Decidability of the Theory of the Totally Unbounded $\omega$-Layered Structure

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Outline

- MSO logics over tree structures
- Layered structures
- The automaton-based approach
- A solution to the decision problem
- Further work
Let $\Lambda = \{1, \ldots, k\}$ be a finite set of edge labels.

We consider tree structures extended with tuples of unary predicates, namely, structures of the form

$$(\mathcal{T}, \bar{V}) = (S, (E_l)_{l \in \Lambda}, (V_i)_{i \in [1,m]})$$

where

- $S = \Lambda^*$ (set of vertices)
- $E_l = \{(v, vl) : v \in S\}$ (l-labeled edges)
- $V_i \subseteq S$ for all $1 \leq i \leq m$ (unary predicates)
Formulas over a tree structure $\mathcal{T}$ are built up from atoms:

- $x_i = x_j$ “$x_i$ and $x_j$ denote the same vertex”
- $X_i \subseteq X_j$ “$X_i$ denotes a subset of $X_j$”
- $E_l(x_i, x_j)$ “$(x_i, x_j)$ denotes a $l$-labeled edge”
- $X_k(x_i)$ “$x_i$ denotes a vertex in $X_k$”

...through connectives $\land$, $\lor$, $\neg$ and quantifiers $\forall$, $\exists$ over first-order and second-order variables.

**Remark:** we can restrict ourselves to an expressively equivalent framework devoid of *first-order variables.*
Free second-order variables $X_1, \ldots, X_m$ will be interpreted by tuples of (unary) predicates $V_1, \ldots, V_m$.

Given a formula $\varphi(\bar{X})$, we write $\mathcal{T} \models \varphi[\bar{V}]$ to say that $\varphi(\bar{X})$ holds in $\mathcal{T}$ by substituting $V_i$ for $X_i$, for all $1 \leq i \leq m$.

The decision problem $MTh(\mathcal{T}, \bar{V})$ is the problem of deciding whether, for a given formula $\varphi(\bar{X})$,

$$\mathcal{T} \models \varphi[\bar{V}]$$
Layered structures: totally unbounded

**Goal:** we want to decide the MSO theory of the totally unbounded $\omega$-layered structure (TULS):

- The structure contains arbitrarily fine/coarse layers
- Arrows map elements of a given layer to elements of the immediately finer layer
- Black vertices denote the elements of a distinguished layer ("layer 0") with an (optional) successor relation $+_0$
Layered structures: downward unbounded

The TULS embeds the **downward unbounded** $\omega$-layered structure (DULS)

(i.e., the structure with a top layer and an infinite number of finer and finer layers):

⇒ The DULS allows one to express properties like 

"$P$ holds true densely in an interval".
...and it embeds the **upward unbounded \( \omega \)-layered structure (UULS)**

(i.e., the structure with a bottom layer and an infinite number of coarser and coarser layers):

\[
\begin{align*}
1 & \quad +0 & \quad 1 \\
2 & \quad 1 & \quad 2 \\
\end{align*}
\]

\[\Rightarrow\] The UULS allows one to express properties like “\( P \) holds at all time points \( 2^i \)”. 
Layered structures: a tree embedding

The totally unbounded $\omega$-layered structure can, in its turn, be embedded into an infinite complete ternary tree $\mathcal{T}_{TULS}$:
The automaton based approach - 1

Any expanded tree structure \((T, \bar{V})\) can be encoded by a (vertex-colored) tree \(T_{\bar{V}}\) (canonical representation).

\[ \Rightarrow \textbf{Idea:} \] to exploit the correspondence between logic over tree structures and Rabin tree automata in order to reduce a decision problem to an acceptance problem.

A **Rabin automaton** works on colored trees in a top-down fashion: it “spreads” its states inside a tree (according to the transition relation) and it verifies that suitable acceptance conditions are met.

We say that a colored tree \(T_{\bar{V}}\) is **accepted** by \(M\) \((T_{\bar{V}} \in \mathcal{L}(M))\) if such conditions are satisfied.
[Rabin’s Theorem] For every formula $\varphi(X)$, there is a Rabin automaton $M$ (and vice versa) such that

$$\mathcal{T} \models \varphi[\bar{V}] \iff \mathcal{T}_{\bar{V}} \in \mathcal{L}(M)$$

$\Rightarrow$ the decision problem $MTh(\mathcal{T}, \bar{V})$ for MSO formulas reduces to an acceptance problem $Acc(\mathcal{T}_{\bar{V}})$ for Rabin automata

$\Rightarrow$ we can restrict our attention to the decidability of the acceptance problem for Rabin tree automata.

**Notation:** Hereafter, we shall drop the subscript $\bar{V}$ from $\mathcal{T}_{\bar{V}}$. 
Proposition: \( \text{Acc}(\mathcal{T}) \) is decidable for any infinite \textbf{regular} tree \( \mathcal{T} \) (i.e., a tree with \textit{only finitely many distinct subtrees}).

However, the colored tree \( \mathcal{T}_{TULS} \) that embeds the TULS is \textbf{not} regular.

\( \Rightarrow \) we look for a larger class of colored trees for which the acceptance problem turns out to be decidable.
The automaton based approach - 4

Idea: Given an automaton $M$, we decide whether $T \in \mathcal{L}(M)$ by reducing it to a simpler problem $T' \in \mathcal{L}(M)$, where $T'$ is a regular tree equivalent to $T$, namely,

$$T \in \mathcal{L}(M) \iff T' \in \mathcal{L}(M)$$

(recall that regular trees enjoys a decidable acceptance problem)

Such a reduction works effectively for several non-regular trees.

In particular, we can reduce the acceptance problem for $T_{TULS}$ to a decidable acceptance problem over an equivalent regular tree.
Given a Büchi automaton $M$, we can define an equivalence $\equiv_M$ over finite words s.t. $u \equiv_M u'$ iff, for every pair of states $r,s$,

$$
\begin{align*}
    r \xrightarrow{u} s & \iff r \xrightarrow{u'} s \\
    r \xrightarrow{o} s & \iff r \xrightarrow{o} s
\end{align*}
$$

**Properties:**

1. $\equiv_M$ has **finite index**

2. $\equiv_M$ is a **congruence** w.r.t. concatenation

3. $\equiv_M$-equivalent factorizations are **indistinguishable** by $M$, namely, if $u_i \equiv_M u'_i$ for all $i \geq 0$, then

$$
u_0u_1u_2\ldots \in \mathcal{L}(M) \iff u'_0u'_1u'_2\ldots \in \mathcal{L}(M)$$
A digression into Büchi automata - 2

[Carton and Thomas] Given an ω-word \( w = u_0u_1u_2 \ldots \), if for any congruence \( \equiv_M \) there are \( p, q \) such that \( \forall i > p. u_i \equiv_M u_{i+q} \)

\[
\begin{align*}
w & \in \mathcal{L}(M) \\
\updownarrow & \\
(u_0 \ldots u_p)(u_{p+1} \ldots u_{p+q})(u_{p+q+1} \ldots u_{p+2q}) \ldots & \in \mathcal{L}(M) \\
\updownarrow & \\
(u_0 \ldots u_p) \cdot (u_{p+1} \ldots u_{p+q})^\omega & \in \mathcal{L}(M)
\end{align*}
\]

\( \Rightarrow \) if such \( p \) and \( q \) are computable for any congruence \( \equiv_M \), then \( \text{Acc}(w) \) can be effectively reduced to a decidable acceptance problem over an ultimately periodic word.

Similar results hold for infinite trees...
A solution to the decision problem - 1

Basic ingredients:

- notion of **tree concatenation** $\mathcal{T}_1 \cdot_c \mathcal{T}_2$
  (defined as the substitution in $\mathcal{T}_1$ of each $c$-colored leaf by $\mathcal{T}_2$)

- notion of **factorization** for infinite trees
  (i.e. infinite concatenation of the form $\mathcal{T}_0 \cdot_{c_0} \mathcal{T}_1 \cdot_{c_1} \ldots$)

- notion of **congruence** $\equiv_M$ w.r.t. tree concatenations
A solution to the decision problem - 2

[Main result] Given an infinite tree $\mathcal{T}$ generated by a factorization $\mathcal{T}_0 \cdot c_0 \mathcal{T}_1 \cdot c_1 \ldots$, if for any congruence $\equiv_M$ there are $p, q$ such that $\forall i > p. \mathcal{T}_i \equiv_M \mathcal{T}_{i+q}$, then:

$$\mathcal{T} \in \mathcal{L}(M)$$

$\Uparrow$

$$\mathcal{T}_0 \cdot c_0 \ldots \mathcal{T}_p \cdot c_p \mathcal{T}_{p+1} \cdot c_{p+1} \ldots \mathcal{T}_{p+q} \cdot c_{p+q} \mathcal{T}_{p+q+1} \cdot c_{p+q+1} \ldots \in \mathcal{L}(M)$$

$\Uparrow$

$$\mathcal{T}_0 \cdot c_0 \ldots \mathcal{T}_p \cdot c_p \mathcal{T}_{p+1} \cdot c_{p+1} \ldots \mathcal{T}_{p+q} \cdot c_{p+q} \mathcal{T}_{p+1} \cdot c_{p+1} \ldots \in \mathcal{L}(M)$$

**Remark.** The last factorization is ultimately periodic and it generates a (decidable) **regular** tree $\mathcal{T}'$. 
A solution to the decision problem - 3

A tree $\mathcal{T}$ is said **residually regular** if we can provide a factorization $\mathcal{T}_0 \cdot_{c_0} \mathcal{T}_1 \cdot_{c_1} \ldots$ that is *effectively ultimately periodic* w.r.t. any congruence $\equiv_M$.

$\Rightarrow$ we solve $Acc(\mathcal{T})$ as follows:

1. we take a factorization $\mathcal{S}$ of $\mathcal{T}$ which is ultimately periodic w.r.t. any congruence $\equiv_M$

2. given automaton $M$, we compute an ultimately periodic factorization $\mathcal{S}'$ that is $\equiv_M$-equivalent to $\mathcal{S}$

3. we know that $\mathcal{S}'$ generates a regular tree $\mathcal{T}'$ and $\mathcal{T}' \in \mathcal{L}(M) \iff \mathcal{T} \in \mathcal{L}(M)$

4. we solve $Acc(\mathcal{T}')$ on automaton $M$

5. we accordingly return *Yes* or *No* to the original problem $Acc(\mathcal{T})$
A solution to the decision problem - 4

In general, residually regular trees are non-regular trees which however exhibit a definite pattern in their structure.

Example. The tree $T_{TULS}$, which embeds the TULS, can be proved to be residually regular:

The sequence of factors is ultimately periodic w.r.t. any equivalence $\equiv_M$

$\Rightarrow$ the tree $T_{TULS}$ (and hence the TULS itself) enjoys a decidable MSO theory.
Conclusions

Results:

- we developed an original automaton-based method to decide the TULS
- as a by-product, we obtained new uniform decidability proofs for the DULS and UULS

Further work:

- to exploit the proposed technique to decide variants of the theories of the DULS and UULS (MSO fragments extended with \textit{equi-level/equi-column} predicates)
- to determine the generality of the proposed method (e.g., to compare it with the transformational approach developed by Caucał)