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ON APÉRY NUMBERS AND GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

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Abstract. Let $p > 3$ be a prime. We derive the following new congruences:

$$\sum_{n=0}^{p-1}(2n+1)A_n \equiv p \pmod{p^4}$$

and

$$\sum_{n=0}^{p-1} D_n \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

where A_n denotes the Apéry number $\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$, D_n stands for the central Delannoy number $\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}$, and $E_0, E_1, E_2, \ldots$ are Euler numbers. We show that the arithmetic means $\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k$ ($n = 1, 2, 3, \ldots$) are always integers and conjecture that $\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \equiv 0 \pmod{n}$ for every $n = 1, 2, 3, \ldots$. We also investigated generalized central trinomial coefficient $T_n(b, c)$ (with $b, c \in \mathbb{Z}$) which is the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$. For any positive integer n we prove that

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(4c-b^2)^{n-1-k} \equiv 0 \pmod{n}$$

and conjecture that

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(b^2-4c)^{n-1-k} \equiv 0 \pmod{n^2}.$$

Our topic is original and many conjectures are raised.

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1. Introduction

In number theory, for an arithmetical function f, analytic number-theorists often study the asymptotical behavoir of the partial sum $\sum_{n\leqslant x} f(n)$. Similarly, for an integer sequence $a_0, a_1, a_2, \ldots$ we may investigate the arithmetic mean $\frac{1}{n}\sum_{k=0}^{n-1} a_k$ $(n = 1, 2, 3, \ldots)$ or the partial sum $\sum_{k=0}^{p-1} a_k$ modulo powers of a prime p. In this paper we initiate the topic for various integer sequences $\{a_k\}_{k\geqslant 0}$ arising naturally from enumeration problems in combinatorics.

Let p be a prime. Partially motivated by H. Pan and Z. W. Sun [PS], Sun and R. Tauraso [ST] proved that

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} C_k \equiv \frac{3(\frac{p}{3})-1}{2} \pmod{p^2},$$

where C_k denotes the Catalan number $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$ and $(-)$ refers to the Legendre symbol. Recently Sun [Su] determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2 for any integer $m \not\equiv 0 \pmod{p}$.

Recall that Apéry numbers are given by

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\})$$

which play a central role in Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (see R. Apéry [Ap] and van der Poorten [Po]). Apéry numbers are related to modular forms and the p-adic Gamma function, see Ken Ono [O, pp.198–203]. The Dedekind eta function in the theory of modular forms is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad \text{with } q = e^{2\pi i\tau},$$

where $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and hence $|q| < 1$. In 1987 F. Beukers [B] conjectured that

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2} \quad \text{for any prime } p > 3,$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.$$

This was finally confirmed by S. Ahlgren and Ono [AO] in 2000.

Let p be an odd prime. Motivated by the author's determination of $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2 for any integer $m \not\equiv 0 \pmod{p}$, we computed $\sum_{k=0}^{p-1} A_k/m^k$ mod p^2 via `Mathematica` and found the following surprising conjecture.

Conjecture 1.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8 \text{ and } p=x^2+2y^2\ (x,y\in\mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8; \end{cases}$$

and

$$\sum_{k=0}^{p-1}(-1)^k A_k \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and } p=x^2+3y^2\ (x,y\in\mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3. \end{cases}$$

Remark 1.1. In number theory, it is well known that if p is an odd prime with $(\frac{-2}{p}) = 1$ (i.e., $p \equiv 1,3 \pmod 8$) then there are unique positive integers x and y such that $p = x^2 + 2y^2$. Also, if p is an odd prime with $(\frac{-3}{p}) = 1$ (i.e., $p \equiv 1 \pmod 3$) then there are unique positive integers x and y such that $p = x^2 + 3y^2$. The reader may consult A. Cox [Co] for these basic facts.

Conjecture 1.1 is also related to modular forms since J. Stienstra and F. Beukers [SB] proved that if we write

$$q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2 = \sum_{n=1}^{\infty} b(n)q^n$$

and

$$q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3 = \sum_{n=1}^{\infty} c(n)q^n,$$

then for any odd prime p we have

$$b(p) = \begin{cases} 4x^2-2p & \text{if } p\equiv 1,3 \pmod 8 \ \&\ p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z}, \\ 0 & \text{if } p\equiv 5,7 \pmod 8, \end{cases}$$

and

$$c(p) = \begin{cases} 4x^2-2p & \text{if } p\equiv 1 \pmod 3 \ \&\ p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z}, \\ 0 & \text{if } p\equiv 2 \pmod 3. \end{cases}$$

Conjecture 1.1 seems very challenging and we are unable to prove it; we also have a similar conjecture for $\sum_{k=0}^{p-1}\varepsilon^k kA_k \bmod p^2$ where $\varepsilon = \pm 1$. Nevertheless we can establish the following novel property of Apéry numbers.

Theorem 1.1. (i) *For any positive integer n we have*

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}. \tag{1.1}$$

If $p>3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p \pmod{p^4}. \tag{1.2}$$

(ii) *Let $\varepsilon \in \{\pm 1\}$ and $m \in \mathbb{Z}^+$, and let p be any prime. Then*

$$\sum_{k=0}^{p-1}(2k+1)\varepsilon^k A_k^m \equiv 0 \pmod{p}. \tag{1.3}$$

Remark 1.2. The values of

$$s_n = \frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k \in \mathbb{Z}$$

with $n=1,\ldots,8$ are

$$1,\ 8,\ 127,\ 2624,\ 61501,\ 1552760,\ 41186755,\ 1131614720$$

respectively. Via the Zeilberger algorithm we obtain the recursion

$$\begin{aligned}&(n+2)^3(n+3)(2n+1)s_{n+3}\\ =&(n+2)(2n+1)(35n^3+193n^2+345n+203)s_{n+2}\\ &-(n+1)(2n+5)(35n^3+122n^2+132n+40)s_{n+1}\\ &+n(n+1)^3(2n+5)s_n\end{aligned}$$

for $n=0,1,2,\ldots$.

For $n \in \mathbb{N}$ we define the *Apéry polynomial* $A_n(x)$ as follows:

$$A_n(x) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 x^k.$$

Obviously $A_n(1)=A_n$. By a slight modification of our proof of (1.1) given in the next section, we see that

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x) = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k}x^k$$

for every $n = 1, 2, 3, \ldots$. Thus, for any odd prime p and integer x we have

$$\sum_{k=0}^{p-1}(2k+1)A_k(x) \equiv p\left(\frac{x}{p}\right) \pmod{p^2},$$

since $p \mid \binom{p+k}{2k+1}$ for every $k = 0, \ldots, (p-3)/2$, and $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \ldots, p-1$.

Based on our computation via `Mathematica`, we raise the following conjecture which has the same flavor with Theorem 1.1.

Conjecture 1.2. *For any $\varepsilon \in \{\pm 1\}$, $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n-1}(2k+1)\varepsilon^k A_k(x)^m \equiv 0 \pmod{n}.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k(x) \equiv p\left(\frac{1-4x}{p}\right) \pmod{p^2}.$$

Also, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(2k+1)A_k(-3) \equiv \sum_{k=0}^{p-1}(2k+1)(-1)^k A_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3}.$$

Remark 1.3. The values of $\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k$ with $n = 1, \ldots, 8$ are

$$1,\ -7,\ 117,\ -2441,\ 57449,\ -1453635,\ 38609845,\ -1061792695$$

respectively.

In contrast with Conjecture 1.1, we have the following conjecture involving the binary quadratic form $x^2 + y^2$.

Conjecture 1.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k A_k(-2)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}\ \&\ p = x^2 + y^2\ (2 \nmid x,\ 2 \mid y), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Provided $p>3$ we also have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^kA_k(-2)\equiv p-\frac{4}{3}p^2q_p(2)\pmod{p^3},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Remark 1.4. As observed by Fermat and proved by Euler, any prime $p\equiv 1\pmod 4$ can be uniquely written in the form x^2+y^2 with x odd and y even. Conjecture 1.3 determines $x^2 \bmod p^2$ via the integer sequence $\{(-1)^kA_k(-2)\}_{k\geqslant 0}$; in Sections 4 and 5 we will present more conjectures in this spirit.

The central Delannoy numbers (see [CHV]) are defined by

$$D_n=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\ (n\in\mathbb{N}).$$

Such numbers arise naturally in many enumeration problems in combinatorics (cf. Sloane [Sl]); for example, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Our second theorem is concerned with central Delannoy numbers.

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k(-1)\equiv\sum_{k=0}^{p-1}D_k\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\pmod{p^3},\tag{1.4}$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers defined by

$$E_0=1\ \text{and}\ \sum_{\substack{k=0\\2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\quad\text{for } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

We also have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^kD_k\equiv p-\frac{7}{12}p^4B_{p-3}\pmod{p^5}\tag{1.5}$$

and

$$\sum_{k=0}^{p-1}(2k+1)D_k\equiv p+2p^2q_p(2)-p^3q_p(2)^2\pmod{p^4},\tag{1.6}$$

where $B_0,B_1,B_2\ldots$ are Bernoulli numbers.

Now we give our fourth conjecture.

Conjecture 1.4. *Let p be any odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2}\equiv 2\left(\frac{-1}{p}\right)E_{p-3}\ (\mathrm{mod}\ p).$$

If $p>3$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k^2\equiv p^2-4p^3q_p(2)-2p^4q_p(2)^2\ (\mathrm{mod}\ p^5).$$

Recall that for a prime p and a rational number x, the *p-adic valuation* of x is given by

$$\nu_p(x)=\sup\{a\in\mathbb{N}:\ x\equiv 0\ (\mathrm{mod}\ p^a)\}.$$

Just like the Apéry polynomial $A_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2x^k$ we define

$$D_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Actually $D_n((x-1)/2)$ coincides with the Legendre polynomial $P_n(x)$ of degree n.

Conjecture 1.5. (i) *For any $n\in\mathbb{Z}$ the numbers*

$$s(n)=\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k\left(\frac{1}{4}\right)$$

and

$$t(n)=\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^kD_k\left(-\frac{1}{4}\right)^3$$

are rational numbers with denominators $2^{2\nu_2(n!)}$ and $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$ respectively. Moreover, the numerators of $s(1),s(3),s(5),\ldots$ are congruent to 1 *modulo* 12 *and the numerators of $s(2),s(4),s(6),\ldots$ are congruent to* 7 *modulo* 12. *If p is an odd prime and $a\in\mathbb{Z}^+$, then*

$$s(p^a)\equiv t(p^a)\equiv 1\ (\mathrm{mod}\ p).$$

For $p=3$ and $a\in\mathbb{Z}^+$ we have

$$s(3^a)\equiv 4\ (\mathrm{mod}\ 3^2)\quad\text{and}\quad t(3^a)\equiv -8\ (\mathrm{mod}\ 3^5).$$

(ii) *Let p be a prime. For any positive integer n and p-adic integer x, we have*

$$\nu_p\left(\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x)\right) \geqslant \min\{\nu_p(n),\, \nu_p(4x-1)\}$$

and

$$\nu_p\left(\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k D_k(x)^3\right) \geqslant \min\{\nu_p(n),\, \nu_p(4x+1)\}.$$

For $n \in \mathbb{N}$, the nth central trinomial coefficient and the nth Motzkin numbers are defined by

$$T_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k} \quad \text{and} \quad M_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k.$$

It is known that T_n coincides with $[x^n](1+x+x^2)^n$, the coefficient of x^n in the expansion of $(1+x+x^2)^n$, and that M_n equals the number of paths from $(0,0)$ to $(n,0)$ in an $n\times n$ grid using only steps $(1,1)$, $(1,0)$ and $(1,-1)$ (cf. Sloane [Sl]). Quite recently H. Q. Cao and Pan [CP] determined $\sum_{k=0}^{p-1}T_k \bmod p$ and $\sum_{k=0}^{p-1}(-1)^kT_k \bmod p^2$, where p is an odd prime.

Our following conjecture seems sophisticated.

Conjecture 1.6. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n}.$$

If p is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}M_kT_k \equiv \frac{4}{3}\left(\frac{p}{3}\right)+\frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{M_kT_k}{(-3)^k}\equiv\frac{p}{2}\left(\left(\frac{p}{3}\right)-1\right)\pmod{p^2}.$$

Given $b,c\in\mathbb{Z}$, we define the *generalized central trinomial coefficients*

$$\begin{aligned}T_n(b,c):=&[x^n](x^2+bx+c)^n=[x^0](b+x+cx^{-1})^n\\ =&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\binom{n}{k}b^{n-2k}c^k\end{aligned}$$

and introduce the *generalized Motzkin numbers*

$$M_n(b,c):=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_kb^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\binom{n}{k}\frac{b^{n-2k}c^k}{k+1}$$

$(n=0,1,2,\ldots)$. Note that

$$T_n=T_n(1,1),\ M_n=M_n(1,1),$$

$$T_n(2,1)=[x^n](x+1)^{2n}=\binom{2n}{n},$$

and

$$M_n(2,1)=\sum_{k=0}^{n}\binom{n}{2k}C_k2^{n-2k}=C_{n+1}.$$

It is also known (cf. [Sl]) that $D_n=T_n(3,2)$. Thus $T_n(b,c)$ can be viewed a natural common extension of central binomial coefficients, central trinomial coefficients and central Delannoy numbers, while $M_n(b,c)$ can be viewed as a natural common extension of Catalan numbers and Motzkin numbers. H. S. Wilf [W, p. 159] observed that

$$\sum_{n=0}^{\infty}T_n(b,c)x^n=\frac{1}{\sqrt{1-2bx+(b^2-4c)x^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b,c)=(2n+1)bT_n(b,c)+(4c-b^2)nT_{n-1}(b,c)\ \ (n\in\mathbb{Z}^+).\quad(1.7)$$

(See also T. D. Noe [N].)

Our third theorem is concerned with generalized central trinomial coefficients and generalized Motzkin numbers.

Theorem 1.3. *Let p be an odd prime and let $b,c,m\in\mathbb{Z}$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{m^k}\equiv\left(\frac{(m-b)^2-4c}{p}\right)\pmod p \tag{1.8}$$

and

$$2c\sum_{k=0}^{p-1}\frac{M_k(b,c)}{m^k}\equiv(m-b)^2-((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right)\pmod p. \tag{1.9}$$

Theorem 1.4. *Let $b,c\in\mathbb{Z}$.*

(i) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(4c-b^2)^{n-1-k}\equiv 0 \pmod n, \tag{1.10}$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(4c-b^2)^{n-1-k}=nT_n(b,c)T_{n-1}(b,c). \tag{1.11}$$

(ii) *Suppose that $b^2-4c=1$ (i.e., $b=2d+1$ and $c=d^2+d$ for some $d\in\mathbb{Z}$). Then*

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)=\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k\in\mathbb{Z} \tag{1.12}$$

for all $n\in\mathbb{Z}^+$. If p is a prime not dividing c, then

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)\equiv p+\frac{b+1}{b-1}p\left(\left(\frac{b+1}{2}\right)^{p-1}-1\right)\pmod{p^3}. \tag{1.13}$$

For any odd prime p we also have

$$\frac{b-1}{2}\sum_{k=0}^{p-1}(2k+1)^2T_k(b,c)\equiv\left(\frac{(b-1)/2}{p}\right)\pmod p. \tag{1.14}$$

Remark 1.5. The author notes that for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k3^{n-1-k}=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-1-k}(k+1)\binom{2k}{k}.$$

If $b,c\in\mathbb{Z}$ with $b^2-4c=1$, then for any prime $p\nmid c$ by (1.13) we have

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)\equiv p \pmod{p^2}.$$

Conjecture 1.7. *Let $b,c\in\mathbb{Z}$.*

(i) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(b^2-4c)^{n-1-k}\equiv 0 \pmod{n^2}.$$

If c is nonzero and p is an odd prime not dividing b^2-4c, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{(b^2-4c)^k}\equiv 1+\frac{b^2}{c}\cdot\frac{(\frac{b^2-4c}{p})-1}{2} \pmod{p}.$$

(ii) *Suppose that $b^2-4c=1$. Then*

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^m\equiv 0 \pmod{n}$$

for all $m,n\in\mathbb{Z}^+$. If p is a prime not dividing c, then

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)^3\equiv p\left(\frac{-2b-1}{p}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)^4\equiv p \pmod{p^2}.$$

Remark 1.6. Note that $D_n=T_n(3,2)$ and $3^2-4\times 2=1$. Thus Conjecture 1.7(i) implies that

$$\sum_{k=0}^{n-1}(2k+1)D_k^2\equiv 0 \pmod{n^2}$$

for all $n\in\mathbb{Z}^+$. The values of $\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)D_k^2$ with $n=1,\ldots,9$ are

$$1,\ 7,\ 97,\ 1791,\ 38241,\ 892039,\ 22092673,\ 571387903,\ 15271248769$$

respectively.

Theorems 1.1 and 1.2 will be proved in the next section. In Section 3 we will prove Theorems 1.3 and 1.4. Sections 4 and 5 contain various conjectures involving generalized central trinomial coefficients and a new kind of numbers respectively. We hope that our conjectures in Sections $1,4$ and 5 will interest number theorists and stimulate further research.

2. Proofs of Theorems 1.1 and 1.2

Lemma 2.1. *Let $k\in\mathbb{N}$. Then, for any $n\in\mathbb{Z}^+$ we have the identity*

$$\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2=\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2. \tag{2.1}$$

Proof. Obviously (2.1) holds when $n=1$.

Now assume that $n>1$ and (2.1) holds. Then

$$\begin{aligned}&\sum_{m=0}^{n}(2m+1)\binom{m+k}{2k}^2\\ =&\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2+(2n+1)\binom{n+k}{2k}^2\\ =&\frac{(n+k+1)^2}{2k+1}\binom{n+k}{2k}^2=\frac{(n+1-k)^2}{2k+1}\binom{(n+1)+k}{2k}^2.\end{aligned}$$

Combining the above, we have proved the desired result by induction. □

Proof of Theorem 1.1. (i) Let n be any positive integer. Then

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)A_m&=\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2\\ &=\sum_{k=0}^{n-1}\binom{2k}{k}^2\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2\\ &=\sum_{k=0}^{n-1}\binom{2k}{k}^2\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2\quad\text{(by (2.1))}\\ &=\sum_{k=0}^{n-1}\frac{(n-k)^2}{2k+1}\binom{n}{k}^2\binom{n+k}{k}^2.\end{aligned}$$

Since

$$(n-k)\binom{n}{k}=n\binom{n-1}{k}\quad\text{for all}\ \ k=0,\dots,n-1,$$

we have

$$\begin{aligned}\frac{1}{n}\sum_{m=0}^{n-1}(2m+1)A_m&=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{n-k}{2k+1}\binom{n}{k}\binom{n+k}{k}^2\\ &=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{n-k}{2k+1}\binom{n+k}{2k}\binom{2k}{k}\binom{n+k}{k}\\ &=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k}\in\mathbb{Z}.\end{aligned}$$

This proves (1.1).

Now we fix a prime $p>3$. By the above, for any $n\in\mathbb{Z}^+$ we have

$$\sum_{m=0}^{n-1}(2m+1)A_m=\sum_{k=0}^{n-1}\frac{n^2}{2k+1}\binom{n-1}{k}^2\binom{n+k}{k}^2. \tag{2.2}$$

Observe that

$$\begin{aligned}
&\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{2k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2\\
=&\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{2k+1}\prod_{0<j\leqslant k}\left(\frac{p^2-j^2}{j^2}\right)^2\\
\equiv&\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{1}{2k+1}=\sum_{k=0}^{(p-3)/2}\left(\frac{1}{2k+1}+\frac{1}{2(p-1-k)+1}\right)\\
=&\sum_{k=0}^{(p-3)/2}\left(\frac{1}{2k+1}+\frac{1}{2p-2k-1}\right)=\sum_{k=0}^{(p-3)/2}\frac{2p}{(2k+1)(2p-2k-1)}\\
\equiv&\sum_{k=0}^{(p-3)/2}\frac{-2p}{(2k+1)^2}=-2p\bigg(\sum_{k=1}^{p-1}\frac{1}{k^2}-\sum_{k=1}^{(p-1)/2}\frac{1}{(2k)^2}\bigg)\pmod{p^2}.
\end{aligned}$$

Since

$$\sum_{k=1}^{p-1}\frac{1}{(2k)^2}\equiv\sum_{k=1}^{p-1}\frac{1}{k^2}\pmod p,$$

we have

$$2\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\pmod p.$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{p-1}(2k+1)A_k\equiv&\frac{p^2}{2(p-1)/2+1}\binom{p-1}{(p-1)/2}^2\binom{p+(p-1)/2}{(p-1)/2}^2\\
=&p\prod_{k=1}^{(p-1)/2}\left(\frac{p^2-k^2}{k^2}\right)^2\equiv p\sum_{k=1}^{(p-1)/2}\left(1-\frac{2p^2}{k^2}\right)\\
\equiv&p\bigg(1-2p^2\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\bigg)\equiv p\pmod{p^4}.
\end{aligned}$$

This concludes the proof of (1.2).

(ii) As $A_0=1$ and $A_1=3$, (1.3) with $p=2$ holds trivially.

Below we assume that $p>2$. If $k\in\{0,1\ldots,p-1\}$, then

$$A_{p-1-k}=\sum_{j=0}^{p-1}\binom{(p-1-k)+j}{2j}^2\binom{2j}{j}^2$$
$$\equiv\sum_{j=0}^{p-1}\binom{j-k-1}{2j}^2\binom{2j}{j}^2=\sum_{j=0}^{k}\binom{j+k}{2j}^2\binom{2j}{j}^2=A_k \pmod p$$

Thus

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m=&\sum_{k=0}^{p-1}(2(p-1-k)+1)\varepsilon^{p-1-k}A_{p-1-k}^m\\ \equiv&-\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m \pmod p\end{aligned}$$

and hence (1.3) follows.

Combining the above we have completed the proof of Theorem 1.1. □

Lemma 2.2. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{n}\binom{x+k-1}{k}=\binom{x+n}{n}. \tag{2.3}$$

Proof. By the Chu-Vandermonde identity (see, e.g., [GKP, p. 169]),

$$\sum_{k=0}^{n}\binom{-x}{k}\binom{-1}{n-k}=\binom{-x-1}{n}$$

which is equivalent to (2.3). □

Lemma 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}\equiv -pE_{p-3} \pmod{p^2}. \tag{2.4}$$

Proof. Observe that

$$\begin{aligned}\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}=&\frac12\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\left(\frac{(-1)^k}{2k+1}+\frac{(-1)^{p-1-k}}{(2(p-1-k)+1)}\right)\\ =&-p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{(2k+1)(2k+1-2p)}\\ \equiv&-\frac{p}{4}\sum_{k=0}^{p-1}(-1)^k\left(k+\frac12\right)^{p-3} \pmod{p^2}.\end{aligned}$$

So we have reduced (2.4) to the following congruence

$$\sum_{k=0}^{p-1}(-1)^k\left(k+\frac{1}{2}\right)^{p-3}\equiv 4E_{p-3}\ (\mathrm{mod}\ p). \tag{2.5}$$

Recall that the Euler polynomial of degree n is defined by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

It is well known that

$$E_n(x)+E_n(x+1)=2x^n.$$

Thus

$$\begin{aligned}&2\sum_{k=0}^{p-1}(-1)^k\left(k+\frac{1}{2}\right)^{p-3}\\ =&\sum_{k=0}^{p-1}\left((-1)^kE_{p-3}\left(k+\frac{1}{2}\right)-(-1)^{k+1}E_{p-3}\left(k+1+\frac{1}{2}\right)\right)\\ =&E_{p-3}\left(\frac{1}{2}\right)-(-1)^pE_{p-3}\left(p+\frac{1}{2}\right)\\ \equiv&2E_{p-3}\left(\frac{1}{2}\right)=2\frac{E_{p-3}}{2^{p-3}}\equiv 8E_{p-3}\ (\mathrm{mod}\ p)\end{aligned}$$

and hence (2.5) follows. We are done. □

Proof of Theorem 1.2. (i) We first show (1.4).

Similar to (2.2), we have

$$\sum_{k=0}^{p-1}(2k+1)A_k(-1)=\sum_{k=0}^{p-1}\frac{p^2}{2k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2(-1)^k.$$

Note that

$$\begin{aligned}&\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2\\ =&\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}\prod_{0<j\leqslant k}\left(\frac{p^2-j^2}{j^2}\right)^2\\ \equiv&\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}\equiv -pE_{p-3}\ (\mathrm{mod}\ p^2)\quad\text{(by (2.4))}\end{aligned}$$

and

$$\binom{p-1}{(p-1)/2}\binom{p+(p-1)/2}{(p-1)/2}=\prod_{j=1}^{(p-1)/2}\frac{p^2-j^2}{j^2}$$
$$\equiv(-1)^{(p-1)/2}\left(1-p^2\sum_{j=1}^{(p-1)/2}\frac{1}{j^2}\right)\equiv\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^3).$$

Therefore,

$$\sum_{k=0}^{p-1}(2k+1)A_k(-1)$$
$$\equiv p^2(-pE_{p-3})+\frac{p^2(-1)^{(p-1)/2}}{2(p-1)/2+1}\left(\frac{-1}{p}\right)^2=p\left(\frac{-1}{p}\right)-p^3E_{p-3}\ (\mathrm{mod}\ p^4).$$

Observe that

$$\sum_{n=0}^{p-1}D_n=\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}=\sum_{k=0}^{p-1}\binom{2k}{k}\sum_{n=k}^{p-1}\binom{n+k}{2k}$$
$$=\sum_{k=0}^{p-1}\binom{2k}{k}\sum_{j=0}^{p-1-k}\binom{j+2k}{j}$$
$$=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{2k+1+p-1-k}{p-1-k}\ \text{(by Lemma 2.2)}$$
$$=\sum_{k=0}^{p-1}\binom{2k}{k}\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\frac{k+1}{2k+1}\binom{2k+1}{k}\binom{p+k}{2k+1}$$

and thus

$$\sum_{n=0}^{p-1}D_n=\sum_{k=0}^{p-1}\frac{k+1}{2k+1}\binom{p+k}{k}\binom{p}{k+1}=p+\sum_{k=1}^{p-1}\frac{p}{2k+1}\binom{p-1}{k}\binom{p+k}{k}.$$

For $k=1,\ldots,p-1$ we clearly have

$$\binom{p-1}{k}\binom{p+k}{k}=\prod_{j=1}^{k}\frac{p^2-j^2}{j^2}\equiv(-1)^k\ (\mathrm{mod}\ p^2).$$

Recall that

$$\binom{p-1}{(p-1)/2}\binom{p+(p-1)/2}{(p-1)/2}\equiv\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^3).$$

Therefore

$$\begin{aligned}\sum_{n=0}^{p-1} D_n &\equiv \sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1} \frac{p}{2k+1}(-1)^k + \left(\frac{-1}{p}\right)\\ &\equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3} \quad \text{(by (2.4))}.\end{aligned}$$

(ii) Now we prove (1.5) and (1.6).

Let n be any positive integer. Then

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)(-1)^m D_m &= \sum_{m=0}^{n-1}(2m+1)(-1)^m \sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}\\ &= \sum_{k=0}^{n-1}\binom{2k}{k}\sum_{m=0}^{n-1}(2m+1)(-1)^m\binom{m+k}{2k}\end{aligned}$$

It is easy to show that

$$\sum_{m=0}^{n-1}(2m+1)(-1)^m\binom{m+k}{2k} = (-1)^n(k-n)\binom{n+k}{2k}.$$

Thus

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)(-1)^m D_m =& (-1)^{n-1}\sum_{k=0}^{n-1}\binom{2k}{k}(n-k)\binom{n+k}{2k}\\ =& (-1)^{n-1}\sum_{k=0}^{n-1}(n-k)\binom{n}{k}\binom{n+k}{k}\\ =& (-1)^{n-1} n\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)D_m &= \sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}\\ &= \sum_{k=0}^{n-1}\binom{2k}{k}\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}\\ &= \sum_{k=0}^{n-1} C_k n(n-k)\binom{n+k}{2k} = \sum_{k=0}^{n-1}\frac{n^2}{k+1}\binom{n-1}{k}\binom{n+k}{k}.\end{aligned}$$

For $k\in\{0,\ldots,p-1\}$, we have

$$\begin{aligned}\binom{p-1}{k}\binom{p+k}{k}=&\prod_{0<j\leqslant k}\left(\frac{p+j}{j}\cdot\frac{p-j}{j}\right)=(-1)^k\prod_{0<j\leqslant k}\left(1-\frac{p^2}{j^2}\right)\\ \equiv&(-1)^k\bigg(1-p^2\sum_{0<j\leqslant k}\frac{1}{j^2}\bigg)\pmod{p^4}.\end{aligned}$$

By a known result (see, e.g., [S, Corollary 5.2(a)]),

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\frac{7}{3}pB_{p-3}\pmod{p^2}.$$

Thus

$$\begin{aligned}&\frac{1}{p}\sum_{m=0}^{p-1}(2m+1)(-1)^mD_m=\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k}{k}\\ \equiv&\sum_{k=0}^{p-1}(-1)^k-p^2\sum_{k=1}^{p-1}\sum_{0<j\leqslant k}\frac{(-1)^k}{j^2}=1-p^2\sum_{j=1}^{p-1}\frac{1}{j^2}\sum_{k=j}^{p-1}(-1)^k\\ \equiv&1-p^2\sum_{i=1}^{(p-1)/2}\frac{1}{(2i)^2}\equiv1-\frac{7}{12}p^3B_{p-3}\pmod{p^4}\end{aligned}$$

and hence (1.5) holds. Similarly,

$$\begin{aligned}&\frac{1}{p}\sum_{m=0}^{p-1}(2m+1)D_m=\sum_{k=0}^{p-1}\frac{p}{k+1}\binom{p-1}{k}\binom{p+k}{k}\\ \equiv&\binom{p+(p-1)}{p-1}+p\sum_{k=0}^{p-2}\frac{(-1)^k}{k+1}\bigg(1-p^2\sum_{0<j\leqslant k}\frac{1}{j^2}\bigg)\pmod{p^5}\\ \equiv&\binom{2p-1}{p-1}-p\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k}\equiv1-p\sum_{j=1}^{(p-1)/2}\frac{1}{j}\pmod{p^3}.\end{aligned}$$

(In the last step we employ Wolstenholme's Congruences $\binom{2p-1}{p-1}\equiv1\pmod{p^3}$ and $\sum_{k=1}^{p-1}\frac{1}{k}\equiv0\pmod{p^2}$). To obtain (1.6) it suffices to apply Lehmer's congruence (cf. [L])

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k}\equiv-2q_p(2)+p\,q_p^2(2)\pmod{p^2}.$$

The proof of Theorem 1.2 is now complete. □

3. Proofs of Theorems 1.3-1.4

Lemma 3.1. *Let p be an odd prime and let $m\in\mathbb{Z}$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m(m-4)}{p}\right)\pmod p \tag{3.1}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{m^k}\equiv\frac m2-\frac{m-4}{2}\left(\frac{m(m-4)}{p}\right)\pmod p. \tag{3.2}$$

Proof. Clearly

$$\binom{2k}{k}=\binom{-1/2}{k}(-4)^k\equiv\binom{(p-1)/2}{k}(-4)^k$$

for all $k\in\mathbb{N}$. Thus

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}\equiv&\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\frac{(-4)^k}{m^k}=\left(1-\frac4m\right)^{(p-1)/2}\\ =&\frac{(m(m-4))^{(p-1)/2}}{m^{p-1}}\equiv\left(\frac{m(m-4)}{p}\right)\pmod p.\end{aligned}$$

This proves (3.1).

Observe that

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}+\binom{2k}{k+1}}{m^k}\\ =&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k+1}{k}}{m^k}=\frac{\binom{p}{(p-1)/2}}{m^{(p-1)/2}}+\frac12\sum_{k=0}^{(p-3)/2}\frac{\binom{2k+2}{k+1}}{m^k}\\ \equiv&\frac m2\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}-\frac m2\pmod p.\end{aligned}$$

Hence

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k+1}}{m^k}\equiv\left(\frac m2-1\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}-\frac m2\pmod p$$

and

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{C_k}{m^k}=&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}-\binom{2k}{k+1}}{m^k}\\ \equiv&\frac m2+\left(2-\frac m2\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}\\ \equiv&\frac m2-\frac{m-4}{2}\left(\frac{m(m-4)}{p}\right)\pmod p.\end{aligned}$$

So (3.2) also holds.

Proof of Theorem 1.3. In the case $c\equiv 0 \pmod p$, as $T_k(b,c)\equiv b^k \pmod c$ for all $k\in\mathbb{N}$, we have

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{m^k}\equiv\sum_{k=0}^{p-1}\frac{b^k}{m^k}\equiv\left(\frac{(m-b)^2}{p}\right) \pmod p.$$

So (1.8) holds if $p\mid c$. Note that (1.9) is trivial when $p\mid c$.

Suppose that $c\not\equiv 0 \pmod p$. Note that for any $n\in\mathbb{N}$ we have

$$T_n(b,c)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k\equiv\begin{cases}\binom{n}{n/2}c^{n/2} \pmod b & \text{if } 2\mid n,\\ 0 \pmod b & \text{if } 2\nmid n.\end{cases}$$

In the case $b\equiv 0 \pmod p$, by applying Lemma 3.1 we obtain

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{m^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}c^k}{m^{2k}}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(m^2c^{p-2})^k}\equiv\left(\frac{m^2-4c}{p}\right) \pmod p$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{M_k(b,c)}{m^k}\equiv&\sum_{k=0}^{(p-1)/2}\frac{C_kc^k}{m^{2k}}\equiv\sum_{k=0}^{(p-1)/2}\frac{C_k}{(m^2c^{p-2})^k}\\ \equiv&\frac{m^2}{2c}-\frac{m^2-4c}{2c}\left(\frac{m^2-4c}{p}\right) \pmod p.\end{aligned}$$

So (1.8) and (1.9) hold when $p\mid b$.

Below we assume that $p\nmid bc$. Observe that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n(b,c)}{m^n}=&\sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k\\ =&\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{c^k}{b^{2k}}\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}\end{aligned}$$

and

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n}=\sum_{k=0}^{(p-1)/2}C_k\frac{c^k}{b^{2k}}\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}$$

in a similar way.

Now we consider the case $m \equiv b \pmod p$. For $k \in \{0, 1, \dots, (p-1)/2\}$ we have

$$\sum_{k=0}^{p-1} \frac{b^n}{m^n}\binom{n}{2k} \equiv \sum_{n=2k}^{p-1}\binom{n}{2k} = \sum_{j=0}^{p-1-2k}\binom{2k+j}{j} = \binom{p}{2k+1} \pmod p$$

by Lemma 2.2. Thus, by the above,

$$\sum_{n=0}^{p-1} \frac{T_n(b,c)}{m^n} \equiv \binom{p-1}{(p-1)/2}\frac{c^{(p-1)/2}}{b^{p-1}} \equiv \left(\frac{-c}{p}\right) = \left(\frac{(m-b)^2-4c}{p}\right) \pmod p$$

and

$$\sum_{n=0}^{p-1} \frac{M_n(b,c)}{m^n} \equiv C_{(p-1)/2}\frac{c^{(p-1)/2}}{b^{p-1}} \equiv 2\left(\frac{-c}{p}\right) = 2\left(\frac{(m-b)^2-4c}{p}\right) \pmod p.$$

So (1.8) and (1.9) are true.

Below we consider the remaining case $m \not\equiv b \pmod p$. Observe that

$$\begin{aligned}
&\sum_{n=0}^{p-1} \frac{b^n}{m^n}\binom{n}{2k} = [x^{2k}]\sum_{n=0}^{p-1}\frac{b^n}{m^n}(1+x)^n \\
\equiv&[x^{2k}]\sum_{n=0}^{p-1}(b+bx)^n m^{p-1-n} = [x^{2k}]\frac{(b+bx)^p - m^p}{b+bx-m} \\
=&[x^{2k}]\frac{(b+bx)^p-m^p}{-(m-b)^p}\cdot\frac{(bx)^p-(m-b)^p}{bx-(m-b)} \\
\equiv&[x^{2k}]\frac{b^p+b^px^p-m^p}{-(m-b)^p}\sum_{j=0}^{p-1}(bx)^j(m-b)^{p-1-j} \equiv \frac{b^{2k}}{(m-b)^{2k}} \pmod p.
\end{aligned}$$

Therefore, with the help of Lemma 3.1,

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{T_n(b,c)}{m^n} \equiv& \sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{c^k}{b^{2k}}\cdot\frac{b^{2k}}{(m-b)^{2k}} \\
\equiv& \left(1-\frac{4c}{(m-b)^2}\right)^{(p-1)/2} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod p.
\end{aligned}$$

This proves (1.8)

In a similar way,

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n} \equiv \sum_{k=0}^{(p-1)/2} C_k\frac{c^k}{(m-b)^{2k}} \equiv \sum_{k=0}^{(p-1)/2}\frac{C_k}{M^k} \pmod p,$$

where $M := (m-b)^2c^{p-2}$. Applying Lemma 3.1 we get the desired (1.9). □

Lemma 3.2. *For any $d \in \mathbb{Z}$ we have*

$$T_n(2d+1, d^2+d) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} d^k. \tag{3.3}$$

Proof. The Legendre polynomial of degree n is defined by

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

It is well known that

$$\sum_{n=0}^{\infty} P_n(t)x^n = \frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if we set $b = 2d+1$ and $c = d^2+d$ then

$$\sum_{n=0}^{\infty} P_n(b)x^n = \frac{1}{\sqrt{1-2bx+(b^2-4c)x^2}} = \sum_{n=0}^{\infty} T_n(b,c)x^n$$

and hence

$$T_n(b,c) = P_n(b) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} d^k.$$

This proves (3.3). □

Lemma 3.3. *For $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1} (2m+1)^2 \binom{m+k}{2k} = (4n^2-1)\frac{n-k}{2k+3}\binom{n+k}{2k}. \tag{3.4}$$

Proof. Observe that

$$\begin{aligned}
&(4n^2-1)\frac{n-k}{2k+3}\binom{n+k}{2k} + (2n+1)^2\binom{n+k}{2k} \\
=&(4n^2+8n+3)\frac{n+1-k}{2k+3}\binom{n+k}{2k} \\
=&(4(n+1)^2-1)\frac{n+1-k}{2k+3}\binom{n+1+k}{2k}.
\end{aligned}$$

So we can easily prove (3.4) by induction on n. □

Proof of Theorem 1.4. (i) We first prove (1.11) by induction.

When $n = 1$, both sides of (1.11) are equal to b.

Now assume that (1.11) holds for a fixed integer $n \geqslant 1$. Then

$$\begin{aligned}
& b\sum_{k=0}^{(n+1)-k}(2k+1)T_k(b,c)^2(4c-b^2)^{(n+1)-1-k} \\
=& b(2n+1)T_n(b,c)^2+(4c-b^2)b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(4c-b^2)^{n-1-k} \\
=& b(2n+1)T_n(b,c)^2+(4c-b^2)nT_n(b,c)T_{n-1}(b,c) \\
=& (n+1)T_{n+1}(b,c)T_n(b,c) \quad \text{(by (1.7))}.
\end{aligned}$$

This concludes the induction step.

Now we fix a positive integer n and want to show (1.10). Recall that

$$T_n(b,c) \equiv \begin{cases} \binom{n}{n/2}c^{n/2} \pmod b & \text{if } 2 \mid n, \\ 0 \pmod b & \text{if } 2 \nmid n. \end{cases}$$

When $b \neq 0$, b divides $T_n(b,c)$ or $T_{n-1}(b,c)$ since n or $n-1$ is odd, therefore (1.10) follows from (1.11).

Now it remains to consider the case $b = 0$. Note that $T_k(0,c) = 0$ for $k = 1, 3, 5, \ldots$, and $T_k(0,c) = \binom{k}{k/2}c^{k/2}$ for $k = 0, 2, 4, \ldots$. Thus

$$\begin{aligned}
& \sum_{k=0}^{n-1}(2k+1)T_k(0,c)^2(4c-0^2)^{n-1-k} \\
=& \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(4k+1)\left(\binom{2k}{k}c^k\right)^2(4c)^{n-1-2k} \\
=& (4c)^{n-1}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(4k+1)\frac{\binom{2k}{k}^2}{16^k}.
\end{aligned}$$

By induction, for any $m \in \mathbb{N}$ we have the identity

$$\sum_{k=0}^{m}(4k+1)\frac{\binom{2k}{k}^2}{16^k} = \frac{(m+1)^2}{16^m}\binom{2m+1}{m}^2 = \frac{(2m+1)^2}{16^m}\binom{2m}{m}^2,$$

which was pointed out to the author by R. Tauraso. It follows that

$$4^{n-1}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(4k+1)\frac{\binom{2k}{k}^2}{16^k} = n^2\binom{n-1}{\lfloor n/2\rfloor}^2.$$

Therefore

$$\sum_{k=0}^{n-1}(2k+1)T_k(0,c)^2(4c-0^2)^{n-1-k}\equiv 0 \pmod{n^2}$$

and hence (1.10) holds when $b=0$.

(ii) We prove (1.12) by induction. (1.12) is obvious when $n=1$.

Now suppose the validity of (1.12) for a fixed $n\in\mathbb{Z}^+$. Observe that

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1+k}{k}\left(\frac{b-1}{2}\right)^k-n\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}b^k\\=&\sum_{k=0}^{n}\left((n+1+k)\binom{n+1}{k+1}-n\binom{n}{k+1}\right)\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k\\=&(2n+1)\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k.\end{aligned}$$

Therefore, by Lemma 3.2 and the induction hypothesis, we have

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1+k}{k}\left(\frac{b-1}{2}\right)^k\\=&\sum_{k=0}^{n-1}(2k+1)T_k(b,c)+(2n+1)T_n(b,c)=\sum_{k=0}^{n}(2k+1)T_k(b,c).\end{aligned}$$

This proves (1.12) with n replaced by $n+1$.

As $b^2-4c=1$, for some $d\in\mathbb{Z}$ we have $b=2d+1$ and $c=d^2+d$. Let p be a prime not dividing $c=d(d+1)$. In light of (1.12),

$$\begin{aligned}&\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)T_k(b,c)=\sum_{k=0}^{p-1}\binom{p}{k+1}\binom{p+k}{k}d^k\\=&\binom{2p-1}{p-1}d^{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}\binom{p+k}{k}d^k\\\equiv&d^{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}d^k=d^{p-1}+\frac{(d+1)^p-d^p-1}{d}\\\equiv&1+\frac{(d+1)^p-(d+1)}{d}=1+\frac{b+1}{b-1}\left(\left(\frac{b+1}{2}\right)^{p-1}-1\right)\pmod{p^2}\end{aligned}$$

and hence (1.13) follows.

Now we fix an odd prime p and show (1.14). Let $d = (b-1)/2$. In view of Lemmas 3.2 and 3.3,

$$\begin{aligned}&\sum_{m=0}^{p-1}(2m+1)^2T_m(b,c)\\&=\sum_{m=0}^{p-1}(2m+1)^2\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}d^k\\&=\sum_{k=0}^{p-1}\binom{2k}{k}d^k\sum_{m=0}^{p-1}(2m+1)^2\binom{m+k}{2k}\\&=(4p^2-1)\sum_{k=0}^{p-1}\frac{p-k}{2k+3}\binom{p+k}{2k}\binom{2k}{k}d^k\\&=(4p^2-1)\sum_{k=0}^{p-1}\frac{p-k}{2k+3}\binom{p}{k}\binom{p+k}{k}d^k.\end{aligned}$$

Since $p \mid \binom{p}{k}$ and $\binom{p+k}{k} \equiv 1 \pmod{p}$ for $k = 1, \ldots, p-1$, from the above we obtain

$$\begin{aligned}&\sum_{k=0}^{p-1}(2k+1)^2T_k(b,c)\\&\equiv(4p^2-1)\frac{(p+3)/2}{p}\binom{p}{(p-3)/2}\binom{p+(p-3)/2}{(p-3)/2}d^{(p-3)/2}\\&\equiv-\binom{p-1}{(p+1)/2}d^{(p-3)/2}\equiv(-1)^{(p-1)/2}d^{(p-3)/2}\pmod{p}\end{aligned}$$

and hence

$$d\sum_{k=0}^{p-1}(2k+1)^2T_k(b,c)\equiv(-d)^{(p-1)/2}\equiv\left(\frac{-d}{p}\right)\pmod{p}$$

as desired.

In view of the above, we have completed the proof of Theorem 1.4. □

4. More conjectures on generalized central trinomial coefficients

Those integers

$$S_n=\sum_{k=0}^{n}\binom{n+k}{2k}C_k=\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}\ (n\in\mathbb{N})$$

are called Schröder numbers. It is known that S_n coincides with the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0), (0,1)$ and $(1,1)$ that never rise above the line $y = x$ (see, e.g., [St]).

Conjecture 4.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1} D_kS_k \equiv 1+4pq_p(2)-2p^2q_p(2)^2 \pmod{p^3},$$

and

$$\sum_{k=1}^{(p-1)/2} D_kS_k \equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1 \pmod 4 \text{ and } p=x^2+y^2\ (2\nmid x), \\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

Conjecture 4.2. *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}(2k+1)^2T_k(7,12) \equiv \left(\frac{p}{3}\right) - 4p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)^2D_k \equiv \left(\frac{-1}{p}\right) - 2p + (2-E_{p-3})p^2 \pmod{p^3}.$$

Conjecture 4.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}T_k^2 \equiv \sum_{k=0}^{p-1}\frac{T_k^2}{9^k} \equiv \left(\frac{-1}{p}\right) \pmod p, \quad \sum_{k=0}^{p-1}\frac{T_k^2}{(-3)^k} \equiv \left(\frac{p}{3}\right) \pmod p,$$

$$\sum_{k=0}^{p-1}\frac{M_k^2}{9^k} \equiv 6\left(\frac{p}{3}\right) - 20 \pmod p, \quad \sum_{k=1}^{p-1}\frac{M_k^2}{k} \equiv \frac{1}{2} - \left(\frac{p}{3}\right) \pmod p,$$

$$\sum_{k=0}^{p-1}\frac{kT_k^2}{(-3)^k} \equiv -\frac{1}{2}\left(\frac{p}{3}\right) \pmod p, \quad \sum_{k=0}^{p-1}\frac{kM_k^2}{(-3)^k} \equiv 5 \pmod p,$$

$$\sum_{k=0}^{p-1}k^2M_k^2 \equiv \left(\frac{-1}{p}\right) - \left(\frac{p}{3}\right) \pmod p, \quad \sum_{k=0}^{p-1}\frac{k^2M_k^2}{(-3)^k} \equiv 3\left(\frac{p}{3}\right) - 11 \pmod p.$$

We also have

$$\sum_{k=0}^{p-1}kM_kT_k \equiv \left(\frac{-1}{p}\right) - \frac{5}{3}\left(\frac{p}{3}\right) \pmod p, \quad \sum_{k=0}^{p-1}\frac{kM_kT_k}{(-3)^k} \equiv 2\left(\frac{p}{3}\right) \pmod p,$$

$$\sum_{k=0}^{p-1}\frac{M_kT_k}{9^k} \equiv -4\left(\frac{p}{3}\right) \pmod p, \quad \sum_{k=0}^{p-1}\frac{kM_kT_k}{9^k} \equiv 3\left(\frac{-1}{p}\right) + 7\left(\frac{p}{3}\right) \pmod p.$$

Conjecture 4.4. *Let p be an odd prime.*

(i) *If $b,c\in\mathbb{Z}$ and $b^2-4c\not\equiv 0\pmod p$, then*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{(b^2-4c)^k}\equiv\left(\frac{c(b^2-4c)}{p}\right)\pmod p.$$

(ii) *We have*

$$\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{8^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{T_k(2,-3)^2}{16^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}\quad\text{if } p>3.$$

Remark 4.1. By Theorem 1.3, if p is an odd prime not dividing b^2-4c with $b,c\in\mathbb{Z}$ then

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{(b^2-4c)^k}\equiv\left(\frac{(b^2-4c)((b-1)^2-4c)}{p}\right)\pmod p.$$

Conjecture 4.5. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}T_k(1,2)^2\equiv\sum_{k=0}^{p-1}\frac{T_k(2,-2)^2}{(-4)^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{(-8)^k}$$
$$\equiv\begin{cases}2x\pmod p & \text{if } p\equiv 1\pmod 4\ \&\ p=x^2+y^2\ (4\mid x-1),\\ 0\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}T_k(1,-1)^2\equiv\sum_{k=0}^{p-1}\frac{T_k(2,2)^2}{4^k}$$
$$\equiv\begin{cases}(\frac{2}{p})2x\pmod p & \text{if } p\equiv 1\pmod 4\ \&\ p=x^2+y^2\ (4\mid x-1),\\ 0\pmod p & \text{if } p\equiv 3\pmod 4,\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{T_k(2,2)^2}{4^k}-\sum_{k=0}^{p-1}\frac{T_k(2,1)^2}{8^k}\equiv\begin{cases}0\pmod{p^3} & \text{if } p\equiv 1\pmod 4,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Conjecture 4.6. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(8k+7)T_k(3,1)^2\equiv 0 \pmod{n}$$

and

$$\sum_{k=0}^{n-1}(k+1)T_k(3,1)^2 4^{n-1-k}\equiv 0 \pmod{n}.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}T_k(3,1)^2\equiv\left(\frac{-1}{p}\right) \pmod{p},$$

$$\sum_{k=0}^{p-1}(8k+7)T_k(3,1)^2\equiv 5p\left(\frac{p}{5}\right) \pmod{p^2}.$$

Also,

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k}\equiv\left(\frac{-1}{p}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(k+1)\frac{T_k(4,1)^2}{4^k}\equiv\frac{3}{4}\left(\frac{3}{p}\right)p \pmod{p^2}.$$

Conjecture 4.7. *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(8k+9)T_k(5,1)^2 9^{n-1-k}\equiv 0 \pmod{n}.$$

If $p>5$ is a prime, then

$$\sum_{k=0}^{p-1}\frac{T_k(5,1)^2}{9^k}\equiv\left(\frac{-1}{p}\right) \pmod{p}$$

and

$$\sum_{k=0}^{p-1}(8k+9)\frac{T_k(5,1)^2}{9^k}\equiv 7p\left(\frac{p}{21}\right) \pmod{p^2}.$$

5. Conjectures on a new kind numbers

Motivated by central trinomial coefficients and Apéry numbers, for $b,c\in\mathbb{Z}$ we introduce a new kind of numbers:

$$W_n(b,c):=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n-k}{k}^2 b^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}^2\binom{2k}{k}^2 b^{n-2k}c^k\ \ (n\in\mathbb{N}).$$

Note that $W_n(-b,c)=(-1)^nW_n(b,c)$. For these numbers we have the following conjectures.

Conjecture 5.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}W_k(1,1)$$
$$\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1,3\ (\mathrm{mod}\ 8) \text{ and } p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 5,7\ (\mathrm{mod}\ 8).\end{cases}$$

If $p\equiv 1,3\ (\mathrm{mod}\ 8)$, then

$$\sum_{k=0}^{p-1}(16k+3)W_k(1,1)\equiv 8p\ (\mathrm{mod}\ p^2).$$

When $p\equiv 5,7\ (\mathrm{mod}\ 8)$ and $p\neq 7$, we have

$$\sum_{k=0}^{p-1}\frac{W_k(1,1)}{(-7)^k}\equiv 0\ (\mathrm{mod}\ p^2).$$

Conjecture 5.2. (i) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kW_k(1,-1)$$
$$\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1\ (\mathrm{mod}\ 3) \text{ and } p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 2\ (\mathrm{mod}\ 3).\end{cases}$$

(ii) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(6k+5)(-1)^kW_k(1,-1)\equiv 0\ (\mathrm{mod}\ n).$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1}(6k+5)(-1)^kW_k(1,-1)\equiv p\left(2+3\left(\frac{p}{3}\right)\right)\pmod{p^2}.$$

Remark 5.1. Let $p>3$ be a prime. We also conjecture that

$$\sum_{k=0}^{p-1}\frac{W_k(1,-1)}{(-13)^k}\equiv 0\pmod p\quad\text{if } p\equiv 2\pmod 3,$$

and

$$\sum_{k=0}^{p-1}\frac{W_k(1,-1)}{(-3)^k}\equiv\sum_{k=0}^{p-1}\frac{W_k(1,-1)}{5^k}\equiv 0\pmod p\quad\text{if } p\equiv 3\pmod 4.$$

Conjecture 5.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{W_k(2,-1)}{(-2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 4 \text{ and } p=x^2+y^2\ (2\nmid x),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

If $p\equiv 1\pmod 4$, then

$$\sum_{k=0}^{p-1}(4k+3)\frac{W_k(2,-1)}{(-2)^k}\equiv 0\pmod{p^2}.$$

Conjecture 5.4. (i) *Let p be an odd prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{W_k(2,1)}{(-2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,3\pmod 8 \text{ and } p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

(ii) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(4k+3)W_k(2,-1)(-2)^{n-1-k}\equiv 0\pmod n.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1}(4k+3)\frac{W_k(2,-1)}{(-2)^k}\equiv p\left(2\left(\frac{2}{p}\right)+\left(\frac{-1}{p}\right)\right)\pmod{p^2}.$$

Conjecture 5.5. (i) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{W_k(4,-1)}{(-4)^k}\equiv\sum_{k=0}^{p-1}\frac{W_k(4,-9)}{4^k}\equiv\sum_{k=0}^{p-1}\frac{W_k(4,9)}{16^k}$$
$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2}&\text{if } p\equiv1\pmod 3 \text{ and } p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } p\equiv 2\pmod 8.\end{cases}$$

(ii) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(3k+2)W_k(4,-1)(-4)^{n-1-k}\equiv 0\pmod{2n},$$
$$\sum_{k=0}^{n-1}(3k+2)W_k(4,9)16^{n-1-k}\equiv 0\pmod{2n},$$

and

$$\sum_{k=0}^{n-1}(5k+4)W_k(4,-9)4^{n-1-k}\equiv 0\pmod{2n}.$$

If p is an odd prime, then

$$\sum_{k=0}^{p-1}(3k+2)\frac{W_k(4,-1)}{(-4)^k}\equiv\frac{3(\frac{3}{p})+(\frac{-1}{p})}{2}p\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(3k+2)\frac{W_k(4,9)}{16^k}\equiv 2p\pmod{p^2}.$$

If $p>3$ is a prime, then

$$\sum_{k=0}^{p-1}(5k+4)\frac{W_k(4,-9)}{4^k}\equiv\frac{3(\frac{3}{p})+5(\frac{-1}{p})}{2}p\pmod{p^2}.$$

Conjecture 5.6. (i) *For any prime $p\neq 3,7$, we have*

$$\sum_{k=0}^{p-1}W_k(1,7^4)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,3\pmod 8\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } p\equiv 5,7\pmod 8.\end{cases}$$

(ii) *For all $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(40k+37)W_k(1,7^4)\equiv 0 \pmod{n}.$$

If $p\neq 7$ is a prime, then

$$\sum_{k=0}^{p-1}(40k+37)W_k(1,7^4)\equiv p\left(17\left(\frac{p}{3}\right)+20\right) \pmod{p^2}.$$

Conjecture 5.7. (i) *For any prime $p\neq 7$, we have*

$$\sum_{k=0}^{p-1}(-1)^kW_k(1,-16)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,2,4 \pmod 7\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

(ii) *For all $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(42k+37)(-1)^kW_k(1,-16)\equiv 0 \pmod{n}.$$

If p is a prime, then

$$\sum_{k=0}^{p-1}(42k+37)(-1)^kW_k(1,-16)\equiv p\left(21\left(\frac{p}{7}\right)+16\right) \pmod{p^2}.$$

Remark 5.2. Let p be an odd prime with $(\frac{p}{7})=1$. It is well known that $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$ (see, e.g., [C]).

Conjecture 5.8. (i) *Let $p\neq 2,5$ be a prime. Then we have*

$$\sum_{k=0}^{p-1}W_k(1,-4)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20}\ \&\ p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } p\equiv 3,7 \pmod{20}\ \&\ 2p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 11,13,17,19 \pmod{20}.\end{cases}$$

(ii) *For any* $n \in \mathbb{Z}^+$ *we have*

$$\sum_{k=0}^{n-1}(20k+17)W_k(1,-4) \equiv 0 \pmod{n}.$$

If p *is an odd prime, then*

$$\sum_{k=0}^{p-1}(20k+17)W_k(1,-4) \equiv p\left(10\left(\frac{-1}{p}\right)+7\right) \pmod{p^2}.$$

Remark 5.3. Let $p \neq 2,5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $p \equiv 1,9 \pmod{20}$ then $p = x^2+5y^2$ for some $x,y \in \mathbb{Z}$; if $p \equiv 3,7 \pmod{20}$ then $2p = x^2+5y^2$ for some $x,y \in \mathbb{Z}$.

Conjecture 5.9. (i) *For any prime* $p > 5$*, we have*

$$\sum_{k=0}^{p-1}W_k(1,81)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p \equiv 1,9,11,19 \pmod{40}\ \&\ p = x^2+10y^2,\\ 2p-2x^2 \pmod{p^2} & \text{if } p \equiv 7,13,23,37 \pmod{40}\ \&\ 2p = x^2+10y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-10}{p}) = -1.\end{cases}$$

(ii) *For any* $n \in \mathbb{Z}^+$ *we have*

$$\sum_{k=0}^{n-1}(10k+9)W_k(1,81) \equiv 0 \pmod{n}.$$

If $p > 3$ *is a prime, then*

$$\sum_{k=0}^{p-1}(10k+9)W_k(1,81) \equiv p\left(4\left(\frac{-2}{p}\right)+5\right) \pmod{p^2}.$$

Remark 5.4. Let $p > 5$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $(\frac{-2}{p}) = (\frac{p}{5}) = 1$ then $p = x^2+10y^2$ for some $x,y \in \mathbb{Z}$; if $(\frac{-2}{p}) = (\frac{p}{5}) = -1$ then $2p = x^2+10y^2$ for some $x,y \in \mathbb{Z}$.

Conjecture 5.10. (i) *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} W_k(1,-324)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{13}{p})=(\frac{-1}{p})=1 \ \&\ p=x^2+13y^2, \\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{13}{p})=(\frac{-1}{p})=-1 \ \&\ 2p=x^2+13y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-13}{p})=-1. \end{cases}$$

(ii) *For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(260k+237)W_k(1,-324) \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(260k+237)W_k(1,-324) \equiv p\left(130\left(\frac{-1}{p}\right)+107\right) \pmod{p^2}.$$

Remark 5.5. Let $p > 3$ be a prime. By the theory of binary quadratic forms (see, e.g., [C]), if $(\frac{13}{p}) = (\frac{-1}{p}) = 1$ then $p = x^2 + 13y^2$ for some $x, y \in \mathbb{Z}$; if $(\frac{13}{p}) = (\frac{-1}{p}) = -1$ then $2p = x^2 + 13y^2$ for some $x, y \in \mathbb{Z}$.

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