Reachability via saturation

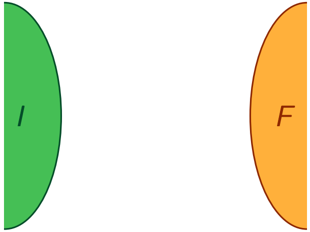
Gabriele Puppis

LaBRI / CNRS

Reachability is semi-decidable

A path connecting two sets, if exists, can be found in finitely many steps.

Forward analysis



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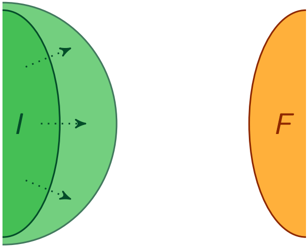

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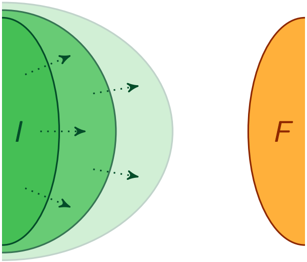

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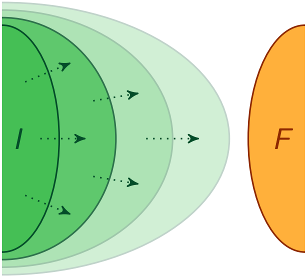

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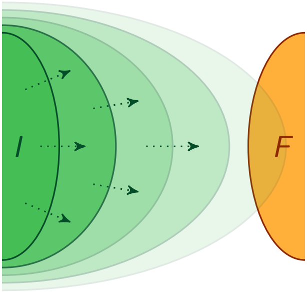

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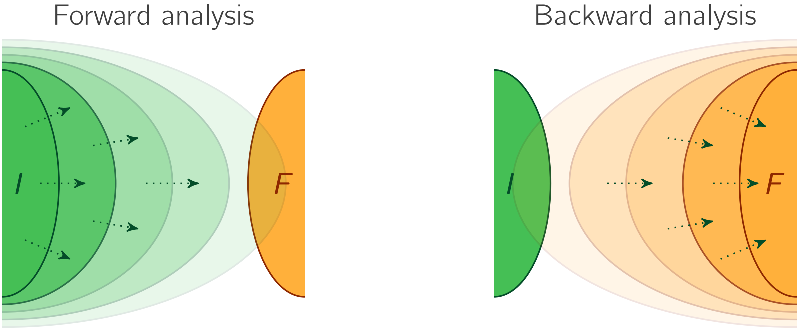

☞ The problem is of course termination, namely, to detect non-reachability...

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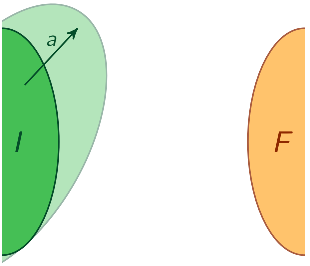

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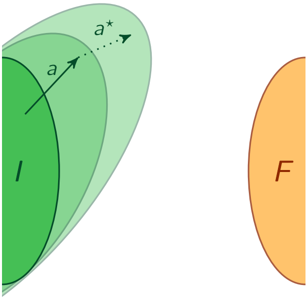

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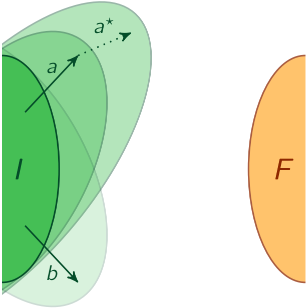

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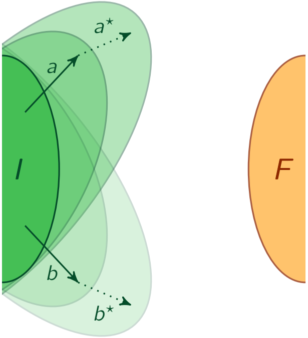

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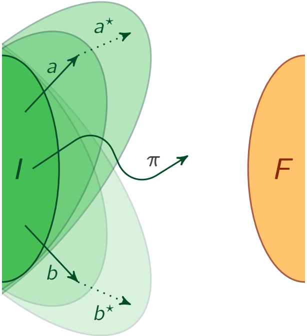

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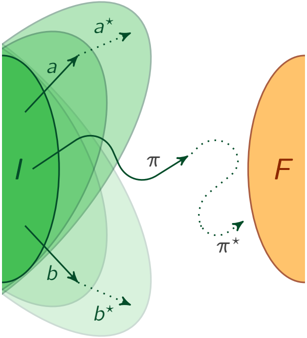

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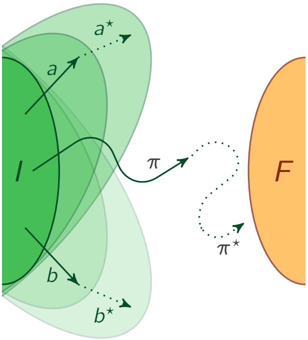
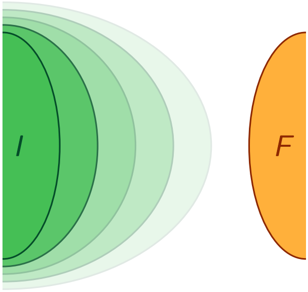


Invariant analysis



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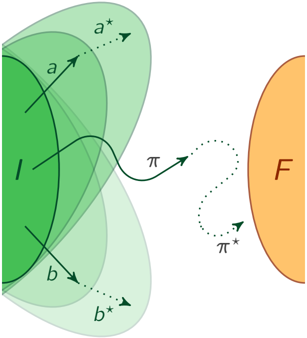

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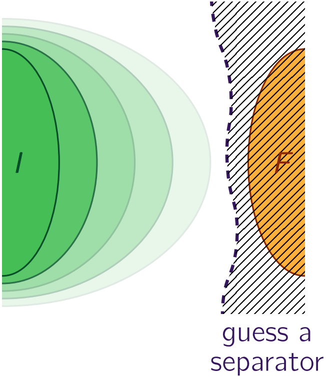

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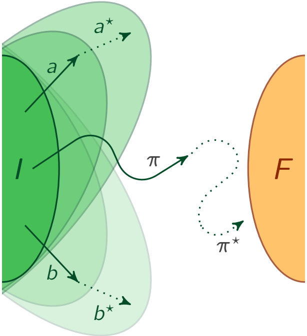

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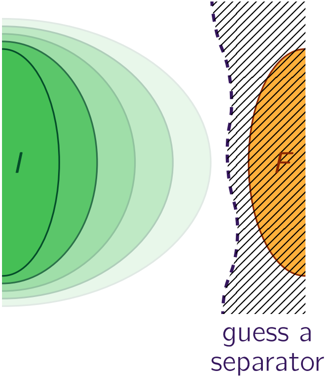

☞ Both approaches require **symbolic representations** of infinite sets

Backward reachability for pushdown systems

Given a pushdown system $\mathcal{P} = (Q, \Sigma, \Gamma, \Delta)$ and a set $B_0 \subseteq Q \cdot \Gamma^\star$ of target configurations, define:

$$B_{n+1} = B_n \cup \left\{ qz \;\middle|\; \exists q'z' \in B_n.\ \exists a \in \Sigma.\ qz \xrightarrow{a} q'z' \right\}$$

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$B_0 = \{q\varepsilon\}$ $\quad B_1 = \{q\varepsilon, q\gamma\}$ $\quad B_2 = \{q\varepsilon, q\gamma, q\gamma\gamma\}$ $\quad \ldots$

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$B_\omega = q\varepsilon^\star$ is indeed regular, but how to efficiently compute it?

"Pump" the changes from B_n to B_{n+1} to obtain
a new sequence $C_0, C_1, \ldots$ that converges more quickly:

(completeness)	$\forall n \in \mathbb{N}.\quad B_n \subseteq C_n$
(soundness)	$\forall n \in \mathbb{N}.\quad C_n \subseteq B_\omega$
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The sets $C_0, C_1, \ldots$ will be defined by automata $\mathcal{A}_0, \mathcal{A}_1, \ldots$ sharing the **same state space**...

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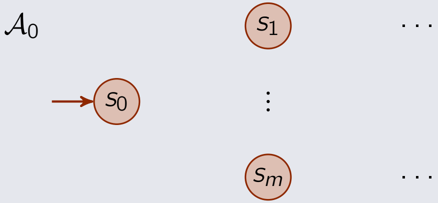

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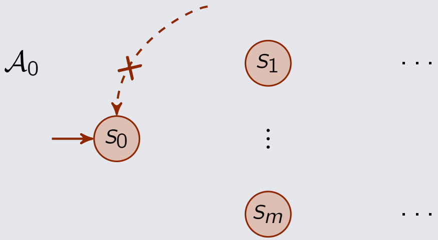

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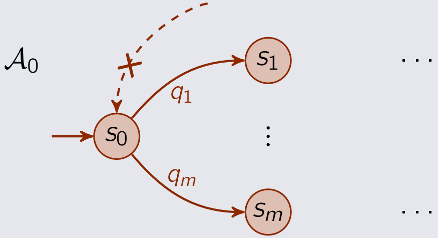

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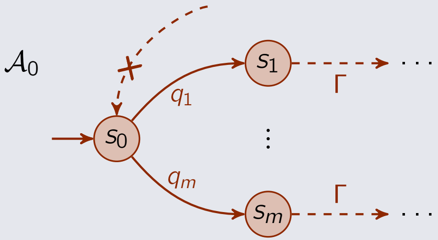
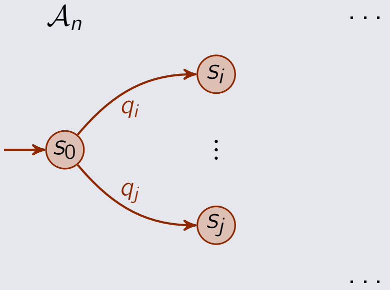
- The other transitions in $\mathcal{A}_0$ are labelled by stack symbols



Saturation procedure

Construct $\mathcal{A}_{n+1}$ from $\mathcal{A}_n$ by adding transitions, as follows:

1. select a transition rule $(q_i\gamma, a, q_j z)$ in the pushdown system $\mathcal{P}$



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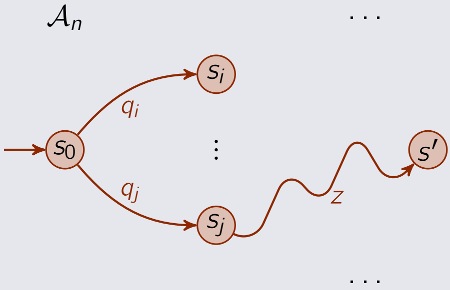

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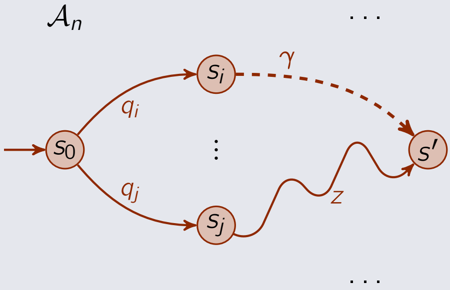

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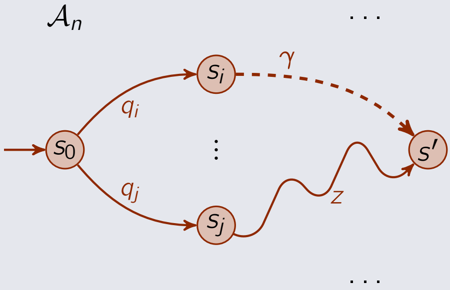

Termination: straightforward

Only polynomially many transitions can be added

(⇒ reachability in PTIME)

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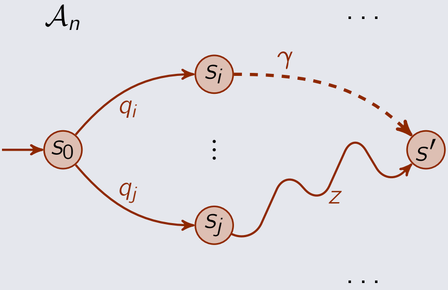

Termination: straightforward

Soundness: by induction on n

$$s_0 \xrightarrow[\mathcal{A}_{n+1}]{q'z'} s'$$

$$\Downarrow$$

$$s_0 \xrightarrow[\mathcal{A}_0]{qz} s' \quad \wedge \quad q'z' \xrightarrow[\mathcal{P}]{\star} qz$$

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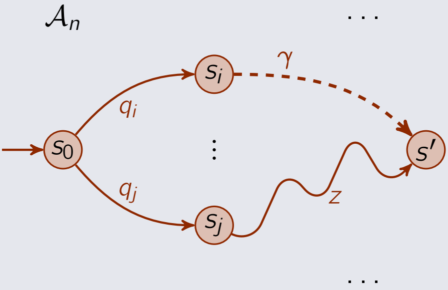

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Completeness:

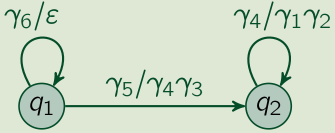
$\forall$ config. $q_i\gamma w \in B_{n+1} \setminus B_n$

$\exists$ trans. $q_i\gamma w \xrightarrow[\mathcal{P}]{a} q_j z w$ with $q_j z w \in B_n$

☞ Select rule $(q_i\gamma, a, q_j z)$ in $\mathcal{P}$ and path $s_0 \xrightarrow[\mathcal{A}_n]{q_j z} s'$ in $\mathcal{A}_n$ to prove that $q_i\gamma w \in \mathscr{L}(\mathcal{A}_{n+1})$

Example

Consider the target set $B_0 = \{q_2\gamma_1\gamma_2\gamma_3\}$ over the pushdown system



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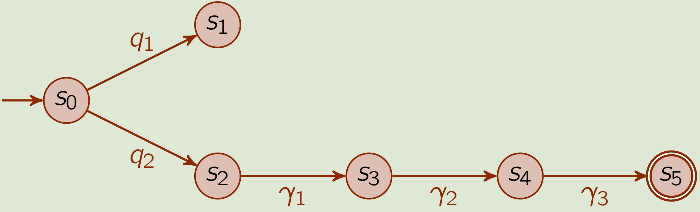

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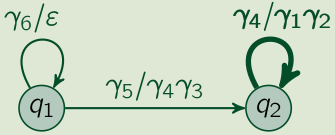

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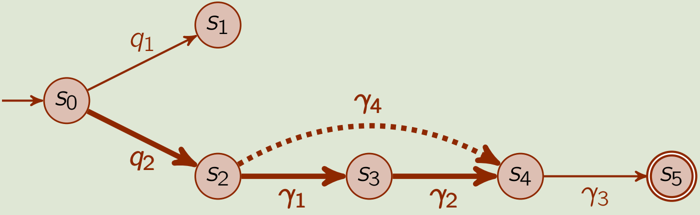

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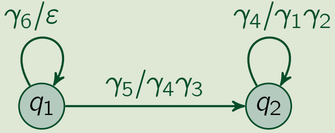

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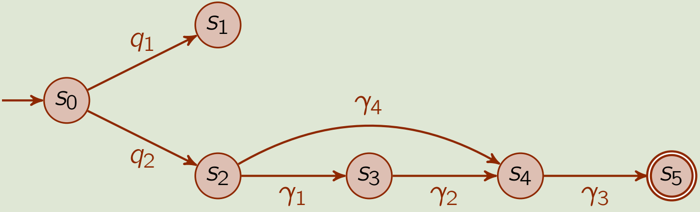

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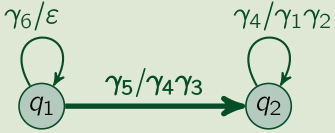

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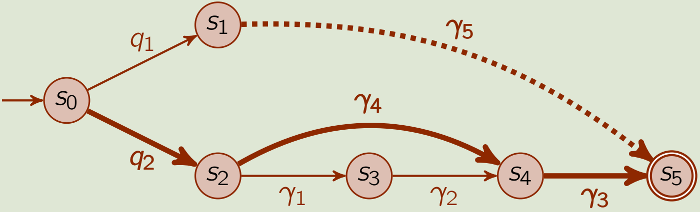

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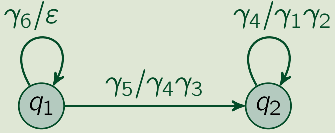

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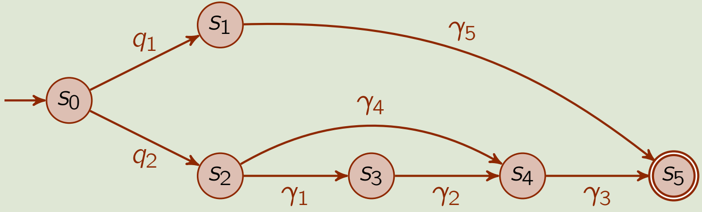

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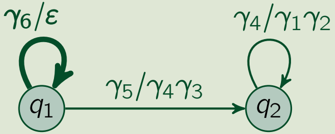

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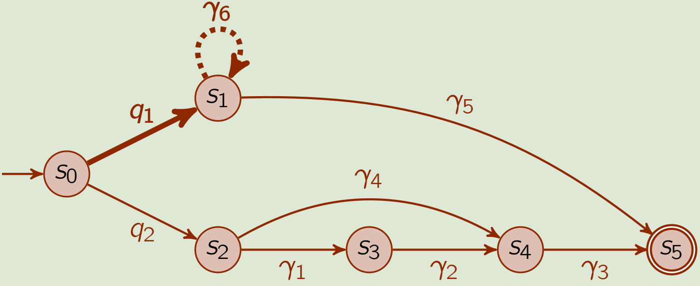

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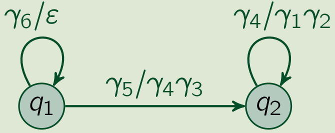
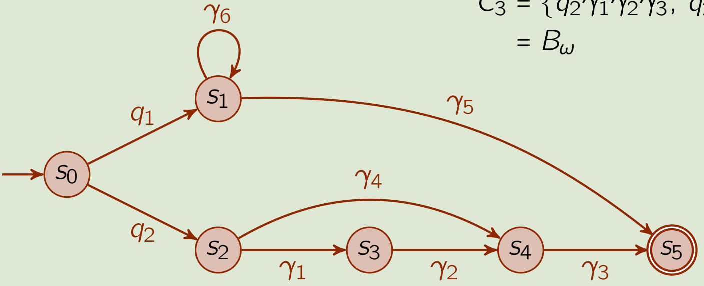


$$C_0 = \{q_2\gamma_1\gamma_2\gamma_3\}$$

$$C_1 = \{q_2\gamma_1\gamma_2\gamma_3,\ q_2\gamma_4\gamma_3\}$$

$$C_2 = \{q_2\gamma_1\gamma_2\gamma_3,\ q_2\gamma_4\gamma_3,\ q_1\gamma_5\}$$

$$C_3 = \{q_2\gamma_1\gamma_2\gamma_3,\ q_2\gamma_4\gamma_3,\ q_1\gamma_6^\star\gamma_5\} = B_\omega$$



Theorem (Bouajjani, Esparza & Maler '97)

Given a pushdown system $\mathcal{P}$ and a regular set B of configurations,
the set of configurations that can reach B
is regular and can be computed in polynomial time.

Theorem (Bouajjani, Esparza & Maler '97)

Given an **alternating** pushdown system $\mathcal{P}$ and a regular set B of conf., the **winning region** for the **B-reachability game** is regular and can be computed in polynomial time.

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Given an **alternating** pushdown system $\mathcal{P}$ and a regular set B of conf., the **winning region** for the **B-reachability game** is regular and can be computed in polynomial time.

Similar generalizations can be proved for:

- **tree rewriting systems**
  (Löding '06, ...)
- reachability games on **higher-order pushdown systems**
  (Bouajjani & Meyer '04, Hague & Ong '07, ...)
- ...

Next we will focus on reachability for systems that use **variables over natural numbers** instead of a stack...

$(x, y) := (0, 0)$

while $(x, y) \neq (0, 1)$ do

if [*input is north west*] then

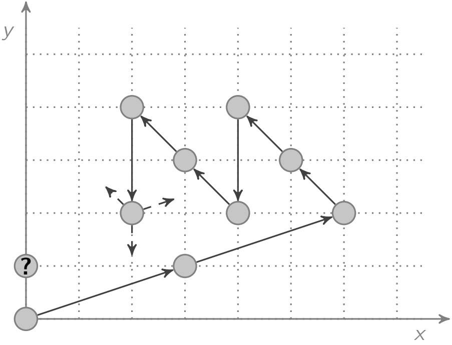
$(x, y) := (x, y) + (1, 3)$

else if [*input is north east*] then

$(x, y) := (x, y) + (-1, 1)$

else if [*input is south*] then

$(x, y) := (x, y) + (0, -2)$



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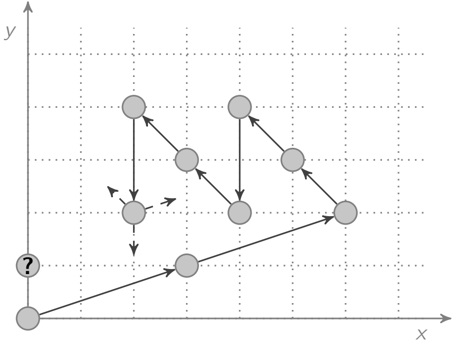

Definition

A **vector addition system** (**VAS**) is a transition system $(\mathbb{N}^k, \Delta)$, where Δ is a finite subset of $\mathbb{Z}^k$ and

$$\bar{x} \longrightarrow \bar{y} \qquad \text{iff} \qquad \begin{cases} \bar{x}, \bar{y} \geq 0 \\ \bar{y} - \bar{x} \in \Delta \end{cases}$$

Definition

A **lossy VAS** is a transition system $(\mathbb{N}^k, \Delta)$, where Δ is a finite subset of $Q \times \mathbb{Z}^k \times Q$ and

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A **VAS with states** is a transition system $(Q \times \mathbb{N}^k, \Delta)$,
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$$(p, \bar{x}) \longrightarrow (q, \bar{y}) \qquad \text{iff} \qquad \begin{cases} \bar{x}, \bar{y} \geq 0 \\ (p,\ \bar{y} - \bar{x},\ q) \in \Delta \end{cases}$$

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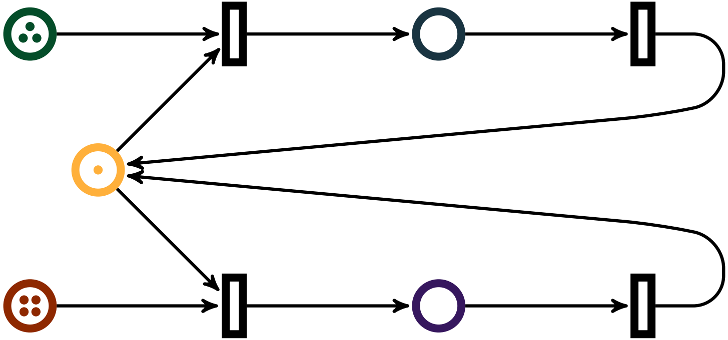
☞ States do not add power, as they can be implemented by counters

e.g. 2 states $=$ 2 additional counters that sum up to 1

$(p, \bar{x}) \longrightarrow (q, \bar{y})$ becomes $(0, 1, \bar{x}) \longrightarrow (1, 0, \bar{y})$

VAS are the same as **Petri nets**:



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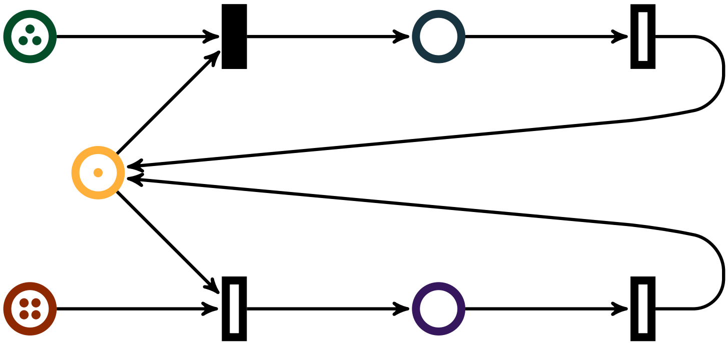

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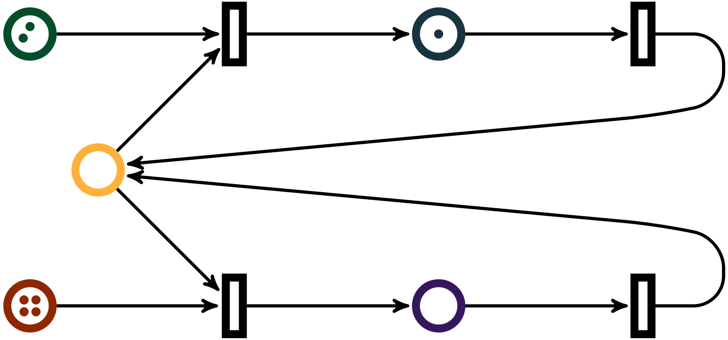

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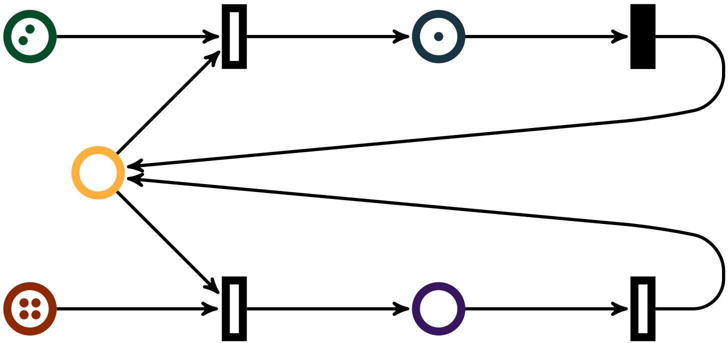

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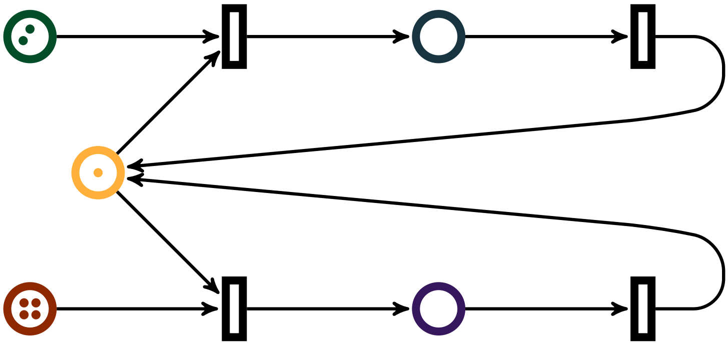

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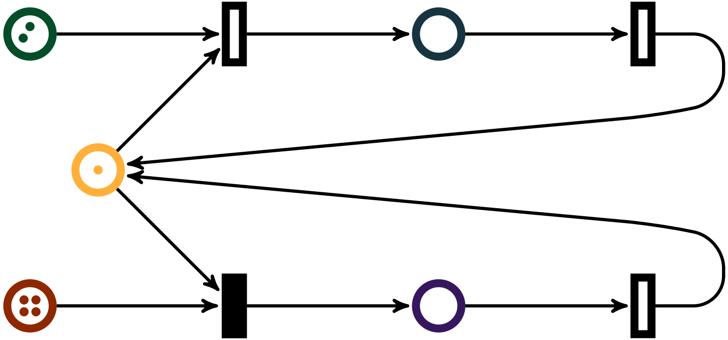

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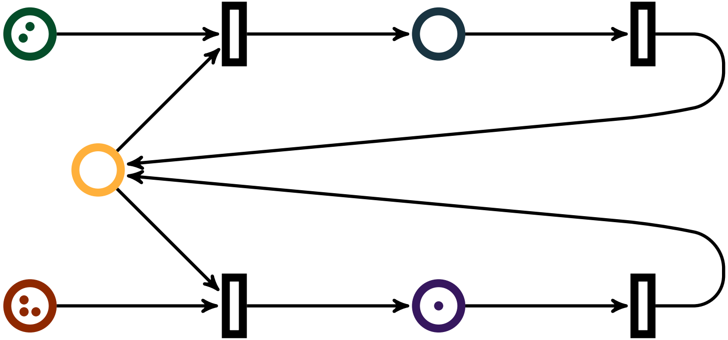

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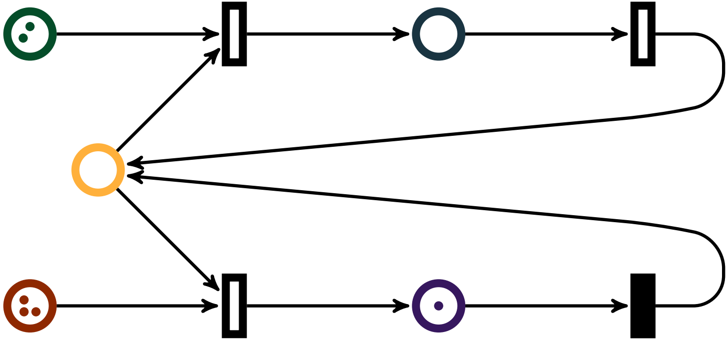

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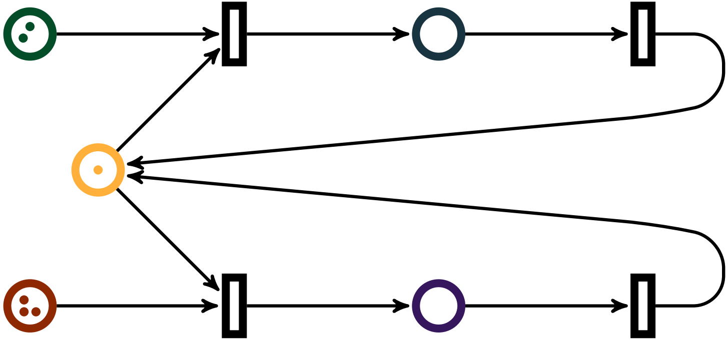

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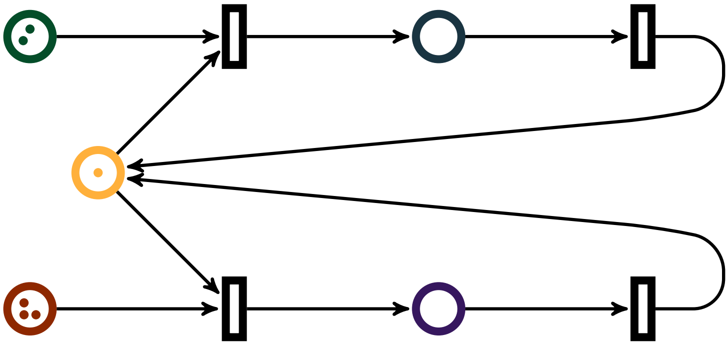
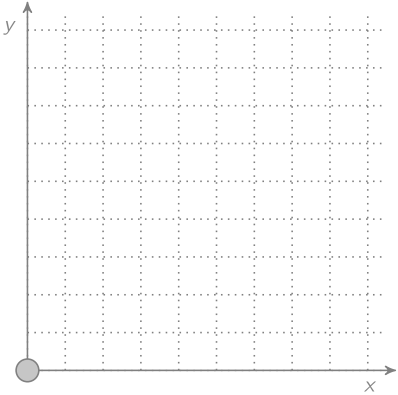


☞ configurations = tokens per location (e.g. $(2,1,3,0,0)$)
transitions = transfers of tokens (e.g. $(0,-1,-1,0,1)$)

We may expect that reachable sets are **linear**...

$(0, 0)$



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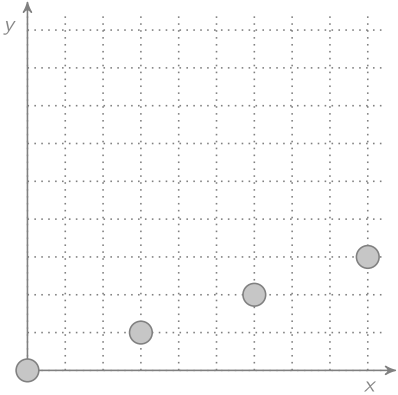
$(0,0) + (3,1)\mathbb{N}$



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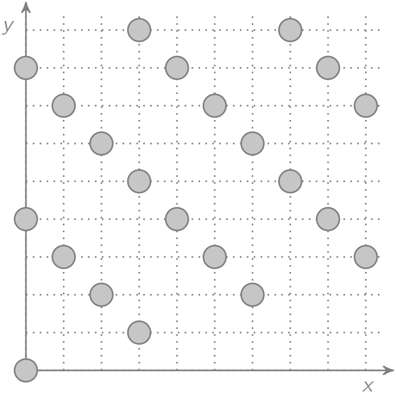

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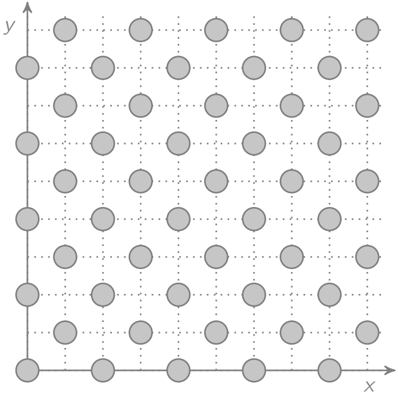

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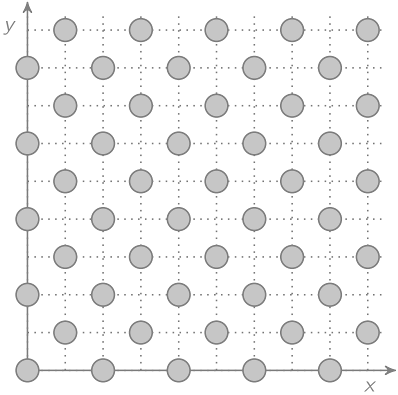
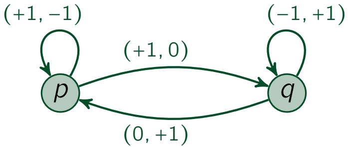
Theorem (Ginsburg '66)

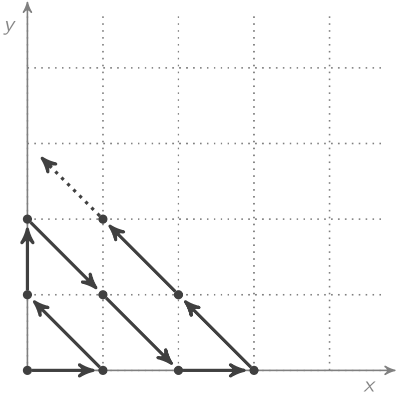
Finite unions of linear sets are precisely the **Presburger sets** i.e. sets definable in $\mathsf{FO}[\mathbb{N},+]$

e.g. $\varphi(x,y) \;=\; \exists z.\ x+y = z+z$



We may expect that reachable sets are **linear**... but they are not!





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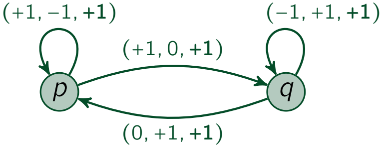

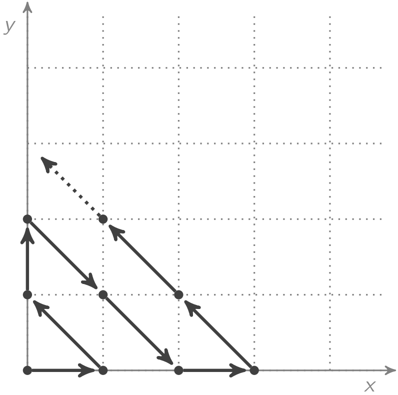

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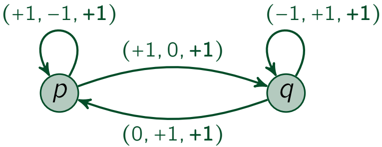

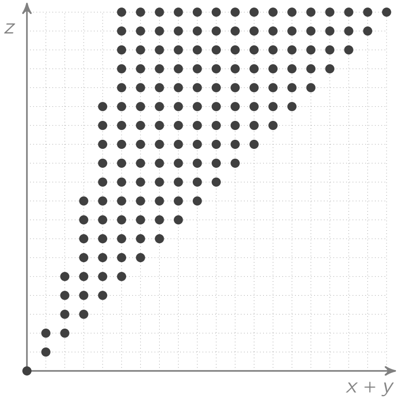

$$(x+y) \;\leq\; z \;\leq\; \mathcal{O}\big((x+y)^2\big)$$

To overcome the problem of representing reachable sets,
we try to **over-approximate by downward closures**:

$$V^{\downarrow} = \left\{ \bar{z} \;\middle|\; \exists\, \bar{y} \in V.\ \bar{z} \leq \bar{y} \right\}$$

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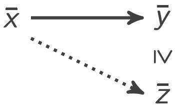

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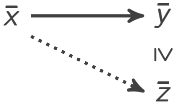
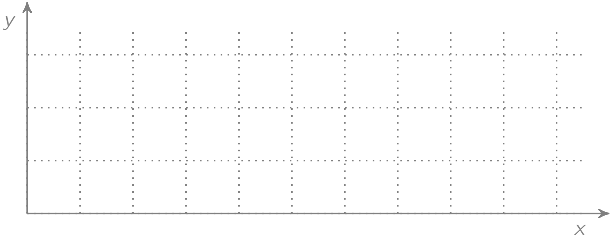


Dickson's Lemma 1913

The pointwise order $\leq$ on $\mathbb{N}^k$ is a **well partial order**
(i.e. all **decreasing chains** and all **antichains** are finite)



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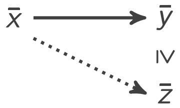

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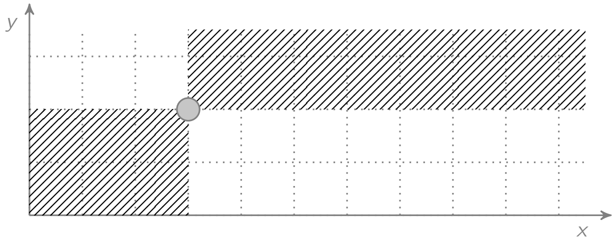

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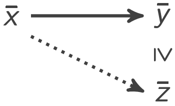

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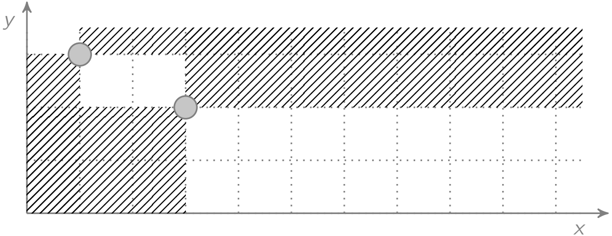

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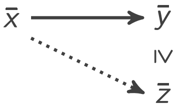

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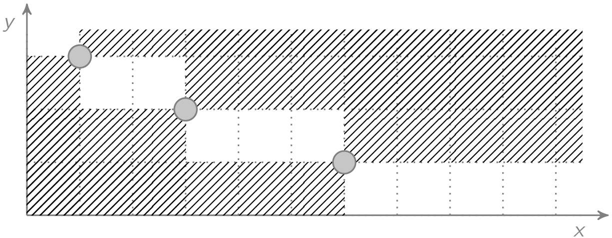

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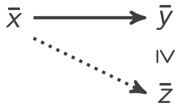

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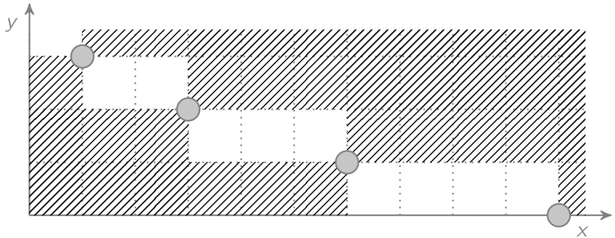

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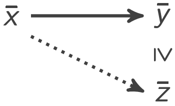

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Lemma

For all subsets V of $(\mathbb{N} \cup \{\infty\})^k$, there is an **antichain** W such that

$$V^{\downarrow} = W^{\downarrow}$$

$\Rightarrow$ we can finitely represent downward-closed sets by antichains

Karp & Miller Algorithm '69

Saturation of downward-closed sets via transition function Δ

- acceleration on emerging dominating sets...

Karp & Miller Algorithm '69

Saturation of downward-closed sets via transition function Δ

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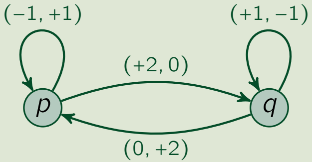
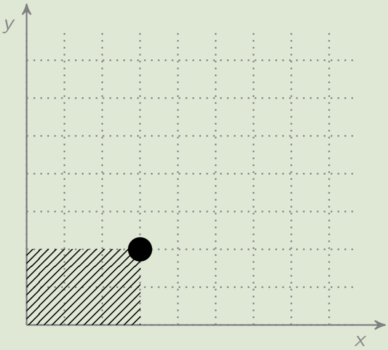
Example





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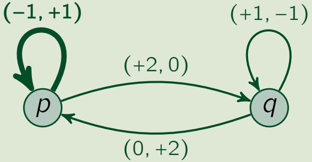

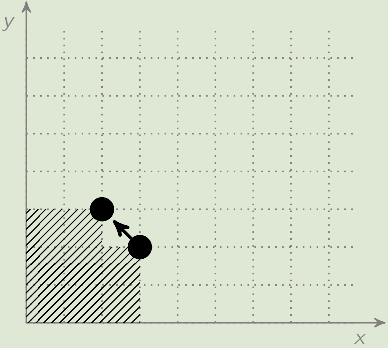

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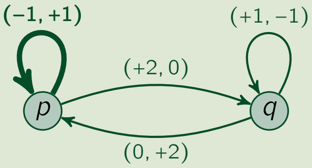

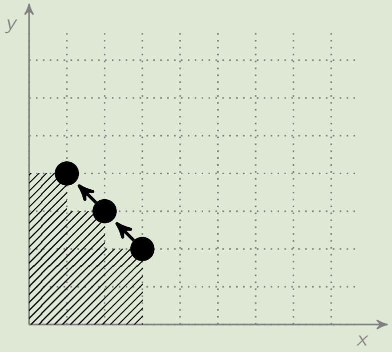

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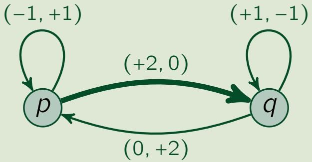

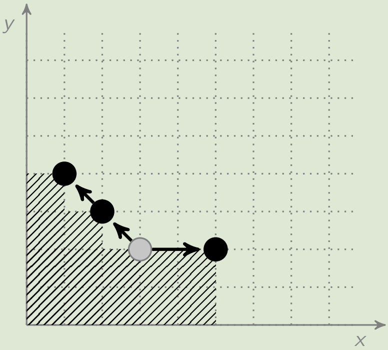

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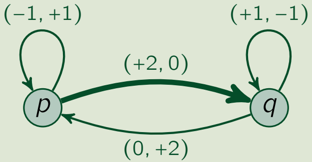

Correctness of acceleration

$\bar{x} \xrightarrow{\star} \bar{x} + \bar{\delta}$ for some $\bar{\delta} \in \mathbb{N}^k$

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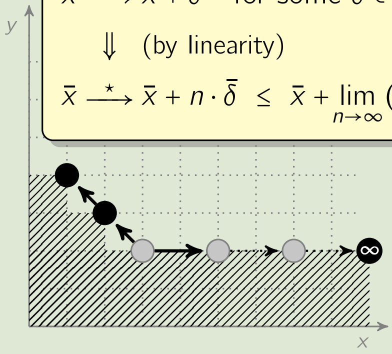
$\bar{x} \xrightarrow{\star} \bar{x} + n \cdot \bar{\delta} \;\leq\; \bar{x} + \lim_{n \to \infty} (n \cdot \bar{\delta})$

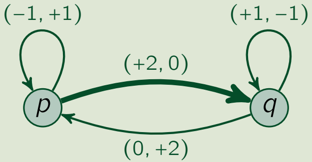


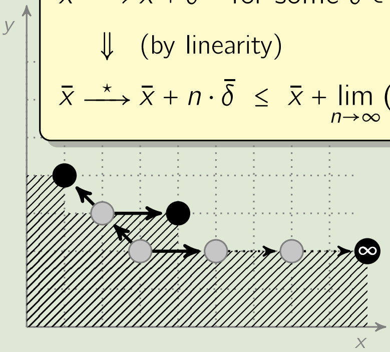
Karp & Miller Algorithm '69

Saturation of downward-closed sets via transition function Δ

- acceleration on emerging dominating sets...

Example





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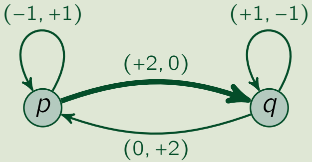

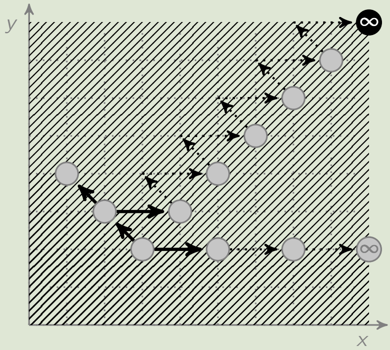

Theorem (Rackoff '78)

Coverability on VAS (i.e. given $\bar{x}, \bar{y}$, tell if $\exists\, \bar{z} \geq \bar{y}.\ \bar{x} \xrightarrow{\star} \bar{z}$) is EXPSPACE-complete.

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Corollary 1

Reachability on lossy VAS is EXPSPACE-complete.

Corollary 2

Control-state reachability on VAS is EXPSPACE-complete.

There are other results similar in spirit...

Theorem (Adbulla, Cerans & Jonsson '96, . . .)

Coverability is decidable (**non-primitive recursive**) on VAS with

- **resets** (e.g. $x := 0$)
- **transfers** (e.g. $x := y + z$)
- **positive guards** (e.g. *if* $[x > 0]$ *then* . . .)

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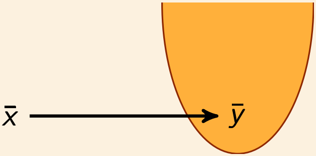
However, we can still exploit Dickson's Lemma with

1. **upward-closed sets**
   they cover more vectors than downward-closed sets!
2. **backward reachability**
   i.e. compute $B_{n+1} = \{ \bar{x} \mid \exists \bar{y} \in B_n .\ \bar{x} \longrightarrow \bar{y} \}$

Lemma

VAS transitions with resets, transfers, and positive guards are **backward-compatible with upward-closures**, i.e.



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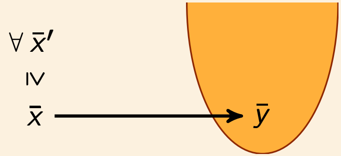

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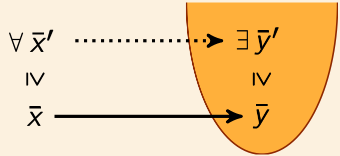

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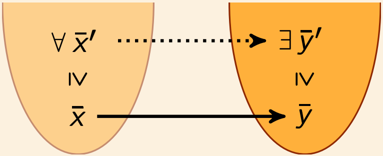

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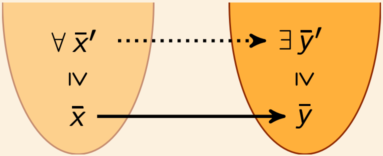
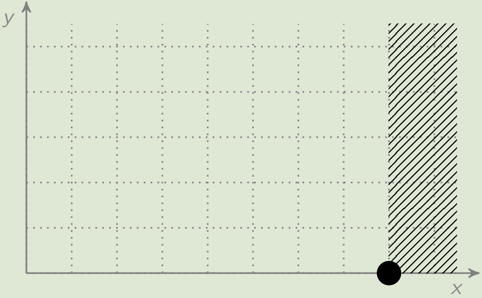


Example of backward coverability analysis



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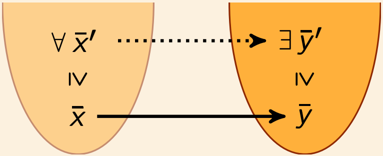

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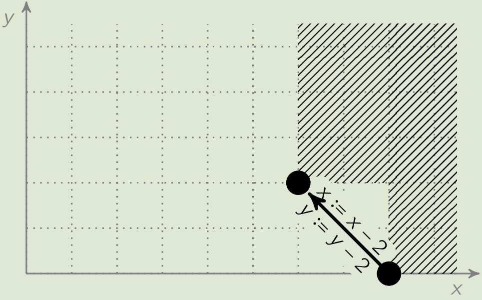

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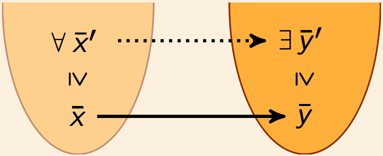

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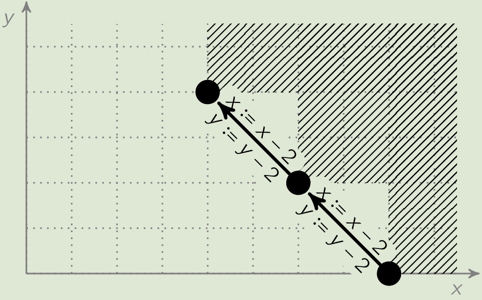

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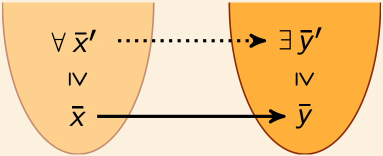

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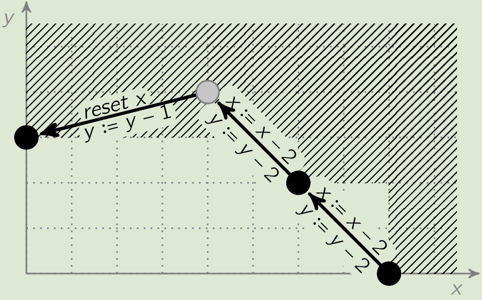

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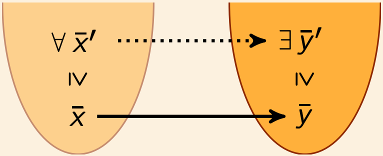

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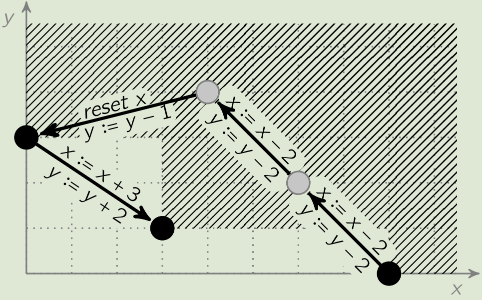

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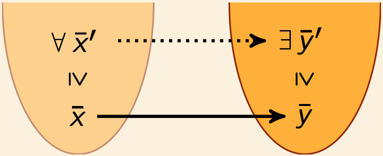

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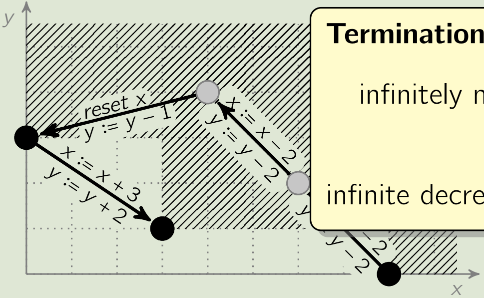

Termination by Dickson's Lemma:

infinitely many emerging points

$\Downarrow$

infinite decreasing chain or antichain

These ideas for coverability analysis can be extended to:

- **Lossy Channel Systems**
  (instead of Dickson's Lemma,
  use Higman's Lemma for the sub-sequence partial order)

- **Timed Petri nets**
  (token have time-stamps, transitions have time constraints)

- **Alternating Finite Memory Automata**
  (finite control states + one register to store
  and compare symbols from an infinite alphabet)

- ...

Now, back to the original reachability problem on VAS...

Separation Theorem (Leroux '92, '09, . . . , '12)

If $\bar{x} \not\xrightarrow{*}_{\Delta} \bar{y}$, then there is a partition (X, Y) of $\mathbb{N}^k$ such that

1. X and Y are **finite unions of linear sets** (or, equally, sets definable in **Presburger logic FO[$\mathbb{N}, +$]**)
2. $\bar{x} \in X$ and $\bar{y} \in Y$
3. X is a **forward invariant**, i.e. $(X + \Delta) \cap \mathbb{N}^k \subseteq X$
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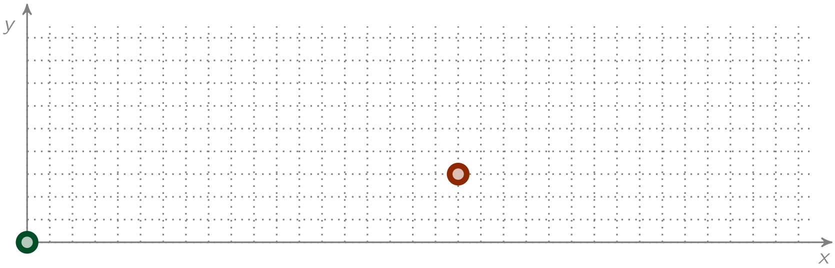

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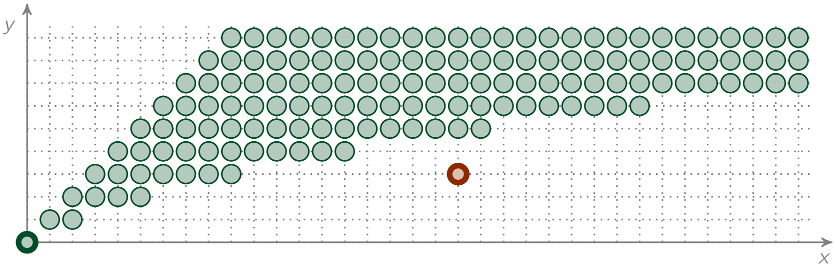

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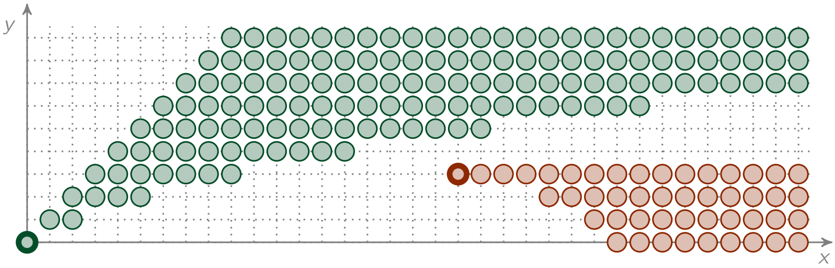

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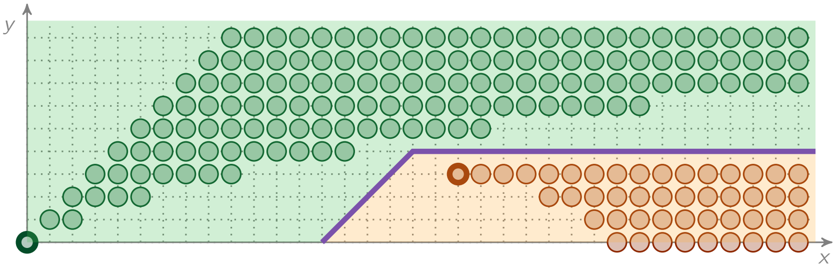

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Enumerate in parallel:

1. the possible finite sequences π of transitions
   (answer positively if $\bar{x} \xrightarrow{\pi} \bar{y}$)

2. the possible Presburger formulas defining partitions (X, Y) of $\mathbb{N}^k$
   (answer negatively if (X, Y) is an invariant separating $\bar{x}$ and $\bar{y}$)

"That's all Folks!"