The q-Harmonic Oscillator and the Al-Salam and Carlitz polynomials

Dedicated to the Memory of Professor Ya. A. Smorodinskiĭ

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Abstract. One more model of a q-harmonic oscillator based on the q-orthogonal polynomials of Al-Salam and Carlitz is discussed. The explicit form of q-creation and q-annihilation operators, q-coherent states and an analog of the Fourier transformation are established. A connection of the kernel of this transform with a family of self-dual biorthogonal rational functions is observed.

Introduction

Recent development in quantum groups has led to the so-called q-harmonic oscillators (see, for example, Refs. [1–7]). Presently known models of q-oscillators are closely related with q-orthogonal polynomials. The q-analogs of boson operators have been introduced explicitly in Refs. [3], [5] and [7], where the corresponding wave functions were constructed in terms of the continuous q-Hermite polynomials of Rogers [8,9], in terms of the Stieltjes–Wigert polynomials [10,11] and in terms of q-Charlier polynomials of Al-Salam and Carlitz [12], respectively. The model related to the Rogers–Szegö polynomials [13] was investigated in [1,6]. Here we introduce the explicit realization of q-creation and q-annihilation operators with the aid of another family of the Al-Salam and Carlitz polynomials [12] when eigenvalues of the corresponding q-Hamiltonian are unbounded. An attempt to unify q-boson operators is also made.

With a great deal of regret we dedicate this paper to the memory of Yacob A. Smorodinskiĭ, who suggested ten years ago that the special case $q = 1$ of this work is interesting and admits a generalization.

1. The Al-Salam and Carlitz Polynomials

The aim of this Letter is to show that the q-orthogonal polynomials $U_n^{(a)}(x;q)$ studied by Al-Salam and Carlitz are closely connected with the q-harmonic oscillator. To emphasize these relations we use the notation $u_n^\mu(x;q) = \mu^{-n} q^{-n(n-1)/2} U_n^{(-\mu)}(x;q)$ for the Al-Salam and Carlitz polynomials. In our notation they can be defined by the *three-term recurrence relation* of the form

$$\mu q^n u_{n+1}^\mu(x;q) + (1-q^n)\, u_{n-1}^\mu(x;q) = (x - (1-\mu)q^n)\, u_n^\mu(x;q)\,, \tag{1}$$

$u_0^\mu(x;q) = 1$, $u_1^\mu(x;q) = \mu^{-1}(x-1+\mu)$. These polynomials are *orthogonal*

$$\int_{-\mu}^{1} u_m^\mu(x;q)\, u_n^\mu(x;q)\, d\alpha(x) = (1+\mu) q^{-n(n-1)/2}\, \frac{(q;q)_n}{\mu^n}\, \delta_{mn} \tag{2}$$

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with respect to a positive measure $d\alpha(x)$, where $\alpha(x)$ is a step function with jumps

$$\frac{q^k}{(-q\mu;q)_\infty (q,-q/\mu;q)_k}$$

at the points $x = q^k, k = 0, 1, \ldots,$ and jumps

$$\frac{\mu q^k}{(-q/\mu;q)_\infty (q,-q\mu;q)_k}$$

at the points $x = -\mu q^k, k = 0, 1, \ldots$ (see, for example, [12,14,15]). Here the usual notations are

$$\begin{aligned}
(a;q)_n &= \prod_{k=0}^{n-1} (1 - aq^k)\,, \\
(a,b;q)_n &= (a;q)_n (b;q)_n\,, \\
(a;q)_\infty &= \lim_{n\to\infty} (a;q)_n\,.
\end{aligned} \tag{3}$$

The orthogonality relation (2) can also be written in terms of the *q-integral* of Jackson,

$$\int_{-\mu}^{1} u_m^\mu(x;q)\, u_n^\mu(x;q)\, \tilde{\rho}(x)\, d_q x = (1-q) d_n^2\, \delta_{mn}\,, \tag{4}$$

where

$$\tilde{\rho}(x) = \frac{(qx, -\mu^{-1}qx;q)_\infty}{(q,-\mu,-q/\mu;q)_\infty}\,; \quad \mu > 0\,,\ 0 < q < 1 \tag{5}$$

and

$$d_n^2 = q^{-n(n-1)/2}\, \frac{(q;q)_n}{\mu^n}. \tag{6}$$

For the definition of the q-integral, see [15]. The "weight function" $\rho(s) = \tilde{\rho}(x)$ in (5) is a solution of the *Pearson-type equation* $\Delta(\sigma\rho) = \rho\tau\nabla x_1$ with $x(s) = q^s$, $\sigma(s) = (1-q^s)(\mu+q^s)$ and $\sigma(s) + \tau(s)\nabla x_1(s) = \mu$. The polynomials $y_n(s) = u_n^\mu(x;q)$ satisfy the *hypergeometric-type difference equation* in self-adjoint form,

$$\frac{\Delta}{\nabla x_1(s)} \left[\sigma(s)\, \rho(s)\, \frac{\nabla y_n(s)}{\nabla x(s)} \right] + \lambda_n\, \rho(s)\, y_n(s) = 0\,,$$

where

$$\lambda_n = q^{3/2}\, \frac{q^{-n} - 1}{(1-q)^2}\,.$$

Here $\Delta f(s) = f(s+1) - f(s) = \nabla f(s+1)$ and $x_1(s) = x(s+1/2)$. (For details, see [16–19].) The orthogonality property (2) or (4) can be proved by using standard Sturm–Liouville-type arguments (cf. [16–19]).

The explicit form of the polynomials $u_n^\mu(x;q)$ is

$$u_n^\mu(x;q) = {}_2\varphi_1\left(q^{-n},\, x^{-1};\, 0\,;\, q\,,\, -\frac{q}{\mu}\, x\right) \tag{7}$$
$$= (-\mu^{-1})^n\, {}_2\varphi_1(q^{-n}, -\mu x^{-1};\, 0\,;\, q\,, qx)\,,\ x = q^s\,.$$

It means $u_n^\mu(x;q) = \left(-\mu^{-1}\right)^n u_n^{1/\mu}(-\mu^{-1}x;q)$. In the limit $q \to 1$ it easy to see from (1) or (7) that

$$\lim_{q\to 1} u_n^{(1-q)\mu}(q^s;q) = {}_2F_0(-n,-s;-;-1/\mu) = c_n^\mu(s)\,, \tag{8}$$

where $c_n^\mu(x)$ are the Charlier polynomials.

2. Model of q-Harmonic Oscillator

The Al-Salam and Carlitz polynomials $u_n^\mu(x;q)$ allow us to consider an interesting model of a q-oscillator (cf. [7]). We can introduce a *q-version of the wave functions* of the harmonic oscillator as

$$\psi_n(s) = \tilde{\psi}_n(x) = d_n^{-1}\left(\tilde{\rho}(x)|x|\right)^{1/2} u_n^\mu(x;q)\,,\ x = q^s\,, \tag{9}$$

where $\tilde{\rho}(x)$ and d_n^2 are defined in (5) and (6), respectively. These q-wave functions satisfy the orthogonality relation

$$(1-q)^{-1}\int_{-\mu}^{1} \tilde{\psi}_n(x)\,\tilde{\psi}_m(x)\,|x|^{-1}\,d_qx = \delta_{nm}\,, \tag{10}$$

which is equivalent to (2) and (4).

The *q-annihilation* b and *q-creation* b^+ operators have the following explicit form

$$b = (1-q)^{-\frac{1}{2}}\left[\mu^{\frac{1}{2}}q^{-s} - \sqrt{(1-q^{s+1})(\mu q^{-1}+q^s)}\,q^{-s}\,e^{\partial_s}\right]\,,$$
$$b^+ = (1-q)^{-\frac{1}{2}}\left[\mu^{\frac{1}{2}}q^{-s} - e^{-\partial_s}\sqrt{(1-q^{s+1})(\mu q^{-1}+q^s)}\,q^{-s}\right]\,, \tag{11}$$

where $\partial_s \equiv \dfrac{d}{ds}$, $e^{\alpha\partial_s}f(s) = f(s+\alpha)$. These operators are adjoint, $(b^+\psi,\chi) = (\psi, b\chi)$, with respect to the scalar product (10). They satisfy the *q-commutation rule*

$$b\,b^+ - q^{-1}b^+b = 1 \tag{12}$$

and act on the q-wave functions defined in (9) by

$$b\,\psi_n = \tilde{e}_n^{1/2}\,\psi_{n-1},\ \ b^+\psi_n = \tilde{e}_{n+1}^{1/2}\psi_{n+1}\,, \tag{13}$$

where

$$\tilde{e}_n = \frac{1-q^{-n}}{1-q^{-1}}\,.$$

The q-*Hamiltonian* $H = b^+ b$ acts on the wave functions (9) as

$$H\psi_n = \tilde{e}_n \psi_n \tag{15}$$

and has the following explicit form

$$H = (1-q)^{-1}\big[\,\mu q^{-2s} + (1-q^s)(\mu + q^s)\, q^{1-2s} - \\ \mu^{\frac{1}{2}} q^{-2s}\sqrt{(1-q^{s+1})(\mu q^{-1} + q^s)}\, e^{\partial_s} - \mu^{\frac{1}{2}} q^{2-2s}\sqrt{(1-q^s)(\mu q^{-1}+q^{s-1})}\, e^{-\partial_s}\big]\,. \tag{16}$$

By factorizing the Hamiltonian (or the difference equation for the Al-Salam and Carlitz polynomials) we arrive at the explicit form (11) for the q-boson operators. The equations (13) are equivalent to the following difference-differentiation formulas

$$\mu q^{-s-1}\Delta u_n^{\mu}(x;q) = (1-q^{-n})\, u_{n-1}^{\mu}(x;q)\,,$$
$$q^{-s}\nabla\,[\,\rho(s)\, u_n^{\mu}(x;q)\,] = \rho(s)\, u_{n+1}^{\mu}(x;q)\,,$$

respectively. Therefore, the main properties of the Al-Salam and Carlitz polynomials admit a simple group-theoretical interpretation in terms of the q-Heisenberg–Weyl algebra (12). The symmetric case $\mu = 1$ in the above formulas corresponds to the *discrete q-Hermite polynomials* $H_n(x;q)$ [12,15].

3. The q-Coherent States

For the model of the q-oscillator under discussion, by analogy with [7] we can construct explicitly the q-*coherent states* $\mid \alpha\rangle$ defined by

$$b \mid \alpha\rangle = \alpha \mid \alpha\rangle\,, \\ \mid \alpha\rangle = f_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n \psi_n(s)}{(\tilde{e}_n!)^{1/2}}\,, \quad \langle \alpha \mid \alpha\rangle = 1\,, \tag{17}$$

where

$$\tilde{e}_n! = \tilde{e}_1\tilde{e}_2\ldots\tilde{e}_n = q^{-n(n-1)/2}\,\frac{(q;q)_n}{(1-q)^n}\,,$$
$$f_\alpha = \left(-(1-q)\mid \alpha\mid^2; q\right)_\infty^{-1/2}\,.$$

By using (9) one can obtain

$$\mid \alpha\rangle = f_\alpha\,(\rho|q^s|)^{\frac{1}{2}} \sum_{n=0}^{\infty} u_n^{\mu}(x;q)\, q^{n(n-1)/2}\,\frac{t^n}{(q;q)_n}\,,\ t = \alpha\mu^{\frac{1}{2}}(1-q)^{\frac{1}{2}}\,. \tag{18}$$

With the aid of the Brenke-type generating function [12,14] for the Al-Salam and Carlitz polynomials,

$$\sum_{n=0}^{\infty} u_n^{\mu}(x;q)\, q^{n(n-1)/2}\, \frac{t^n}{(q;q)_n} = \frac{(-t\,,\, t/\mu;q)_\infty}{(xt/\mu;q)_\infty}\,, \quad \left| t\,\frac{x}{\mu}\right| < 1\,, \tag{19}$$

we arrive at the following *explicit form* for the q-coherent states

$$|\,\alpha\rangle = f_\alpha\,(\rho|q^s|)^{\frac{1}{2}}\, \frac{\left(-\alpha(1-q)^{1/2}\mu^{1/2}, \alpha(1-q)^{1/2}\mu^{-1/2};q\right)_\infty}{\left(\alpha(1-q)^{1/2}\mu^{-1/2}q^s;q\right)_\infty}\,, \tag{20}$$

where $\rho(s) = (q^{1+s}, -\mu^{-1}q^{1+s};q)_\infty/(q,-\mu,-q/\mu;q)_{\subset} nfty$. These coherent states are not orthogonal

$$\langle\alpha\mid\beta\rangle = \frac{(-(1-q)\alpha^*\beta\,;q)_\infty}{(-(1-q)\mid\alpha\mid^2, -(1-q)\mid\beta\mid^2;q)_\infty^{1/2}}\,,$$

where $*$ denotes the complex conjugate.

4. Analog of the Fourier Transformation

To define an analog of the *Fourier transform* we begin, in the spirit of Wiener's approach to the classical Fourier transform [20] (see also [7,21,22]), by deriving the kernel of the form

$$\begin{aligned} K_t(x,y) &= \sum_{n=0}^{\infty} t^n \tilde{\psi}_n(x)\, \tilde{\psi}_n(y) \\ &= (\tilde{\rho}(x)\tilde{\rho}(y)|xy|)^{\frac{1}{2}} \sum_{n=0}^{\infty} u_n^{\mu}(x;q)\, u_n^{\mu}(y;q)\, q^{n(n-1)/2}\, \frac{(\mu t)^n}{(q;q)_n}\,. \end{aligned} \tag{21}$$

The series can be summed with the aid of the bilinear generating function of Al-Salam and Carlitz [12]

$$\begin{aligned} \sum_{n=0}^{\infty} u_n^{\mu_1}(x;q)\, u_n^{\mu_2}(y;q)\, q^{n(n-1)/2}\, \frac{t^n}{(q;q)_n} &= \frac{(-t\,,\, t/\mu_1\,,\, t/\mu_2;q)_\infty}{(tx/\mu_1\,,\, ty/\mu_2;q)_\infty} \\ &\quad \cdot {}_3\varphi_2\left(\begin{matrix} x^{-1}\,,\, y^{-1}\,,\, -qt^{-1} \\ q\mu_1(tx)^{-1},\, q\mu_2(ty)^{-1} \end{matrix} ; q\,,\, q \right) \end{aligned} \tag{22}$$

(the ${}_3\varphi_2$-series is terminating and $max\,(\,|t/\mu_1|\,,\,|t/\mu_2|\,) < 1$). The answer is

$$\begin{aligned} K_t(x,y) &= (\tilde{\rho}(x)\tilde{\rho}(y)|xy|)^{\frac{1}{2}}\, \frac{(t\,,\, t\,,\, -\mu t\,;q)_\infty}{(tx\,,ty\,;q)_\infty} \cdot {}_3\varphi_2\left(\begin{matrix} x^{-1}\,,\, y^{-1}\,,\, -q(\mu t)^{-1} \\ q(tx)^{-1}\,,\, q(ty)^{-1} \end{matrix} ; q\,,\, q \right)\,; \\ &= (\tilde{\rho}(x)\tilde{\rho}(y)|xy|)^{\frac{1}{2}}\, \frac{(t\,,\, t\,,\, -\mu^{-1}t\,;q)_\infty}{(-\mu^{-1}tx, -\mu^{-1}ty\,;q)_\infty} \cdot {}_3\varphi_2\left(\begin{matrix} -\mu x^{-1}, -\mu y^{-1}, -q\mu t^{-1} \\ -q\mu(tx)^{-1}, -q\mu(ty)^{-1} \end{matrix} ; q, q \right) \end{aligned} \tag{23}$$

at $x = q^s$ and at $x = -\mu q^s$ for $s = 0, 1, \ldots$, respectively.

In view of (10) and (21),

$$t^m \tilde{\psi}_m(x) = (1-q)^{-1} \int_{-\mu}^{1} K_t(x,y)\, \tilde{\psi}_m(y)\, |y|^{-1} d_q y\,. \tag{24}$$

Letting $t = i$, we find that the q-wave functions (9) are eigenfunctions of the following "q-Fourier transform",

$$i^m \tilde{\psi}_m(x) = (1-q)^{-1} \int_{-\mu}^{1} K_i(x,y)\, \tilde{\psi}_m(y)\, |y|^{-1} d_q y\,. \tag{25}$$

An easy corollary of (21) or (24) is

$$(1-q)^{-1} \int_{-\mu}^{1} K_t(x,y)\, K_{t'}(x',y)\, |y|^{-1} d_q y = K_{tt'}(x,x')\,. \tag{26}$$

Putting $t = -t' = i$, we obtain the orthogonality relation of the kernel,

$$(1-q)^{-1} \int_{-\mu}^{1} K_i(x,y)\, K_i^*(x',y)\, |y|^{-1} d_q y = \delta_{xx'}\,, \tag{27}$$

which implies the orthogonality of the rational functions (23) and leads to an inversion formula for the q-transformation (25). In view of (8), in the limit $q \to 1^-$ we get one of the "discrete Fourier transforms" considered in [21].

5. Some Biorthogonal Rational Functions

The rational functions (23) have appeared as the kernel of the discrete q-Fourier transform (25). They admit the following extension. With the aid of the bilinear generating function (22) and the orthogonality property of a special case of the q-Meixner polynomials, which are dual to the polynomials (7), we obtain the *biorthogonality relation*,

$$\int_{-\mu_2}^{1} u(x,y)\, v(x',y)\, \tilde{\rho}(y)\, d_q y = (1-q)\, d_x^2\, \delta_{xx'}\,, \tag{28}$$

for the ${}_3\varphi_2$*-rational functions* of the form

$$\begin{aligned} u(x,y) &= {}_3\varphi_2 \left(\begin{matrix} x^{-1}\,,\, y^{-1}\,,\, -qt_1^{-1} \\ q\mu_1(t_1 x)^{-1}\,,\, q\mu_2(t_1 y)^{-1} \end{matrix} ; q,\; q \right) ; \\ &= {}_3\varphi_2 \left(\begin{matrix} -\mu_1 x^{-1}\,,\, -\mu_2 y^{-1}\,,\, -qt_2 \\ -qt_2 x^{-1}\,,\, -qt_2 y^{-1} \end{matrix} ; q,\; q \right) \end{aligned} \tag{29}$$

at $x = q^s$ and at $x = -\mu_1 q^s$ for $s = 0, 1, \ldots$, respectively, and

$$v(x,y) = u(x,y)|_{t_1 \leftrightarrow t_2}\,; \qquad t_1 t_2 = \mu_1 \mu_2\,. \tag{30}$$

Here,

$$\tilde{\rho}(y) = \frac{\left(qy\,,\, -\mu_2^{-1}qy\,;\, q\right)_\infty}{\left(t_1\mu_2^{-1}y\,,\, t_2\mu_2^{-1}y\,;q\right)_\infty}\,;$$
$$= \frac{\left(qy\,,\, -\mu_2^{-1}qy\,;\, q\right)_\infty}{\left(-t_1^{-1}y\,,\, -t_2^{-1}y\,;q\right)_\infty}$$

at $x, x' = \{q^s;\, s = 0, 1, \dots\}$ and at $x, x' = \{-\mu_1 q^s;\, s = 0, 1, \dots\}$, respectively;

$$\tilde{\rho}(y) = \frac{\left(qy\,,\, -\mu_2^{-1}qy\,;\, q\right)_\infty}{\left(t_1\mu_2^{-1}y\,,\, -t_1^{-1}y\,;q\right)_\infty}\,;$$
$$= \frac{\left(qy\,,\, -\mu_2^{-1}qy\,;\, q\right)_\infty}{\left(-t_2^{-1}y\,,\, t_2\mu_2^{-1}y\,;q\right)_\infty}$$

at $x, x' = \{q^s, -\mu_1 q^s\}$ and vice versa, respectively. The squared norm is

$$d_x^2 = \frac{\left(q,\, q, -\mu_1, -\mu_2, -q\mu_1^{-1}, -q\mu_2^{-1}; q\right)_\infty}{\left(-t_1, -t_2,\, t_1\mu_1^{-1},\, t_2\mu_1^{-1},\, t_1\mu_2^{-1},\, t_2\mu_2^{-1}; q\right)_\infty} \cdot \frac{\left(t_1\mu_1^{-1}x,\, t_2\mu_1^{-1}x; q\right)_\infty}{\left(qx, -\mu_1^{-1}qx\,; q\right)_\infty}\, |x|^{-1}\,;$$
$$= \frac{\left(q,\, q, -\mu_1, -\mu_2, -q\mu_1^{-1}, -q\mu_2^{-1}; q\right)_\infty}{\left(-t_1^{-1}, -t_2^{-1},\, \mu_1 t_1^{-1},\, \mu_1 t_2^{-1},\, \mu_2 t_1^{-1},\, \mu_2 t_2^{-1}; q\right)_\infty} \cdot \frac{\left(-t_1^{-1}x,\, -t_2^{-1}x; q\right)_\infty}{\left(qx, -\mu_1^{-1}qx\,; q\right)_\infty}\, |x|^{-1}$$

for $x = q^s$ and for $x = -\mu_1 q^s$, respectively.

The functions (29)–(30) are self-dual and belong to classical biorthogonal rational functions [23–27]. It is interesting to compare the biorthogonality relation (28) with the orthogonality property for the big q-Jacobi polynomials [28], which live at the same terminating ${}_3\varphi_2$-level.

6. Concluding Remarks

In view of (11), it is natural to introduce operators of the form

$$a = \alpha(s) - \beta(s)\, e^{\partial}\,,\quad a^+ = \alpha(s) - e^{-\partial}\beta(s)$$

with two arbitrary functions $\alpha(s)$ and $\beta(s)$ and to satisfy the commutation rule $a\,a^+ - qa^+a = 1$. The result is

$$\alpha(s+1) = q\alpha(s)\,,$$
$$(1-q)\alpha^2(s) + \beta^2(s) - q\beta^2(s-1) = 1$$

and we can choose $\alpha(s) = \varepsilon\, q^s$ and $\beta^2(s) = \varepsilon^2\,(q^{s+1} - \gamma)(q^s - \delta)$ with $(1-q)\gamma\delta\varepsilon^2 = 1$. Since

$$(a^+\psi, \chi) - (\psi, a\chi) = \sum_s \Delta[\beta(s-1)\,\psi^*(s-1)\,\chi(s)]\,,$$

the corresponding operators are adjoint for the two different cases considered in [7] and in this Letter with $0 < q < 1$ and $q > 1$, respectively.

For $\beta = constant$ we can try

$$a = e^{\partial} \left(e^{\partial} - \alpha(s) \right) , \quad a^{+} = \left(e^{-\partial} - \alpha(s) \right) e^{-\partial}$$

and obtain $aa^{+} - qa^{+}a = 1 - q$, when

$$\alpha^2(s+1) = q\alpha^2(s) \, , \quad \alpha(s+2) = q\alpha(s) \, ,$$

which is satisfied for $\alpha = \varepsilon \, q^{s/2}$. This case has been considered in [5].

Finally, the operators

$$a = \varepsilon \, \frac{e^{\gamma\partial}\alpha(s) + e^{-\gamma\partial}\beta(s)}{\alpha(s) - \beta(s)} \, , \quad a^{+} = \varepsilon \, \frac{\alpha(s)e^{-\gamma\partial} + \beta(s)e^{\gamma\partial}}{\alpha(s) - \beta(s)}$$

obey the q-commutation rule provided that $\alpha(s)\beta(s) = \pm 1$ and $\alpha(s + 2\gamma) = q^{-1}\alpha(s)$. Therefore, $\alpha = q^{-s}$ for $\gamma = 1/2$ and $\varepsilon^2 = q^{1/2}(1 - q)^{-1}$ (cf. [3]).

We can also introduce the operators

$$a = \alpha^{-1}(s) - \varepsilon \, \beta^{-1}(s) \, e^{\partial} \, , \quad a^{+} = \alpha(s) - \varepsilon \, e^{\partial}\beta(s)$$

and obtain $aa^{+} - qa^{+}a = 1 - q$ if

$$\alpha(s+1) = q\alpha(s) \, , \quad \beta(s+2) = q\beta(s) \, ,$$

so $\alpha = q^s$ and $\beta = q^{s/2}$. This leads to the Rogers–Szegö polynomials [13] orthogonal on the unit circle (see [1,6]).

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