  $p = 7$ ,  $n = 2$  and  $x = 1/3$ .

**Conjecture 4.5.** *Let  $p \equiv 3 \pmod{4}$  be a prime, and let  $m \in \mathbb{Z}$  with  $p \nmid m(4m+1)$ . Then*

$$\sum_{k=0}^{p-1} \frac{W_k(-m^2)}{(4m+1)^k} \equiv \sum_{k=0}^{p-1} \frac{A_k(-m^2/(4m+1))}{(4m+1)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2p}{k}}{(-16)^k} D_{2k} \left( \frac{1}{4m} \right) \pmod{p^2}.$$

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