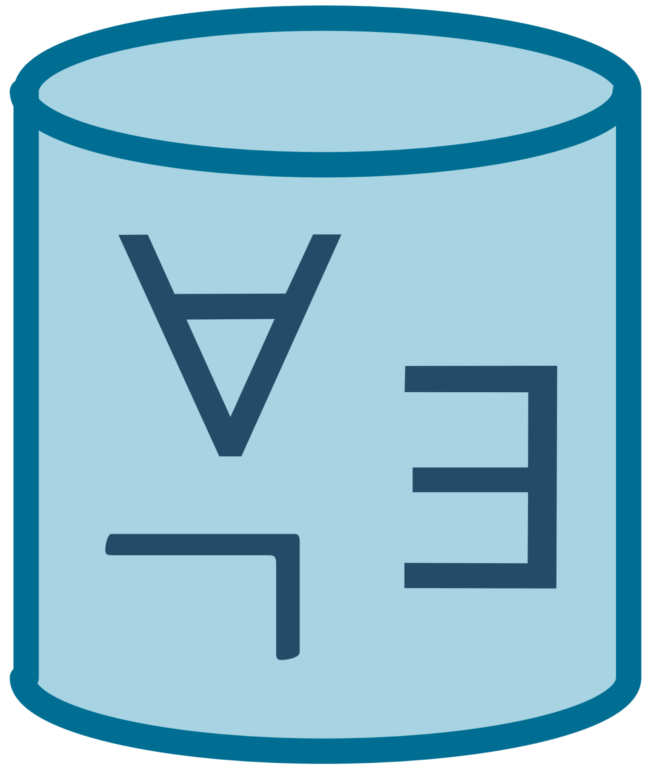
20/7/2015

ECI 2015
Buenos Aires



Fundamentos lógicos de bases de datos
(Logical foundations of databases)

Diego Figueira **Gabriele Puppis**

CNRS LaBRI

0-1 Law

A different perspective: a coarser view on expressiveness...

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How do FO properties distribute among ALL structures?

Or equally, what percentage of graphs verify a given FO sentence?

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$$\mu_\infty(\mathbf{P}) = \lim_{n \to \infty} \mu_n(\mathbf{P})$$

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Theorem. [Glebskii et al. '69, Fagin '76]

For every *FO sentence* ϕ, $\mu_\infty(\phi)$ is either 0 or 1.

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- ϕ = "even number of edges" $\mu_\infty(\phi) = {}^1/_2$
- ϕ = "even number of nodes" $\mu_\infty(\phi)$ not even defined
- ϕ = "more edges than nodes" $\mu_\infty(\phi) = 1$ (yet not FO-definable!)

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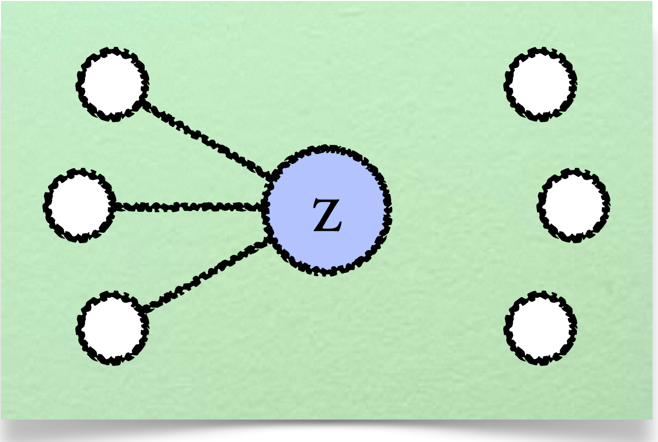
For every *FO sentence* ϕ, $\mu_\infty(\phi)$ is either 0 or 1.

Let k = quantifier rank of ϕ

$$\delta_k = \forall x_1, \ldots, x_k \, \forall y_1, \ldots, y_k \; \exists z \; \bigwedge_{i,j} x_i \neq y_j \wedge E(x_i, z) \wedge \neg E(y_j, z)$$

(Extension Axiom)



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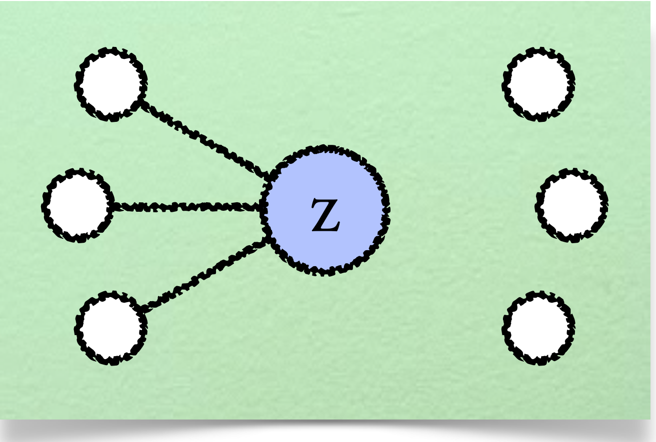

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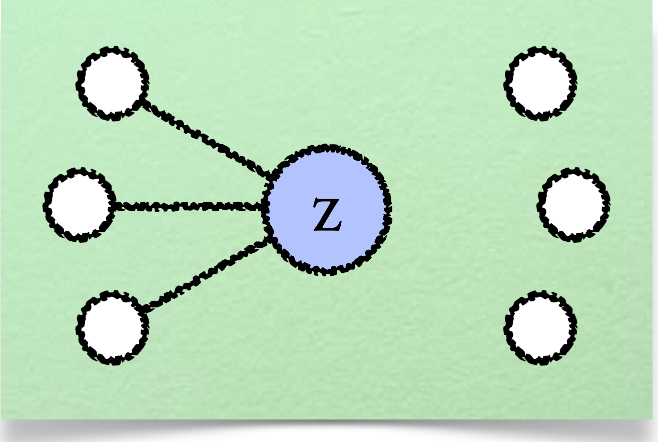

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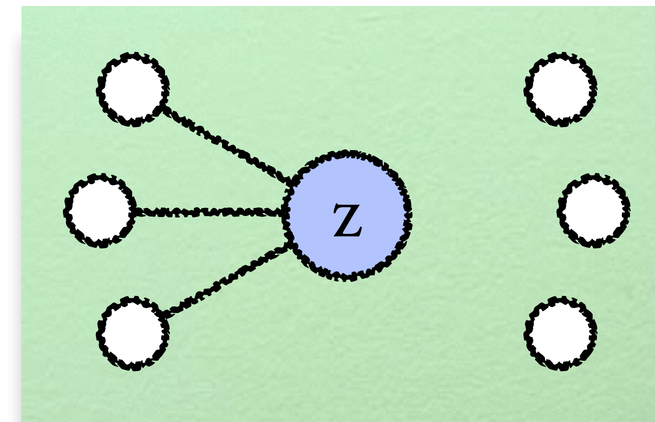

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2 cases

a) There is G $\;G \models \delta_k \wedge \phi \;\Rightarrow$ (by Fact 1) $\forall H$: If $H \models \delta_k$ then $H \models \phi$

Thus, $\mu_\infty(\delta_k) \leq \mu_\infty(\phi)$

$\Rightarrow$ (by Fact 2) $\mu_\infty(\delta_k) = 1$, hence $\mu_\infty(\phi) = 1$

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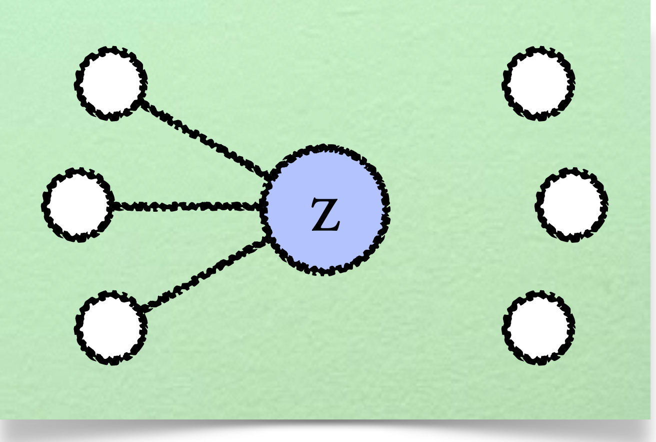

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b) There is no $G \vDash \delta_k \wedge \phi$ $\Rightarrow$ (by Fact 2) there is $G \vDash \delta_k$,

$\Rightarrow$ $G \vDash \delta_k \wedge \neg\phi$ $\Rightarrow$ (by case a) $\mu_\infty(\neg\phi) = 1$

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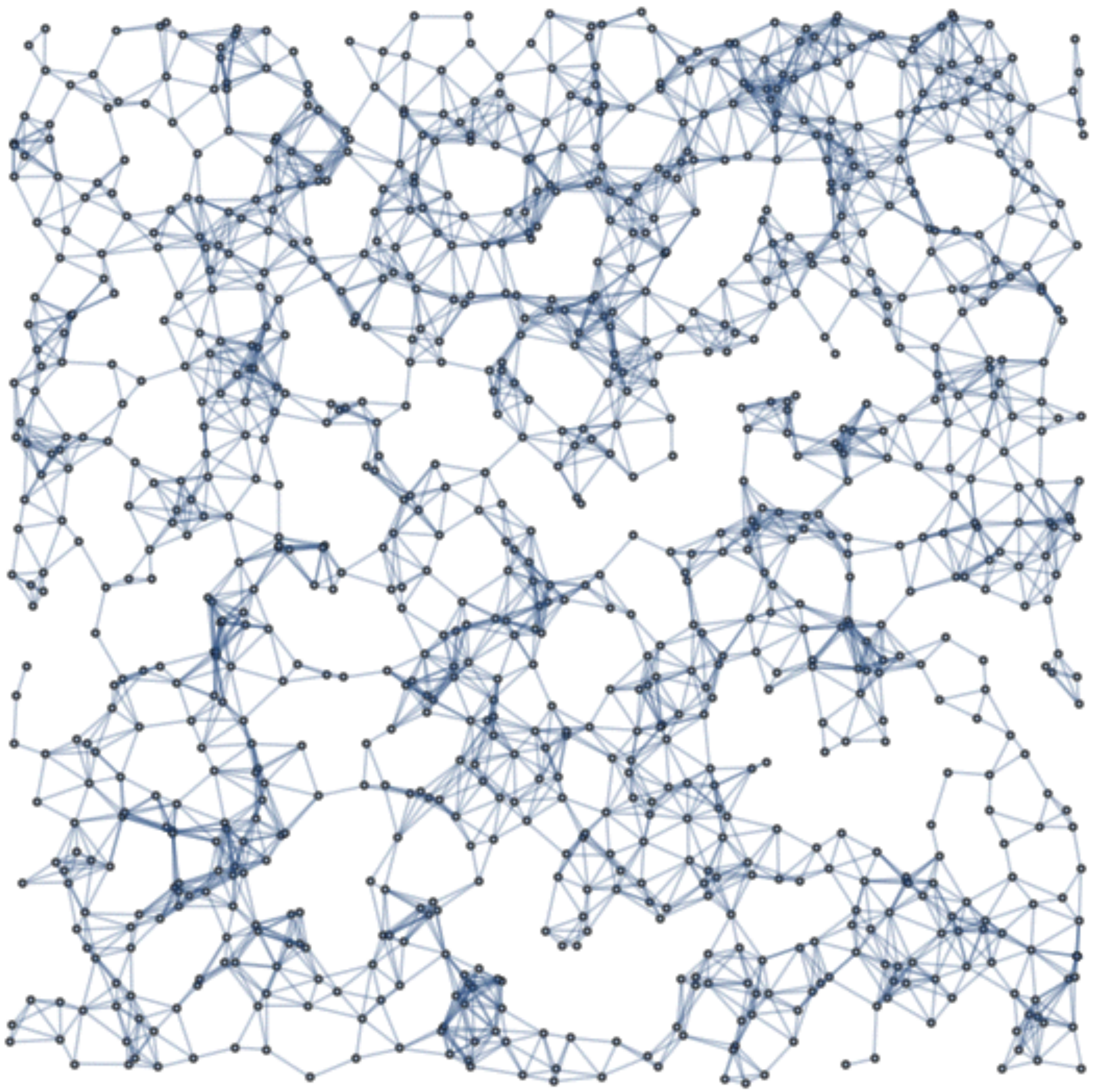
For every *FO sentence* ϕ, $\mu_\infty(\phi)$ is either 0 or 1, *and this depends on whether* $\text{RADO} \vDash \phi$

RADO =



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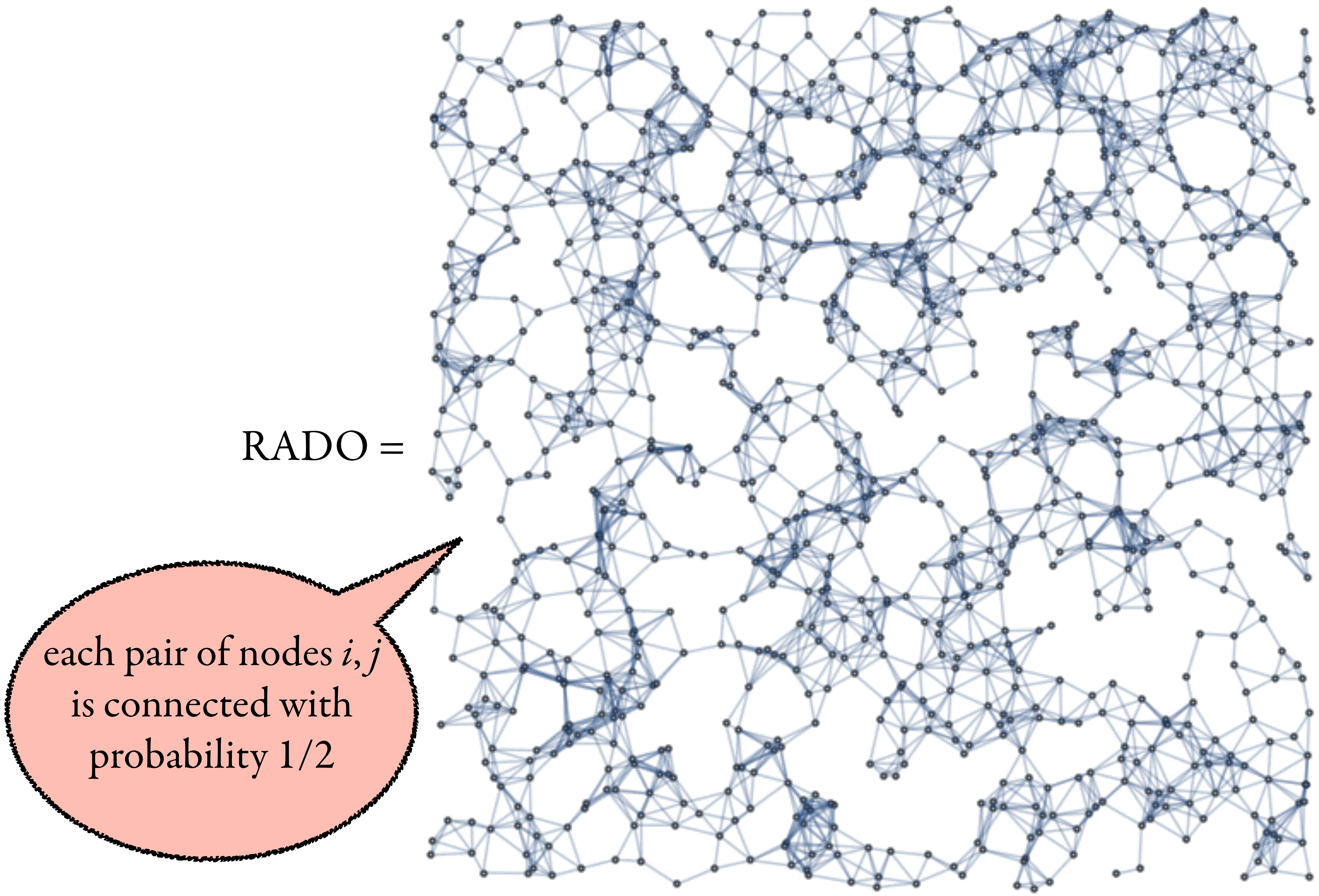

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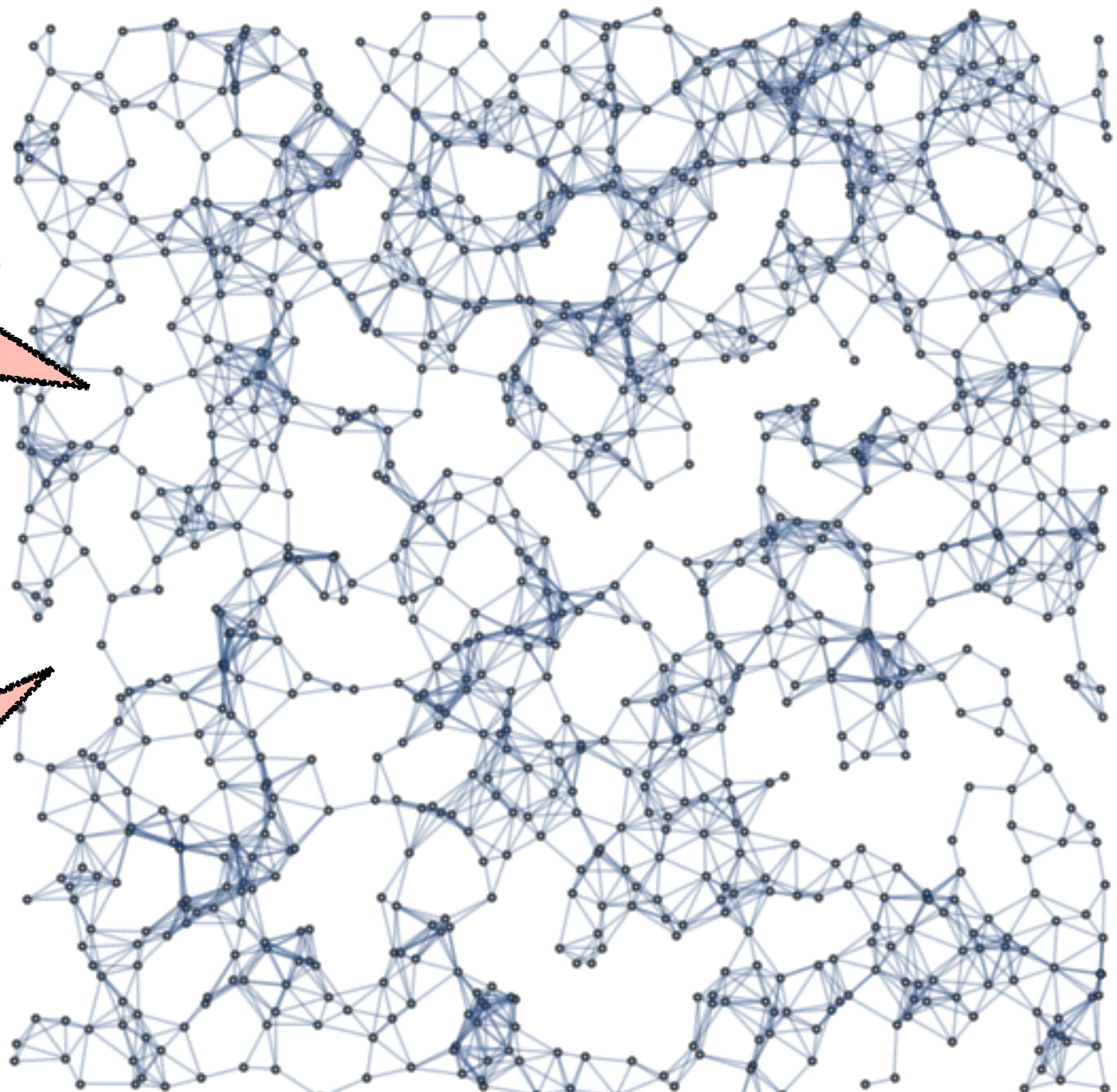

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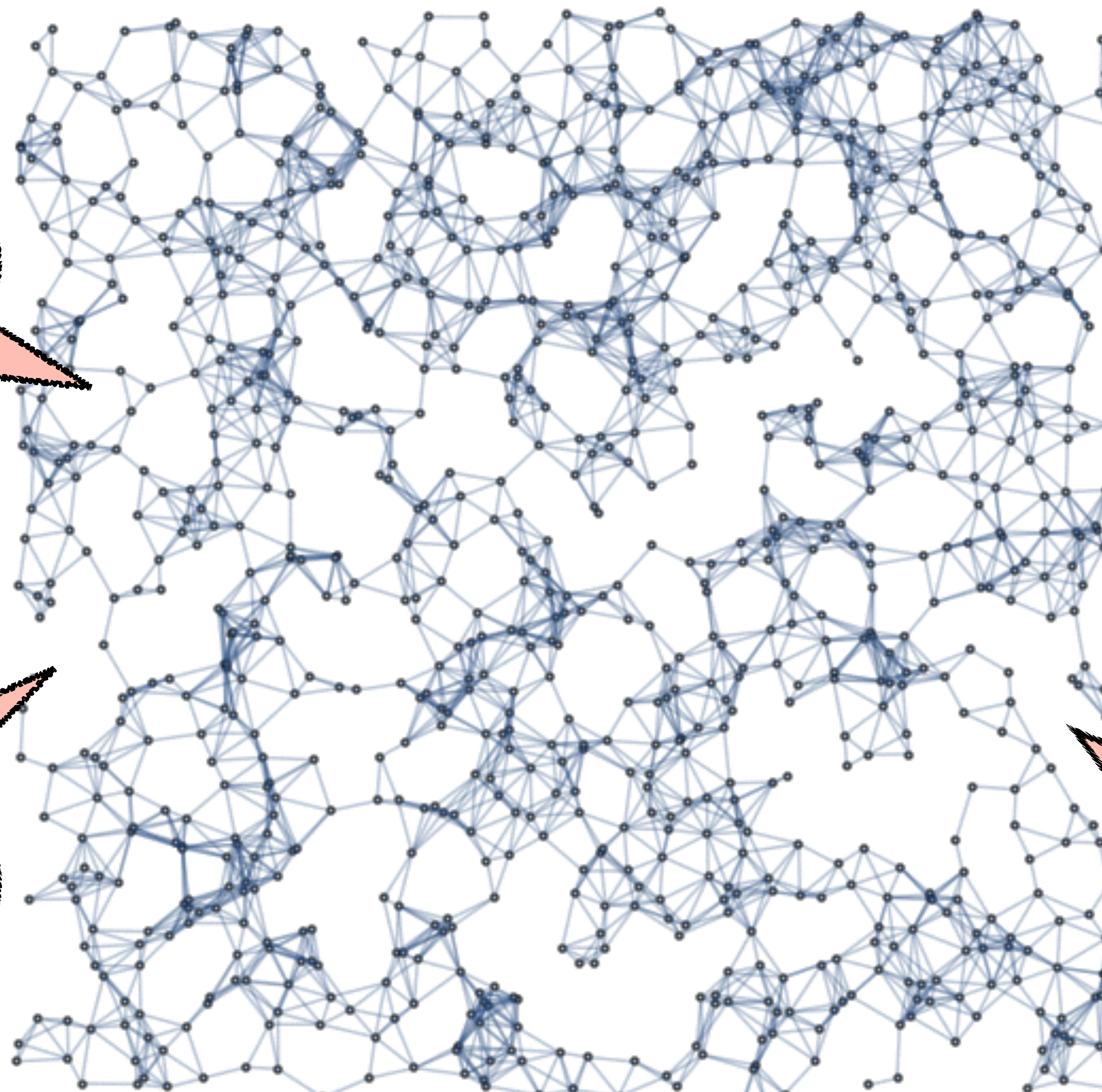

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the unique
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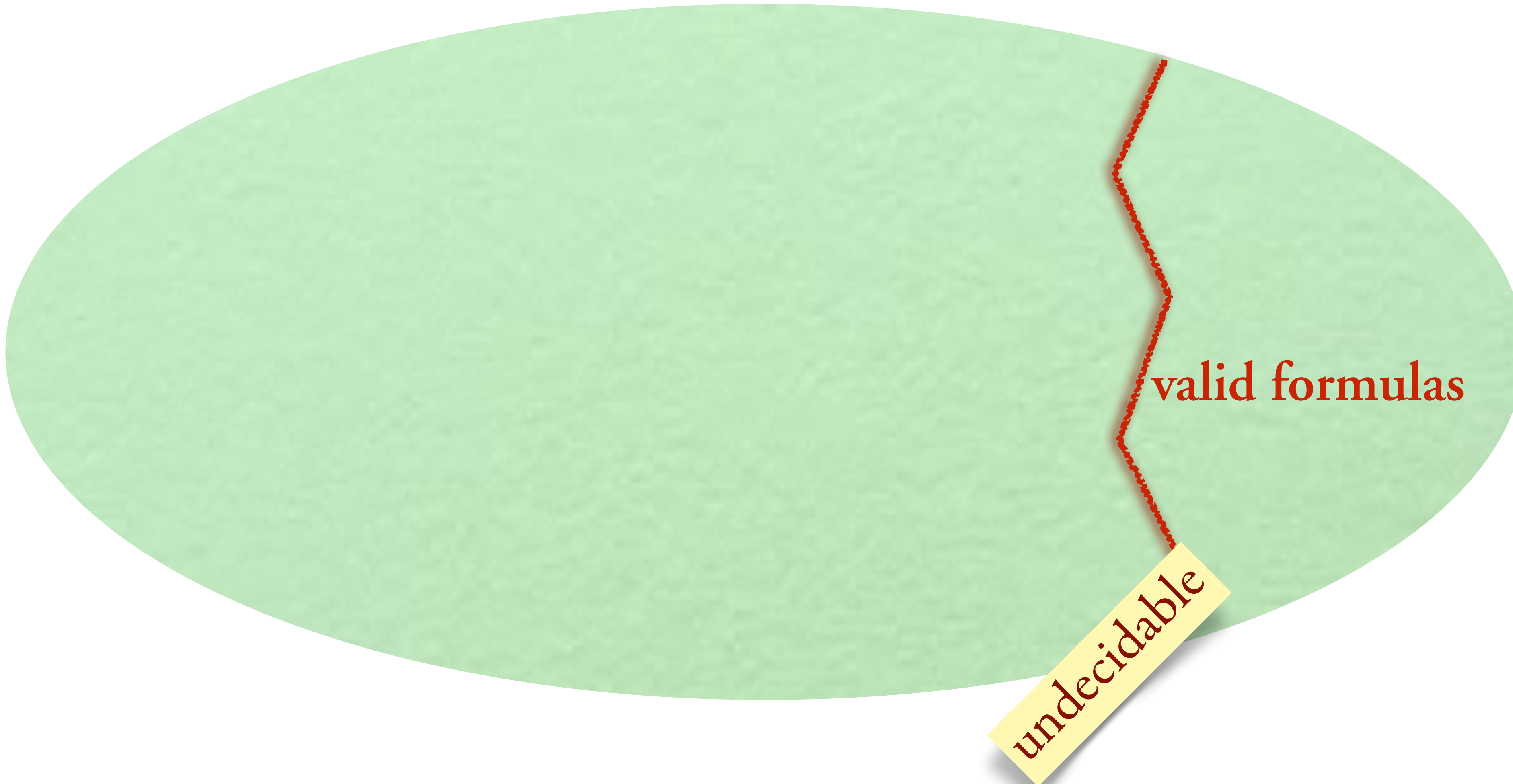

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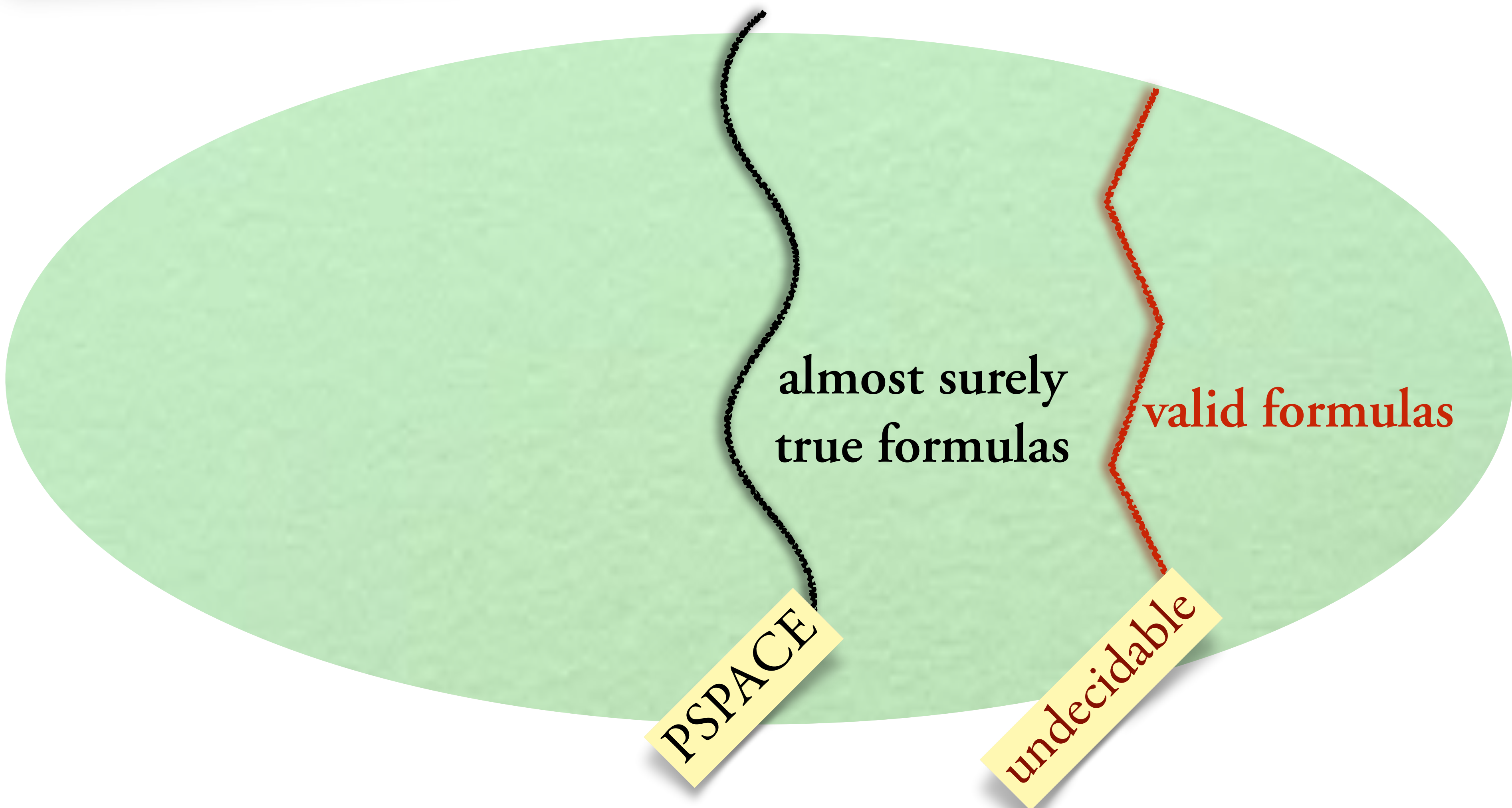

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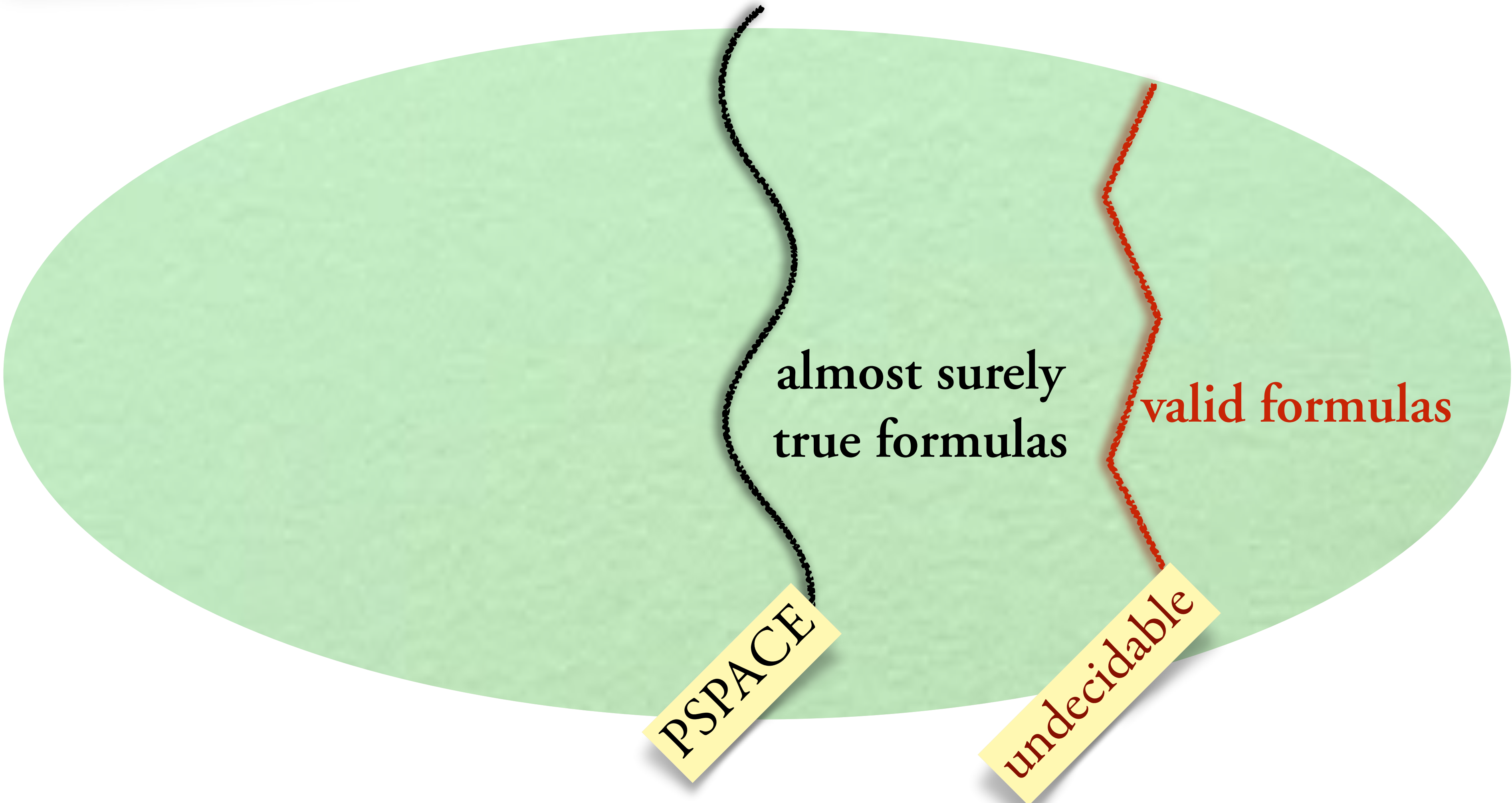
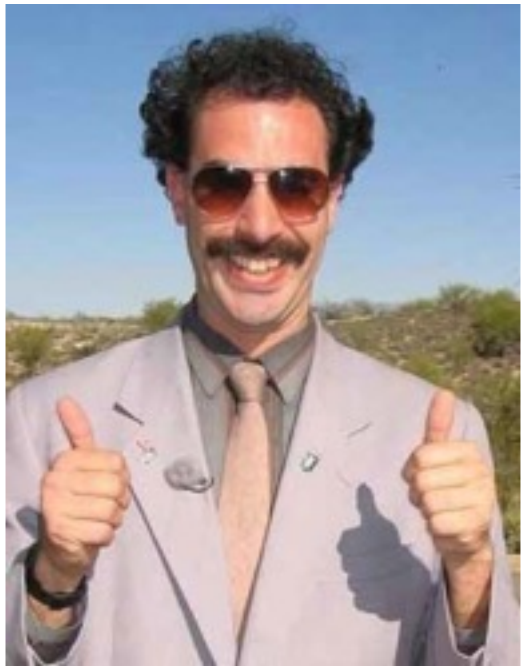


Query evaluation on large databases:

Don't bother evaluating an FO query,
it's either true or false with high probability!



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The 0-1 Law is a tool for proving expressiveness results,
not a statement on the real-life probability of queries being non-empty.

Locality

Idea: First order logic can only express "local" properties

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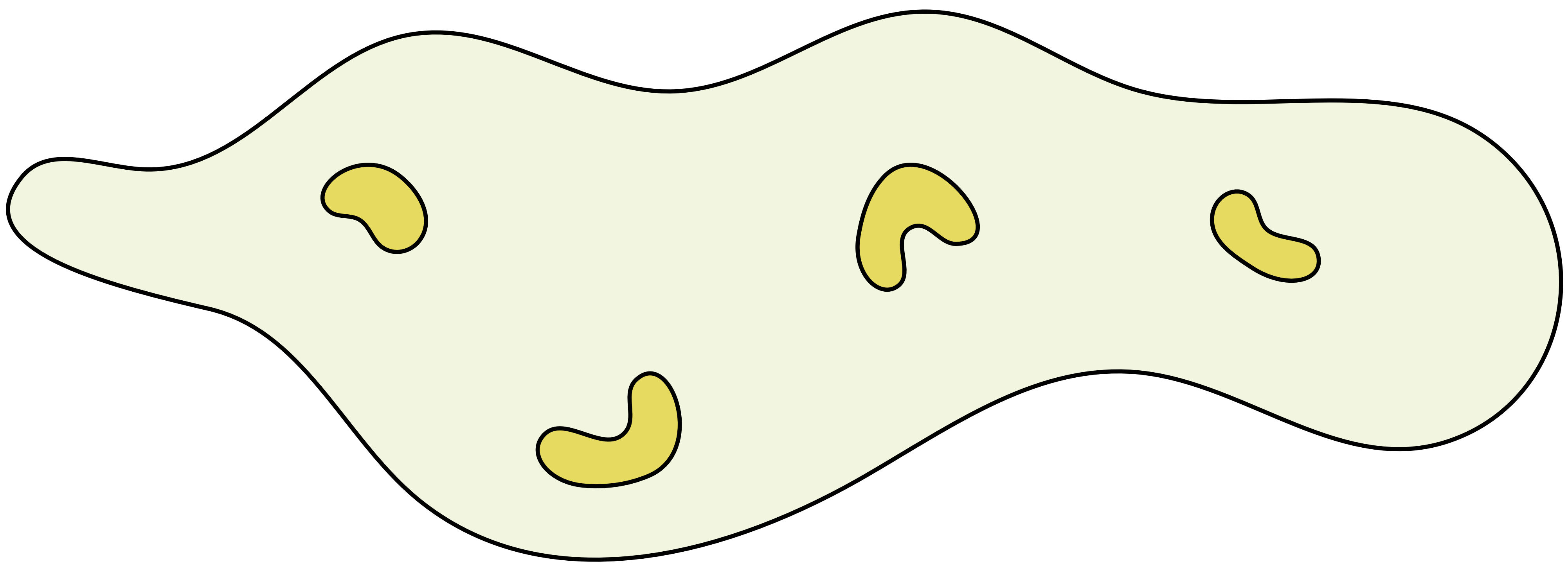
Local = properties of nodes which are close to one another



Hanf locality

Definition. The **Gaifman graph** of a structure $S = (V, R_1, \ldots, R_m)$ is the **undirected** graph

$$G(S) = (V, E) \quad \text{where} \quad E = \{ (u, v) \mid \exists (\ldots, u, \ldots, v, \ldots) \in R_i \text{ for some } i \}$$

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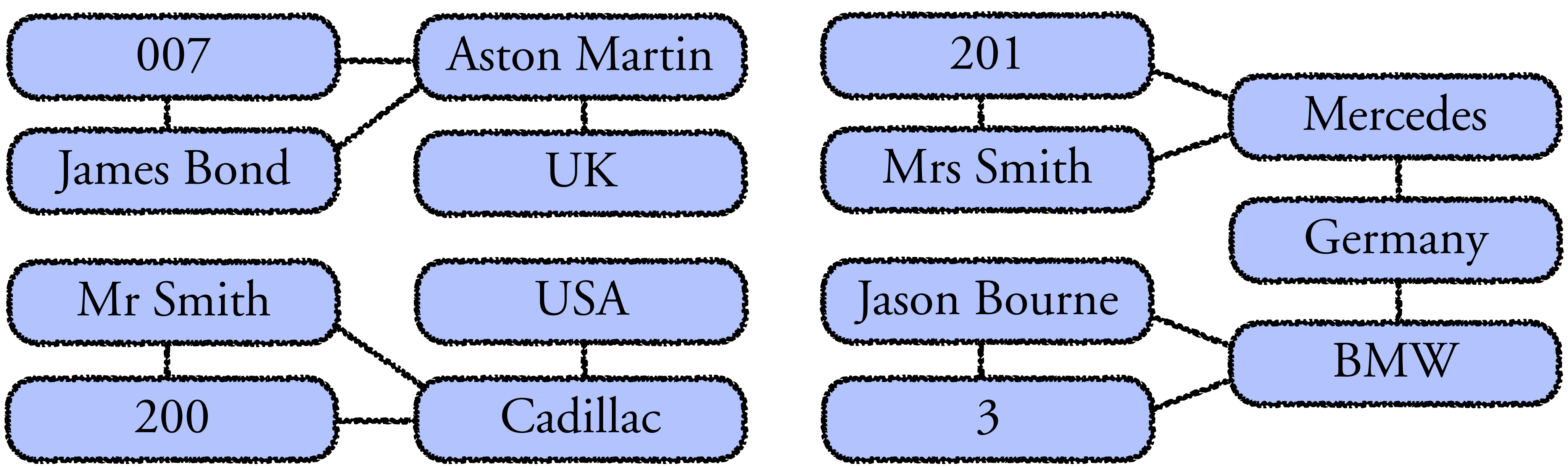

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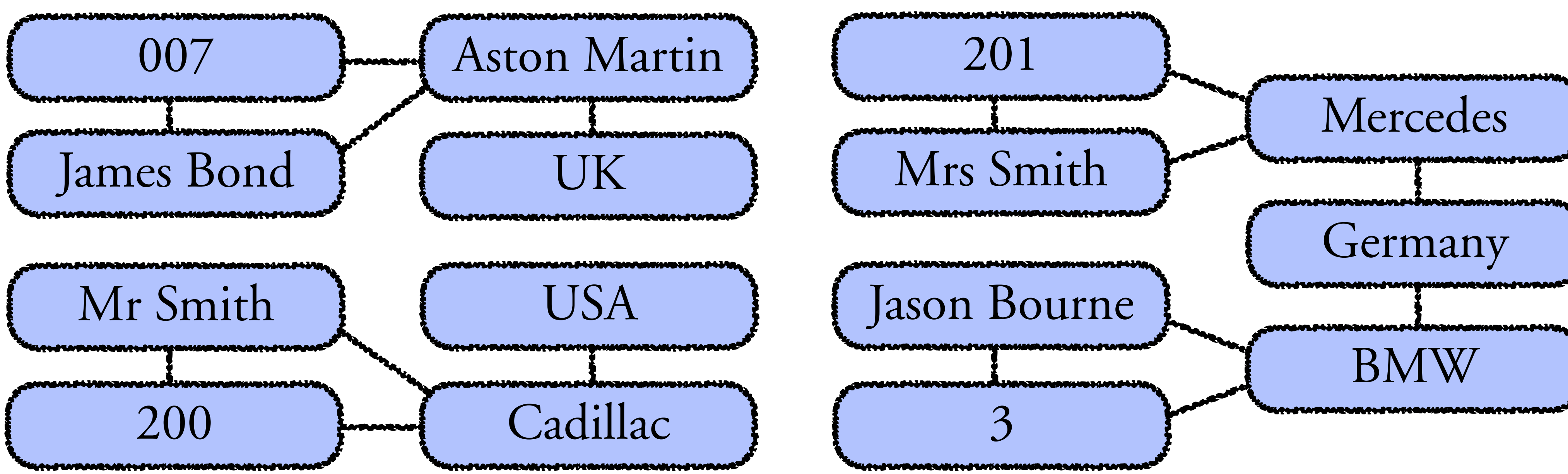
The Gaifman graph of a graph G is the underlying undirected graph.



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- dist (u, v) = *distance* between u and v in the Gaifman graph
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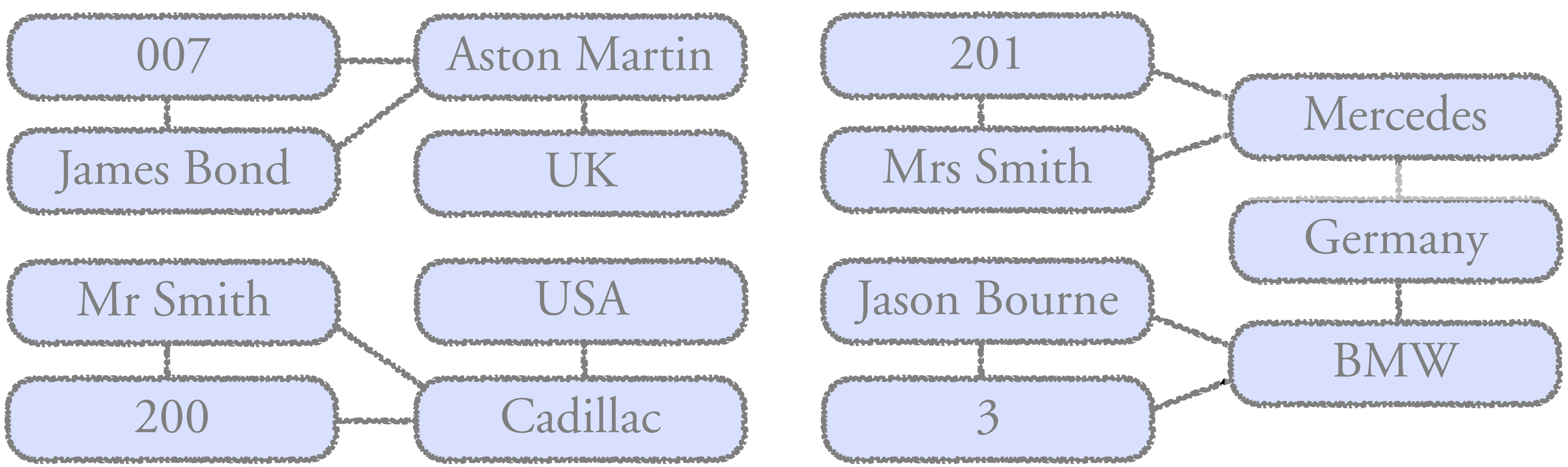

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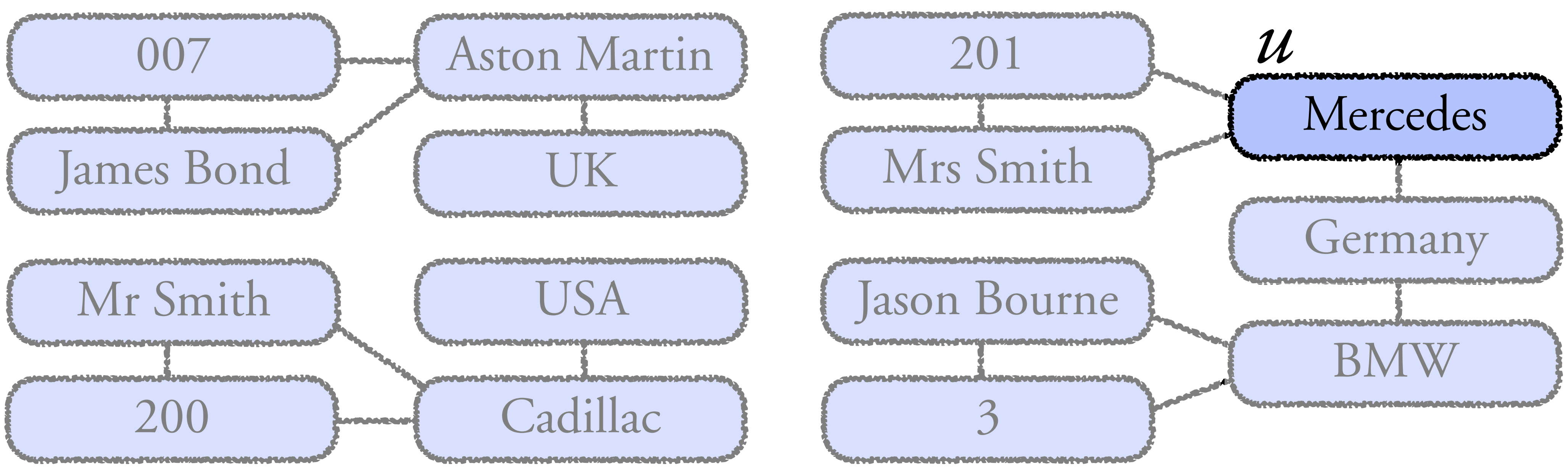

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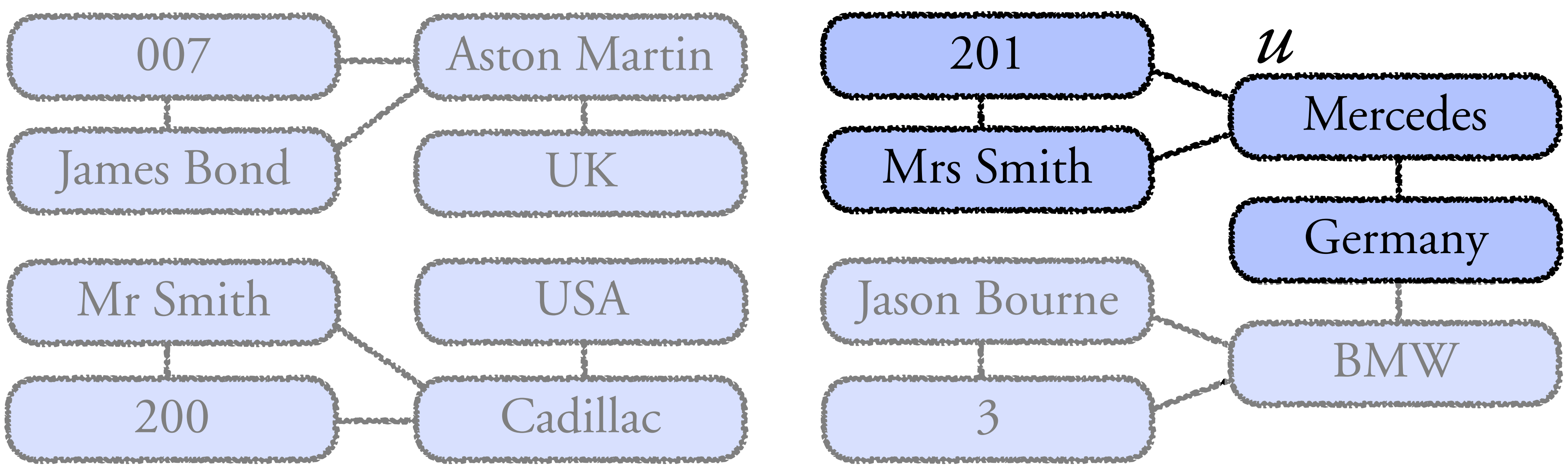

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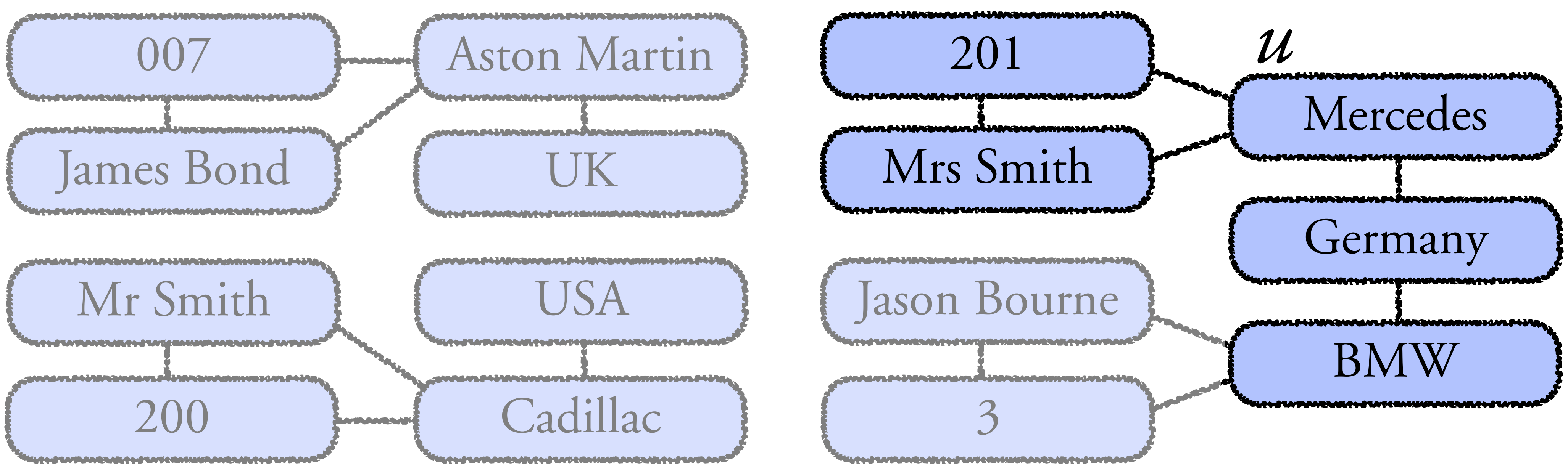

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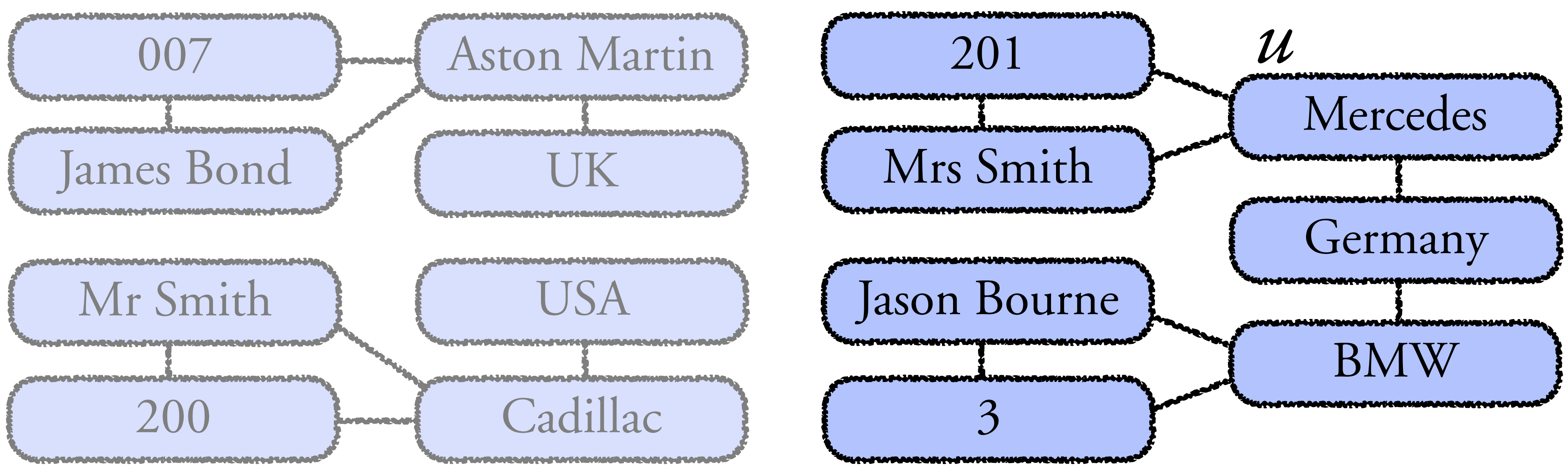

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iff for each structure B, the two numbers

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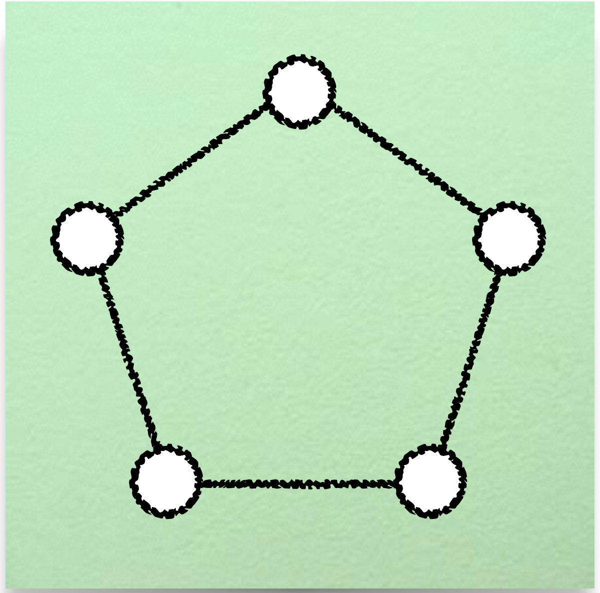
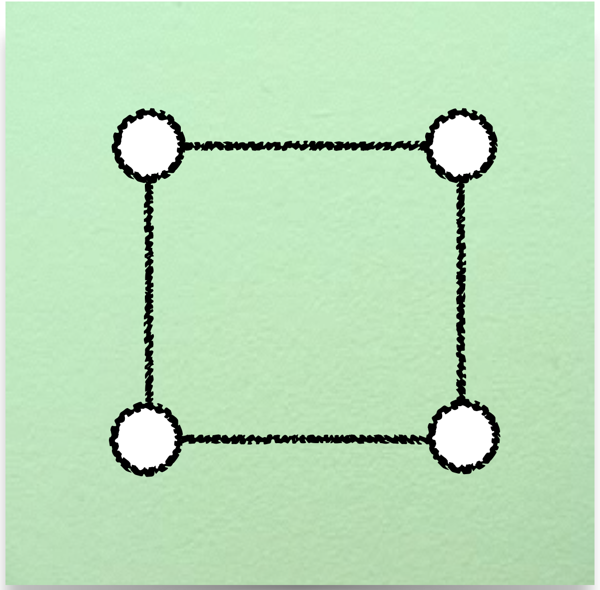
Example. S_1, S_2 are Hanf$(1, 1)$-equivalent iff they have the *same balls* of radius 1





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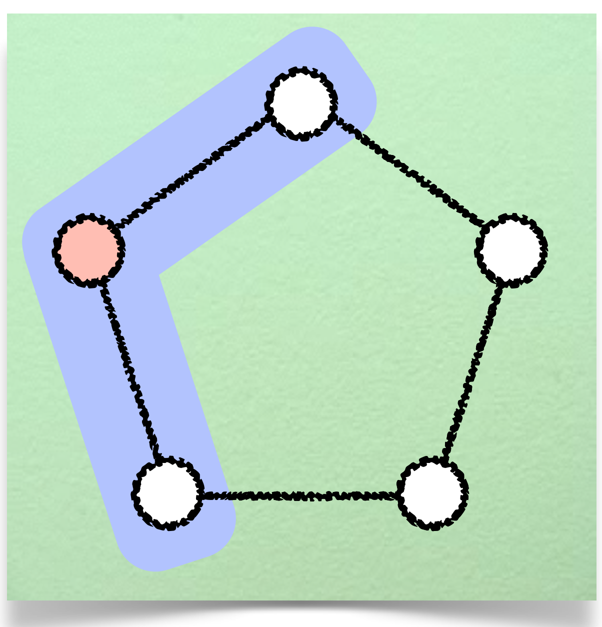

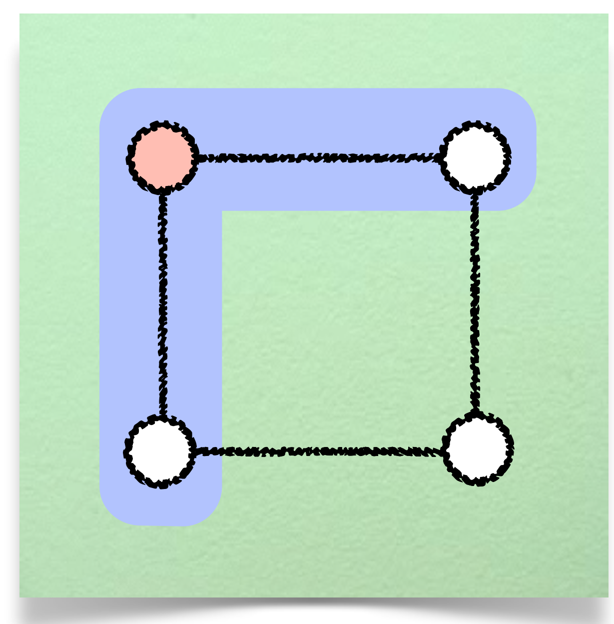

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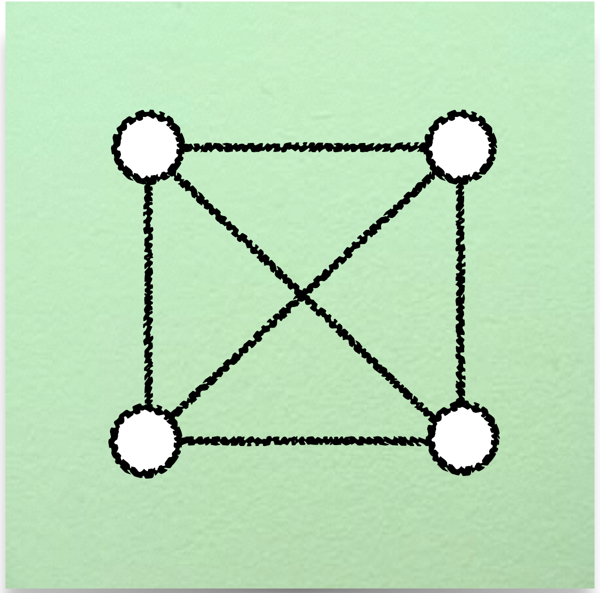

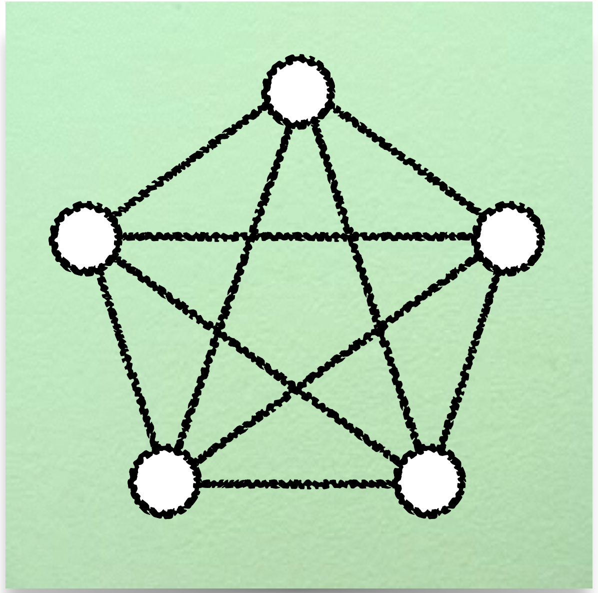

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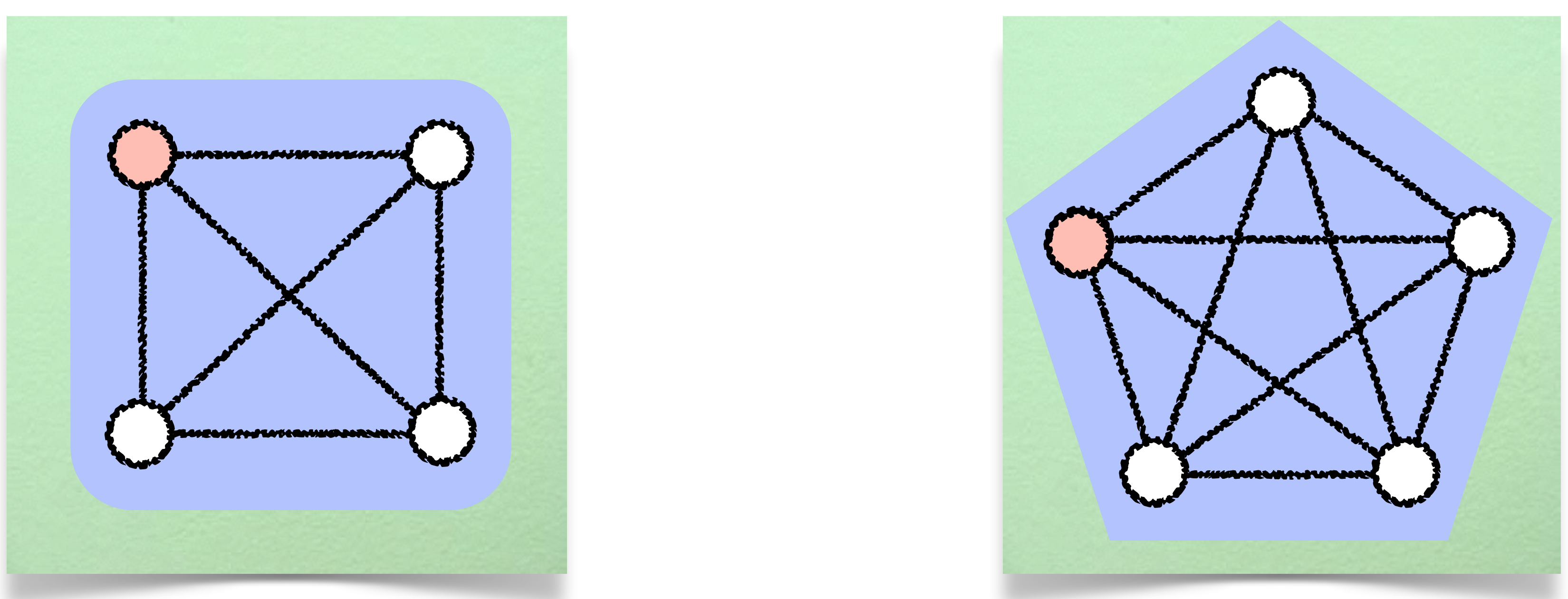

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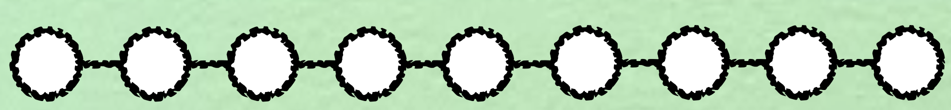

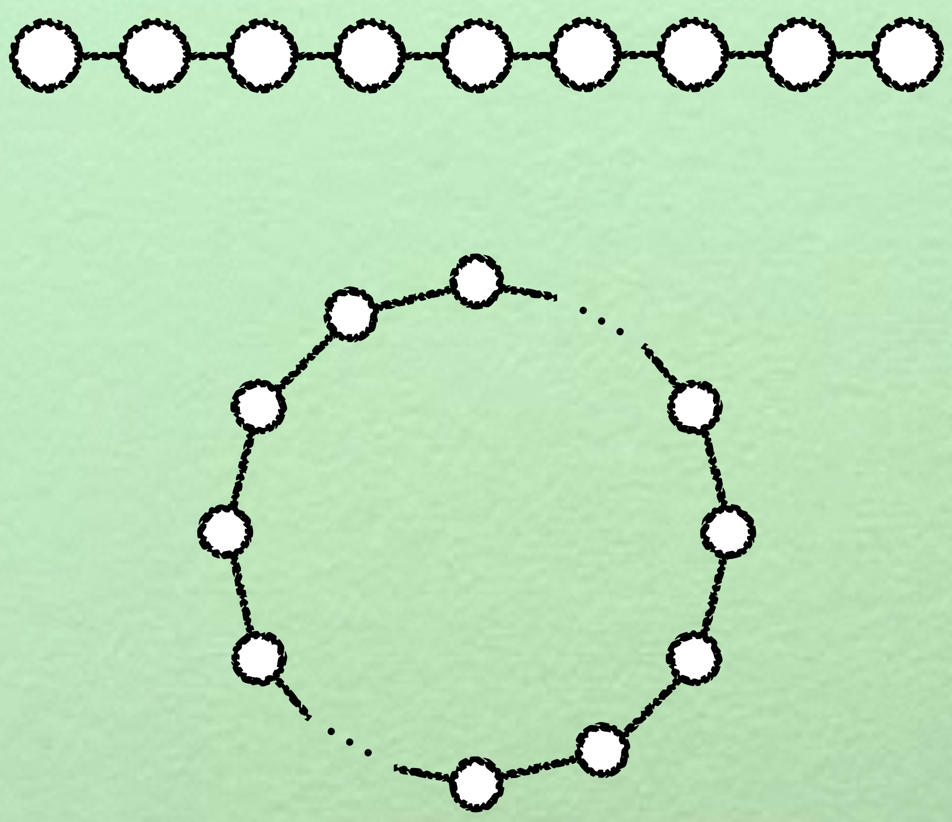

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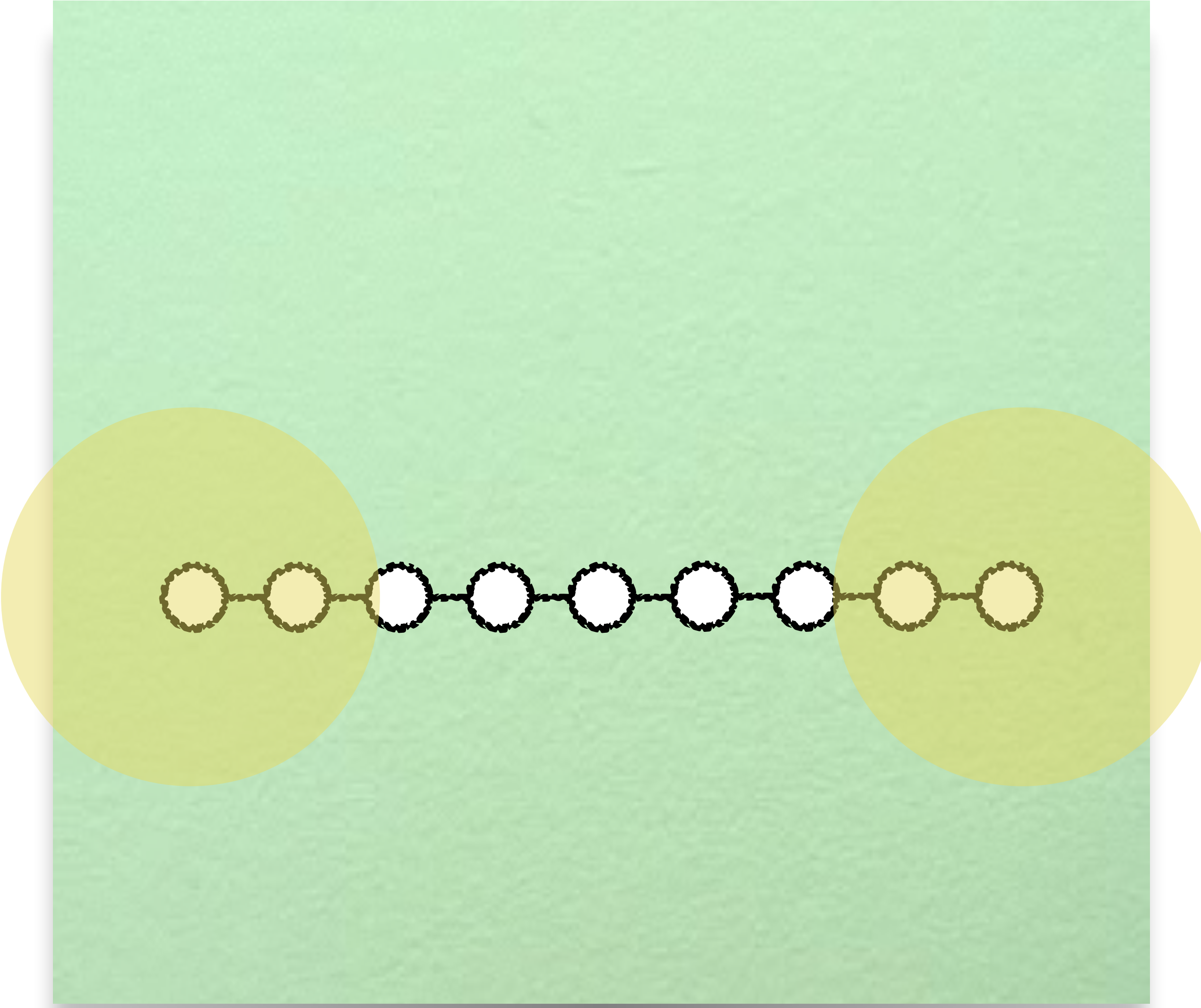

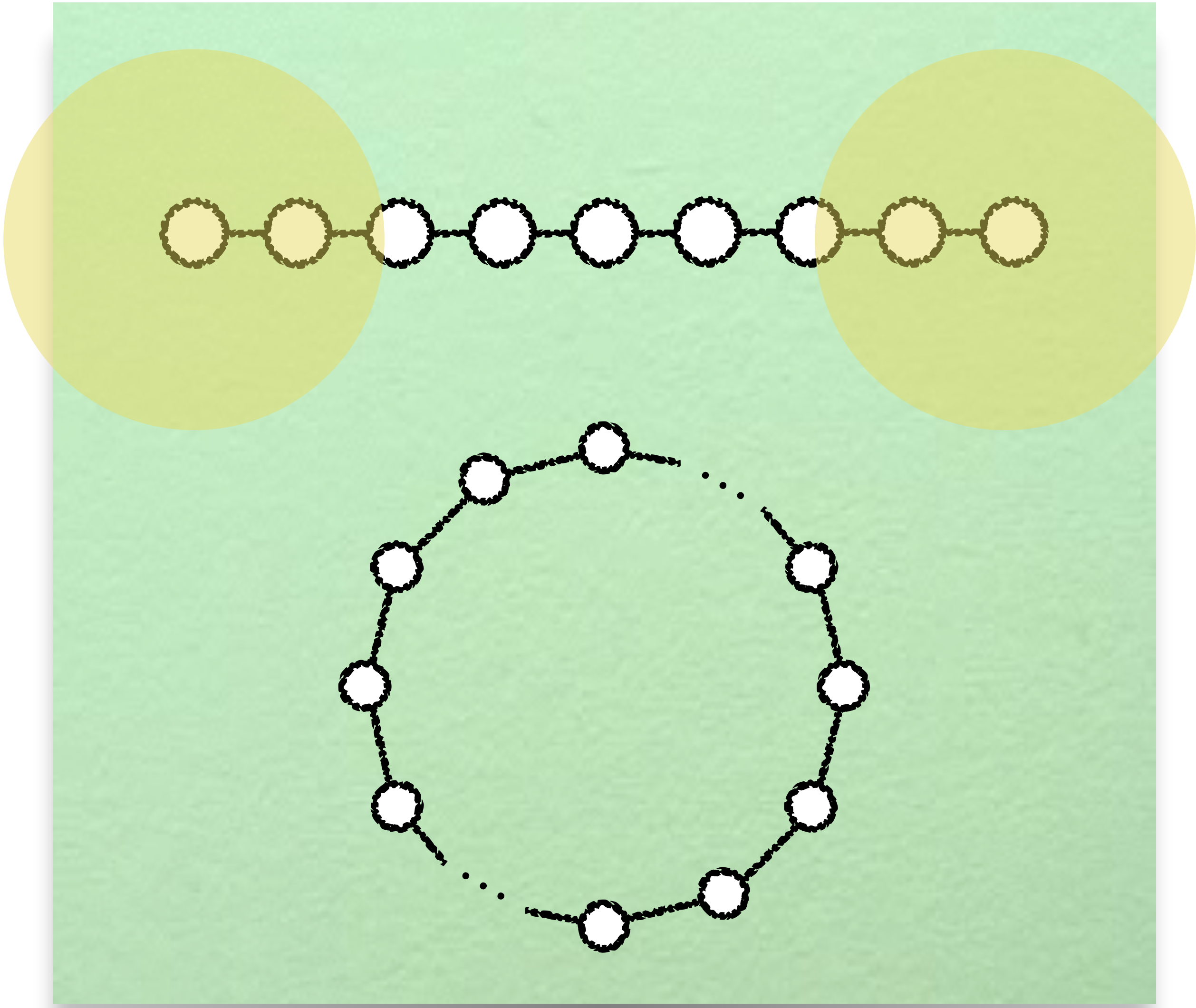

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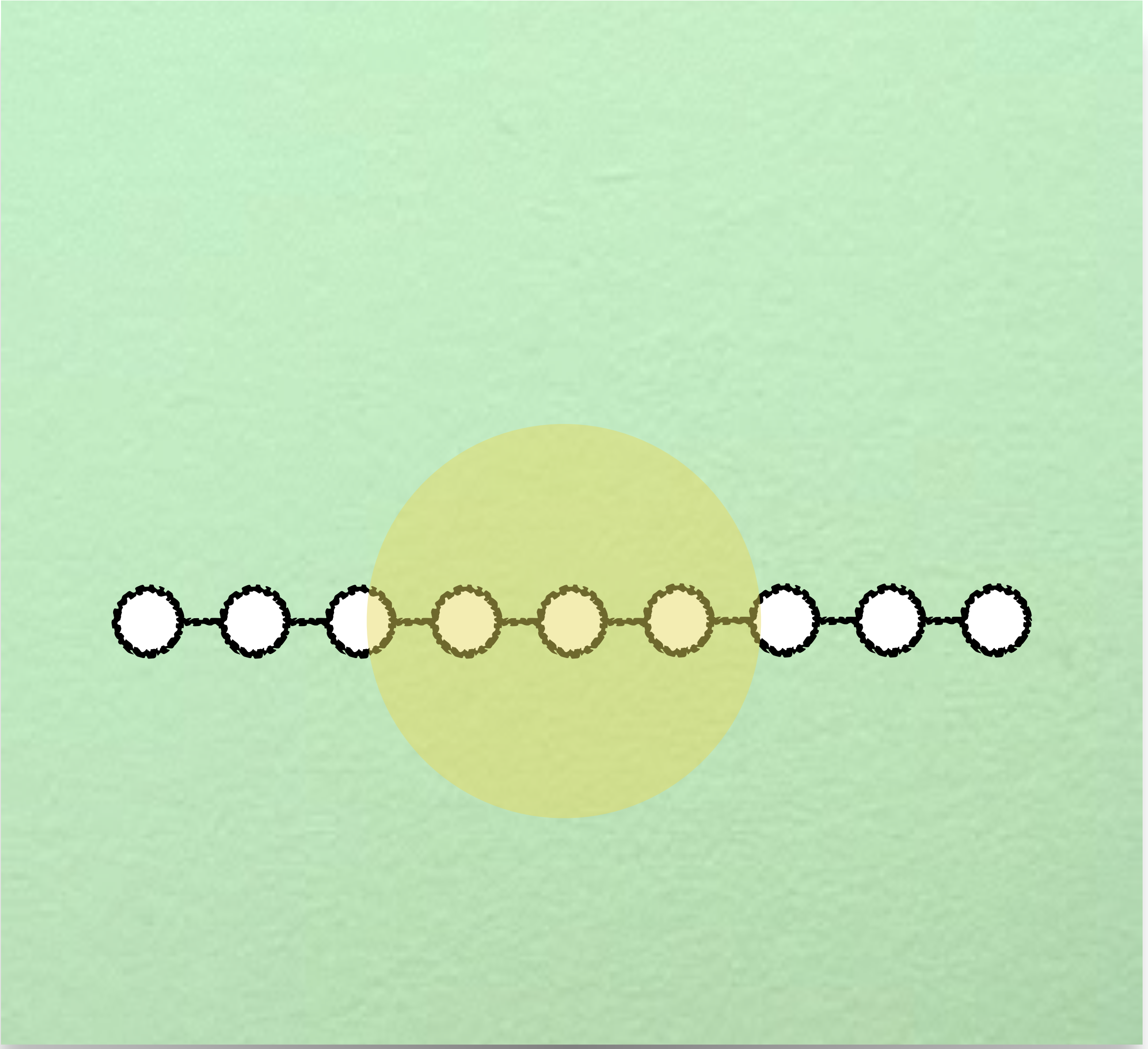

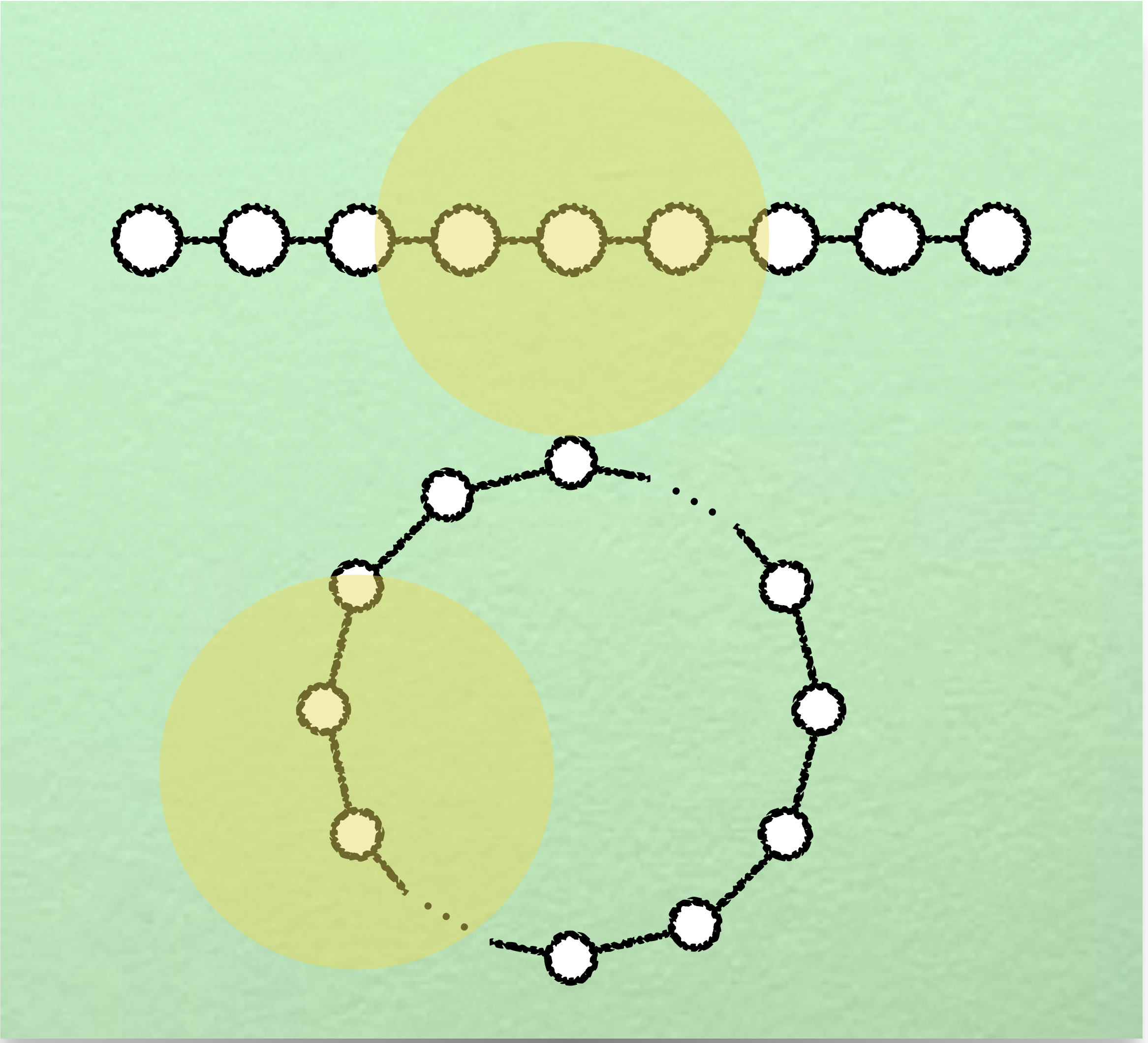

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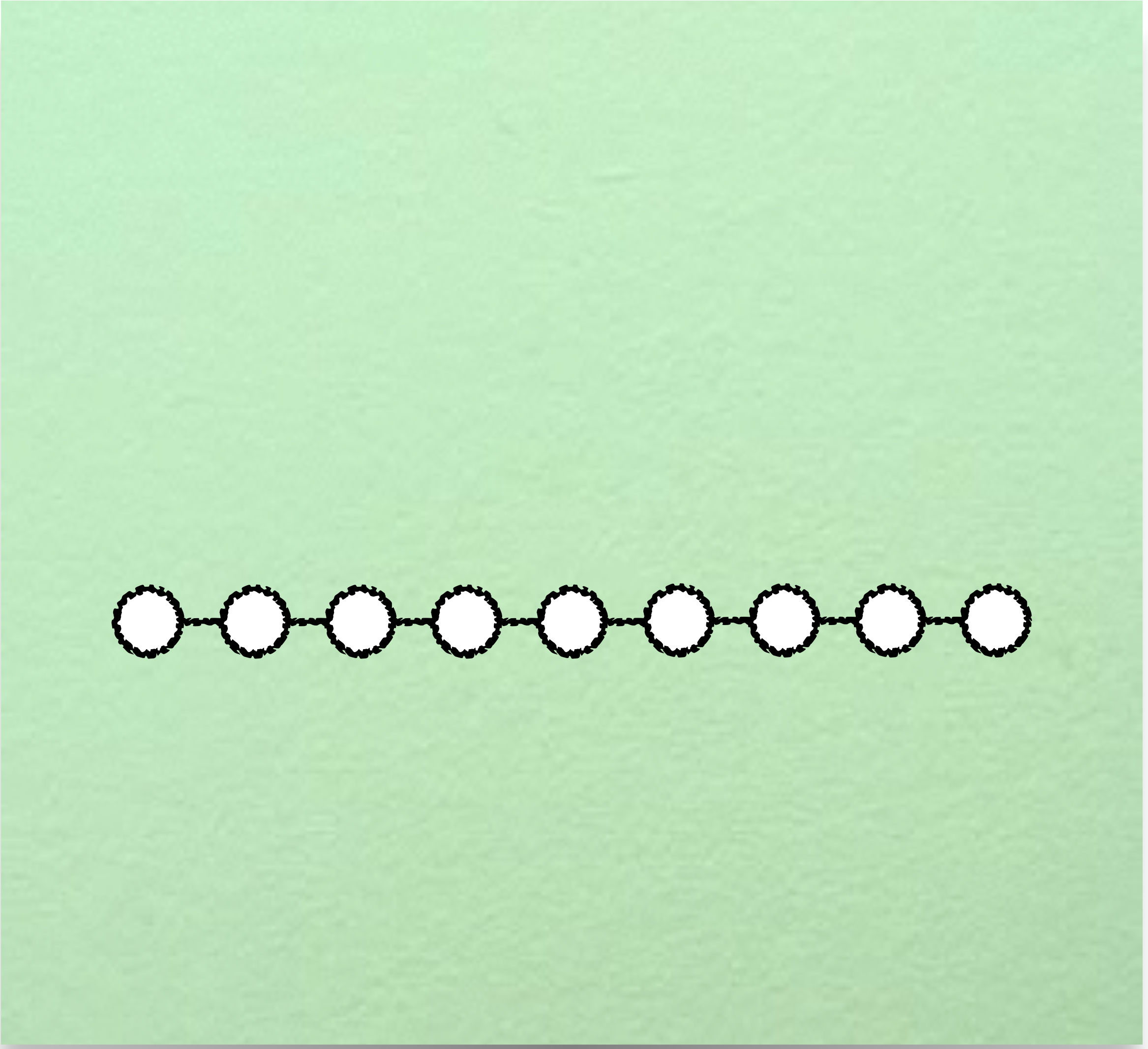

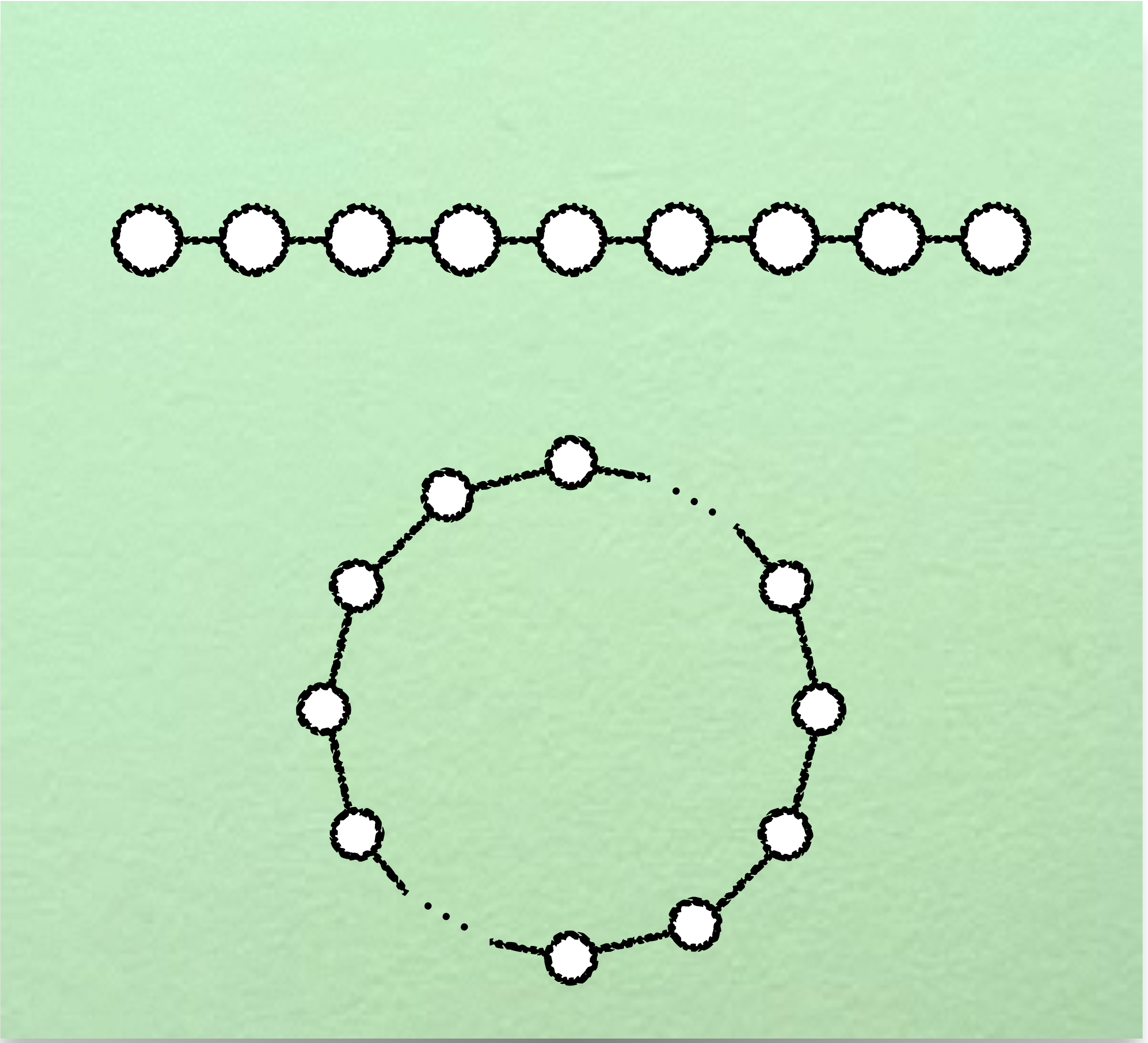

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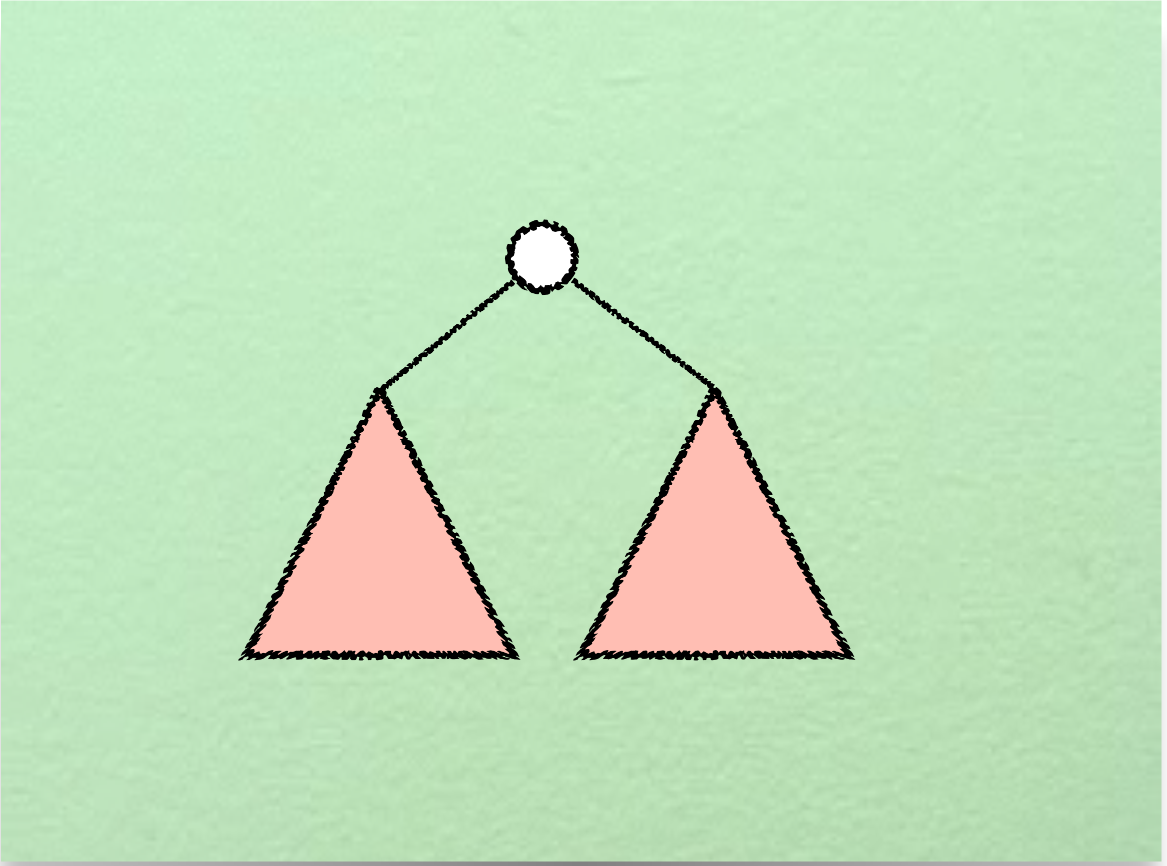

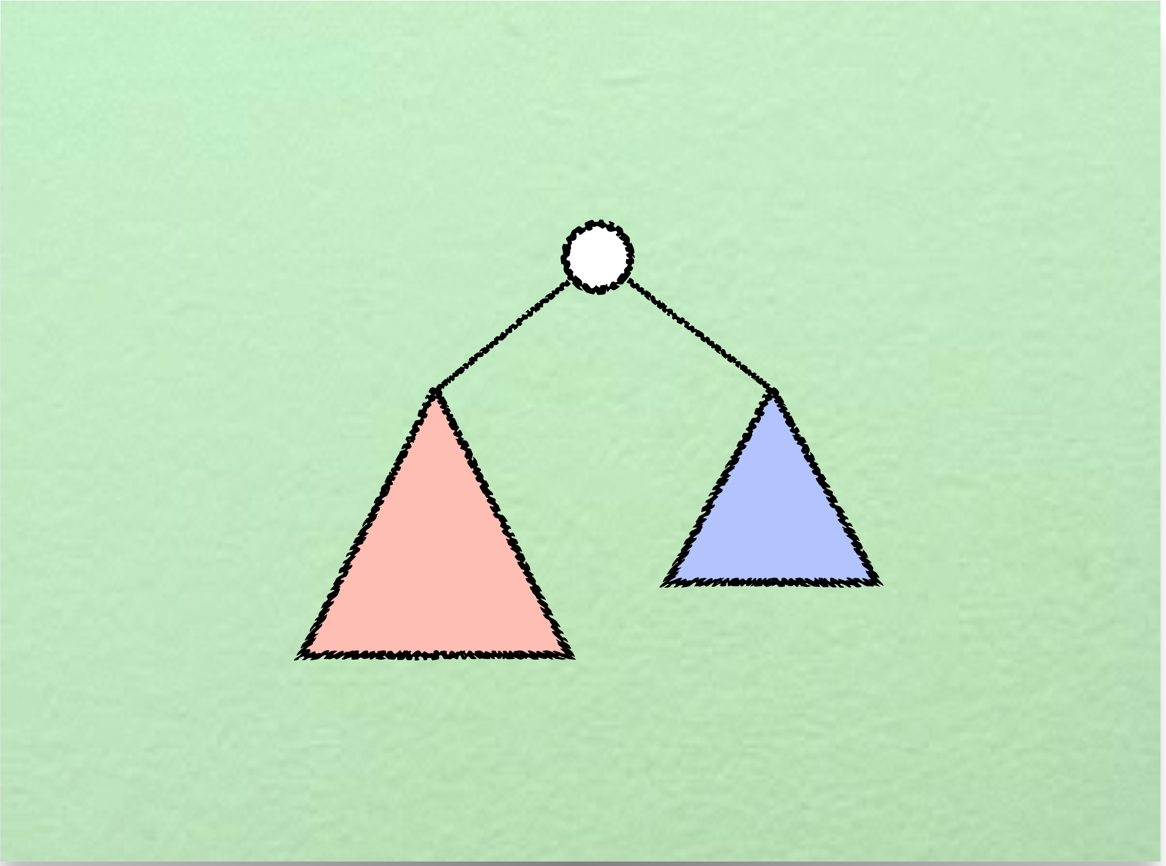

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$2{\cdot}2^n{+}1$ $\qquad\qquad$ $2{\cdot}2^n$

Not (n+2)-equivalent yet they have the same $2^n{-}1$ balls.

Gaifman locality

What about queries?

Eg: Is reachability expressible in FO?

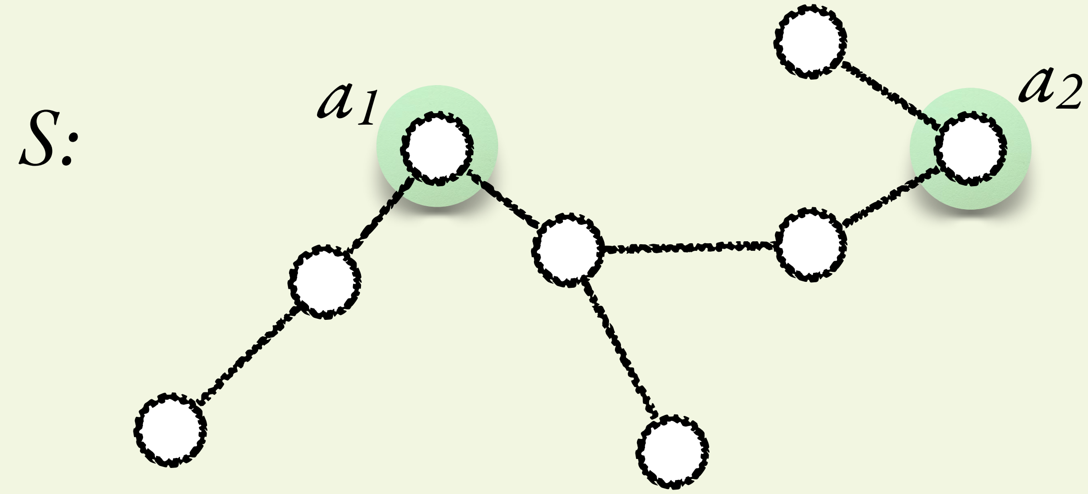
What about equivalence on the same structure?

When are two points indistinguishable?

Gaifman locality

$S[(a_1, \ldots, a_n), r]$ = induced substructure of S

of elements at distance $\leq r$ of some a_i in the Gaifman graph.



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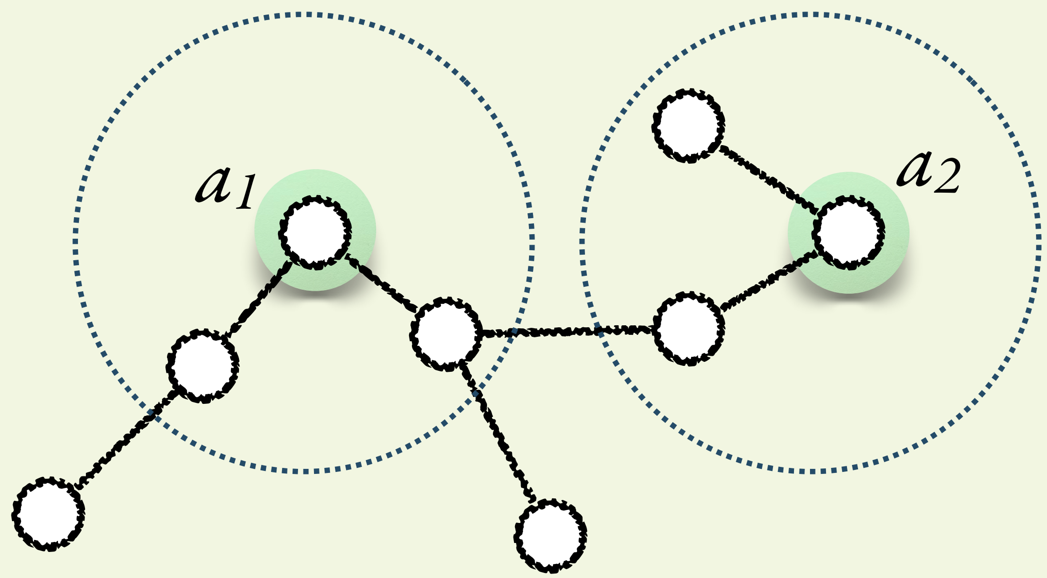

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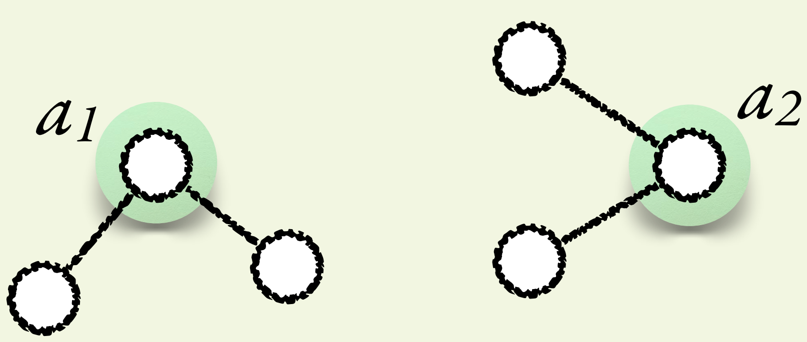

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For any $\phi \in$ FO of quantifier rank k and structure S,

$$S[(a_1, \ldots, a_n), r] \cong S[(b_1, \ldots, b_n), r] \quad \text{for } r = 3^{k+1}$$

implies

$$(a_1, \ldots, a_n) \in \phi(S) \quad \text{iff} \quad (b_1, \ldots, b_n) \in \phi(S)$$

Idea: If the neighbourhoods of two tuples are the same,
the formula cannot distinguish them.

Gaifman locality vs Hanf locality

Difference between Hanf- and Gaifman-locality:

Hanf-locality relates **two different structures,**

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Inside S,

3^{k+1}-balls of $(a_1,\ldots,a_n) = 3^{k+1}$-balls of $(b_1,\ldots,b_n)$

$\Downarrow$

$(a_1,\ldots,a_n)$ indistinguishable from $(b_1,\ldots,b_n)$ through **formulas** of qr $\leq k$

Gaifman locality vs Hanf locality

Difference between Hanf- and Gaifman-locality:

Hanf-locality relates **two different structures**,

S_1 and S_2 have the same # of balls of radius 3^k, **up to threshold *k***

$\Downarrow$

They verify the same **sentences** of qr $\leq k$

Gaifman-locality talks about definability in **one structure**

Inside S,

3^{k+1}-balls of $(a_1,\ldots,a_n) = 3^{k+1}$-balls of $(b_1,\ldots,b_n)$

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$(a_1,\ldots,a_n)$ indistinguishable from $(b_1,\ldots,b_n)$ through **formulas** of qr $\leq k$

Gaifman locality

Schema to show non-expressibility results is, as usual:

A query $Q(x_1,\ldots,x_n)$ is not FO-definable if:

for every $\boldsymbol{k}$ there is a structure S_k and $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ such that

- $S_k[(a_1, \ldots, a_n), 3^{k+1}] \cong S_k[(b_1, \ldots, b_n), 3^{k+1}]$
- $(a_1, \ldots, a_n) \in Q(S_k)$, $(b_1, \ldots, b_n) \notin Q(S_k)$

Proof: If Q were expressible with a formula of quantifier rank k,

then $(a_1, \ldots, a_n) \in Q(S_k)$ iff $(b_1, \ldots, b_n) \in Q(S_k)$. Absurd!

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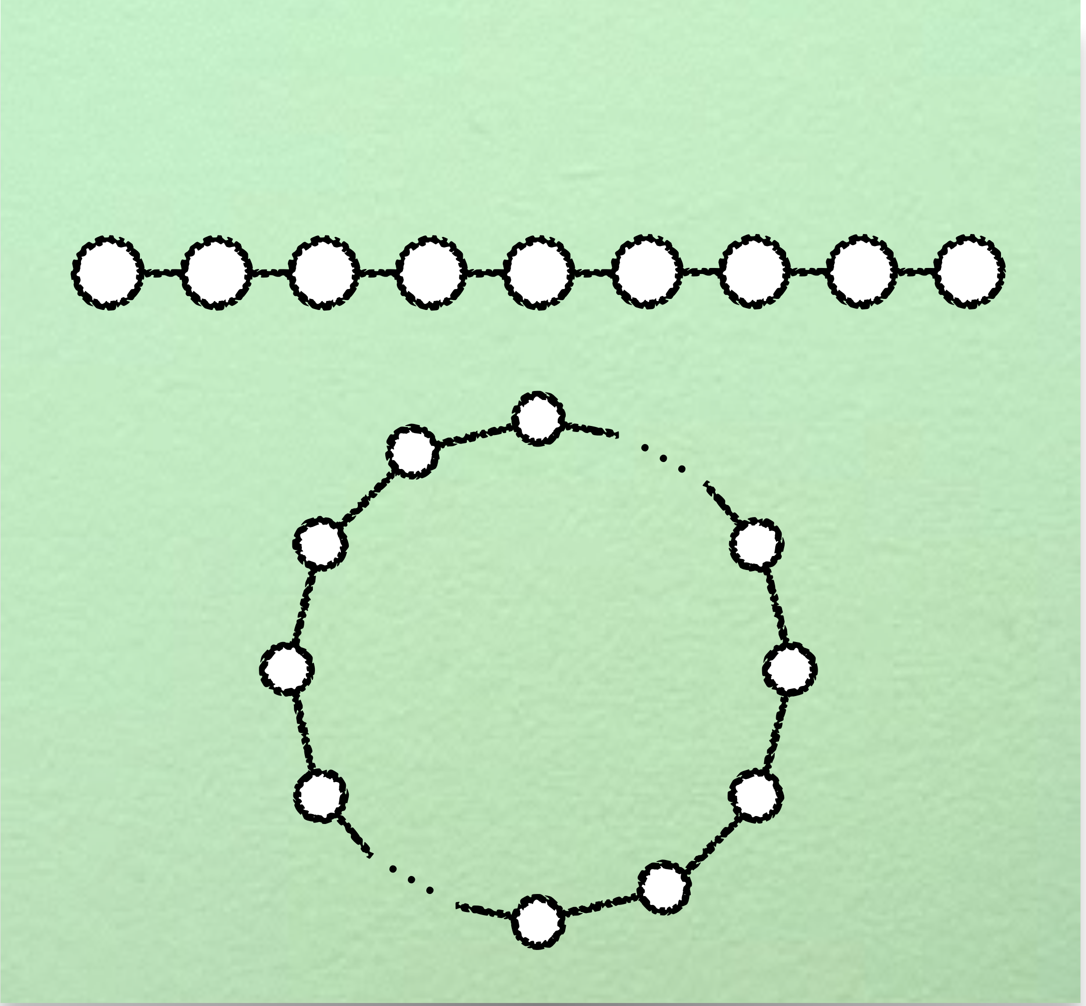

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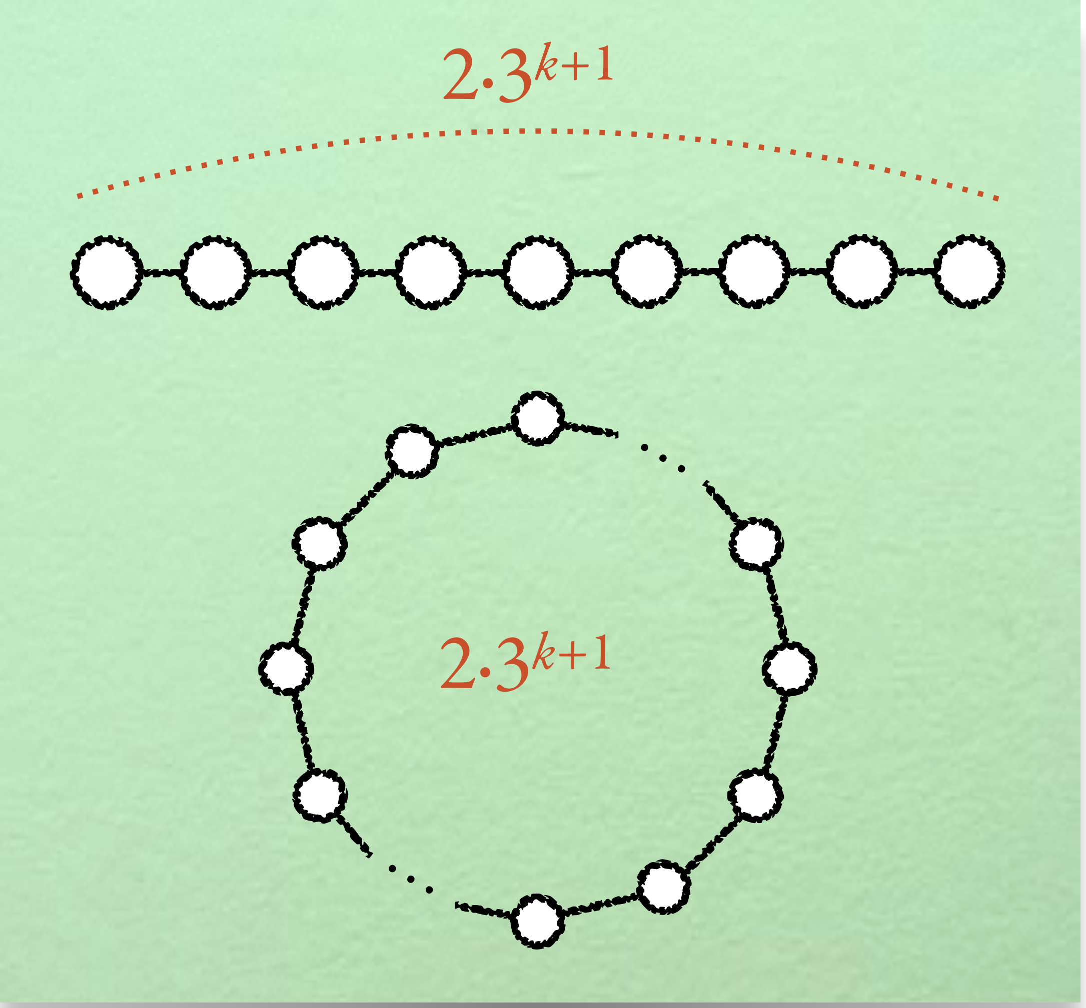

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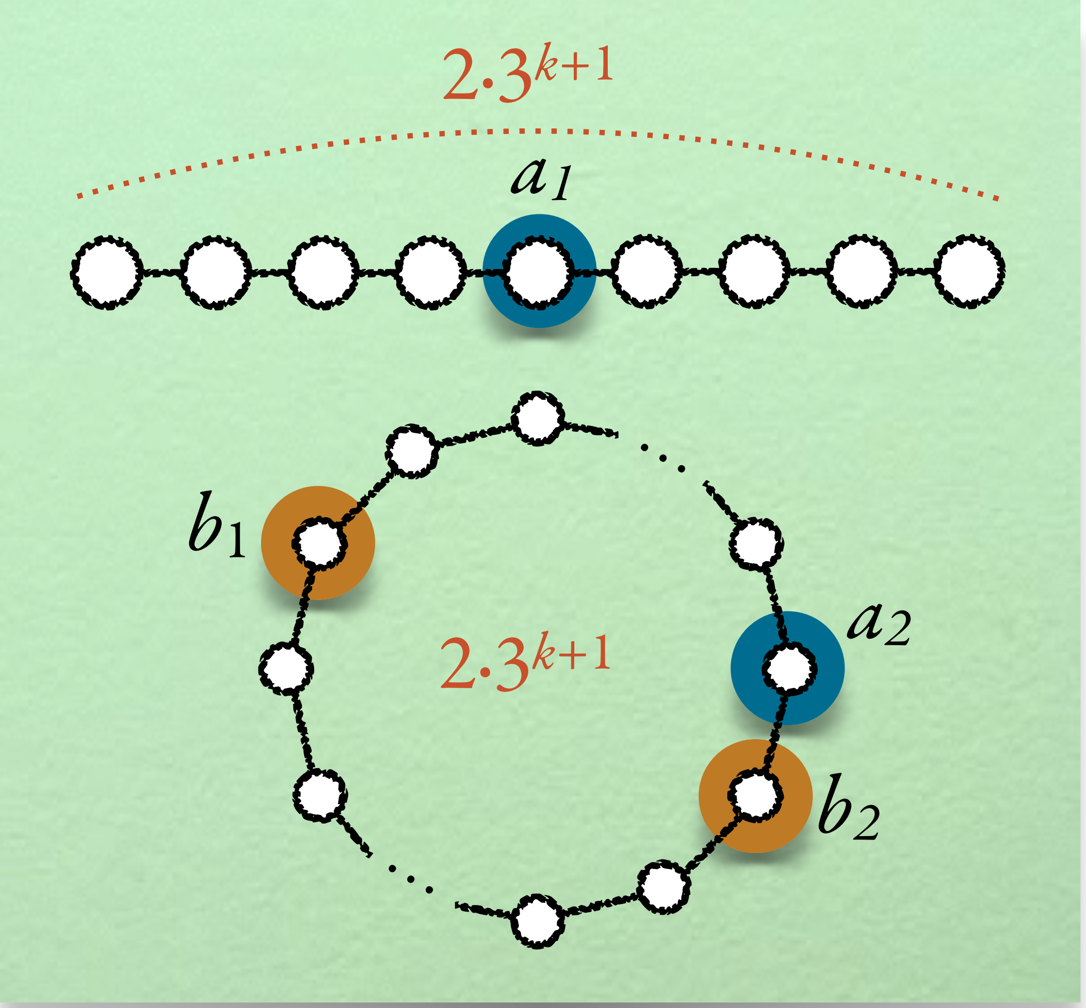

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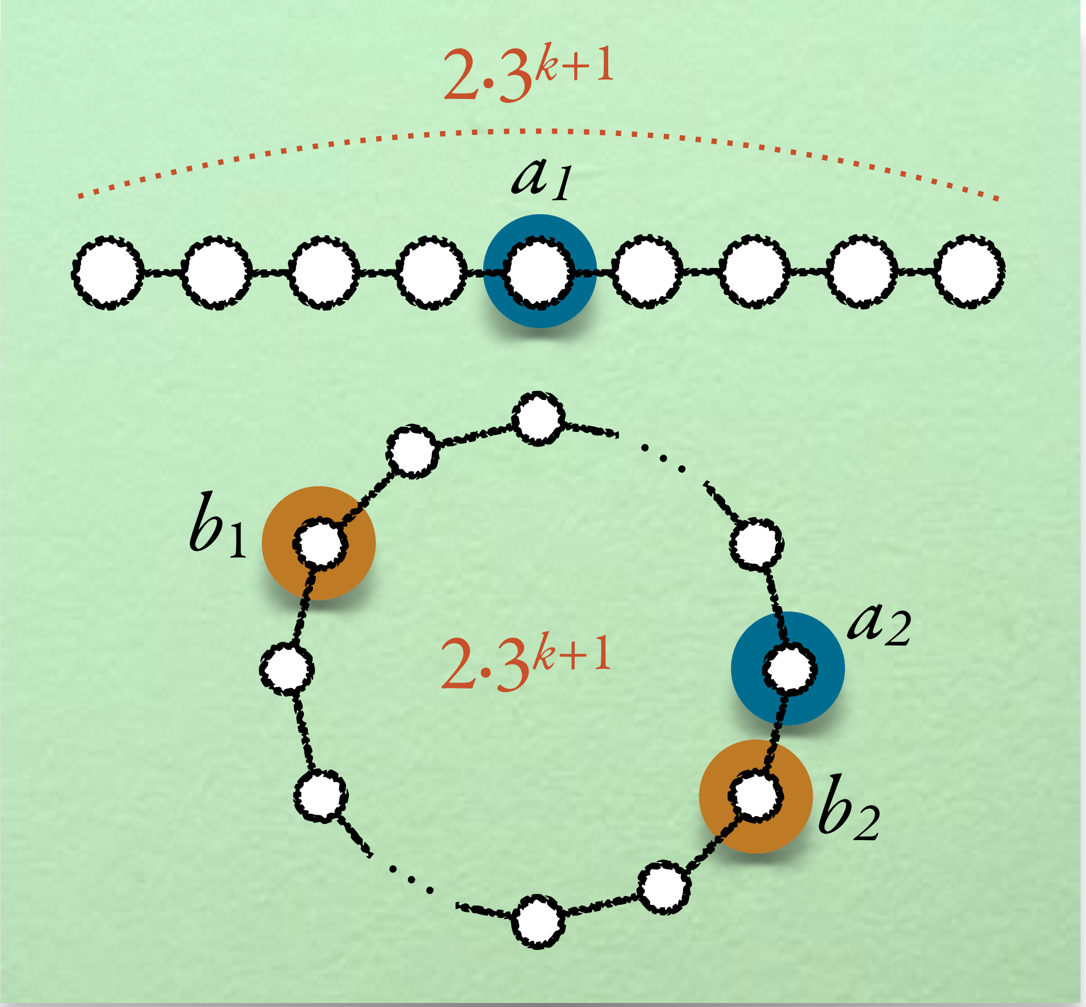

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However,

- b_2 is reachable from b_1,
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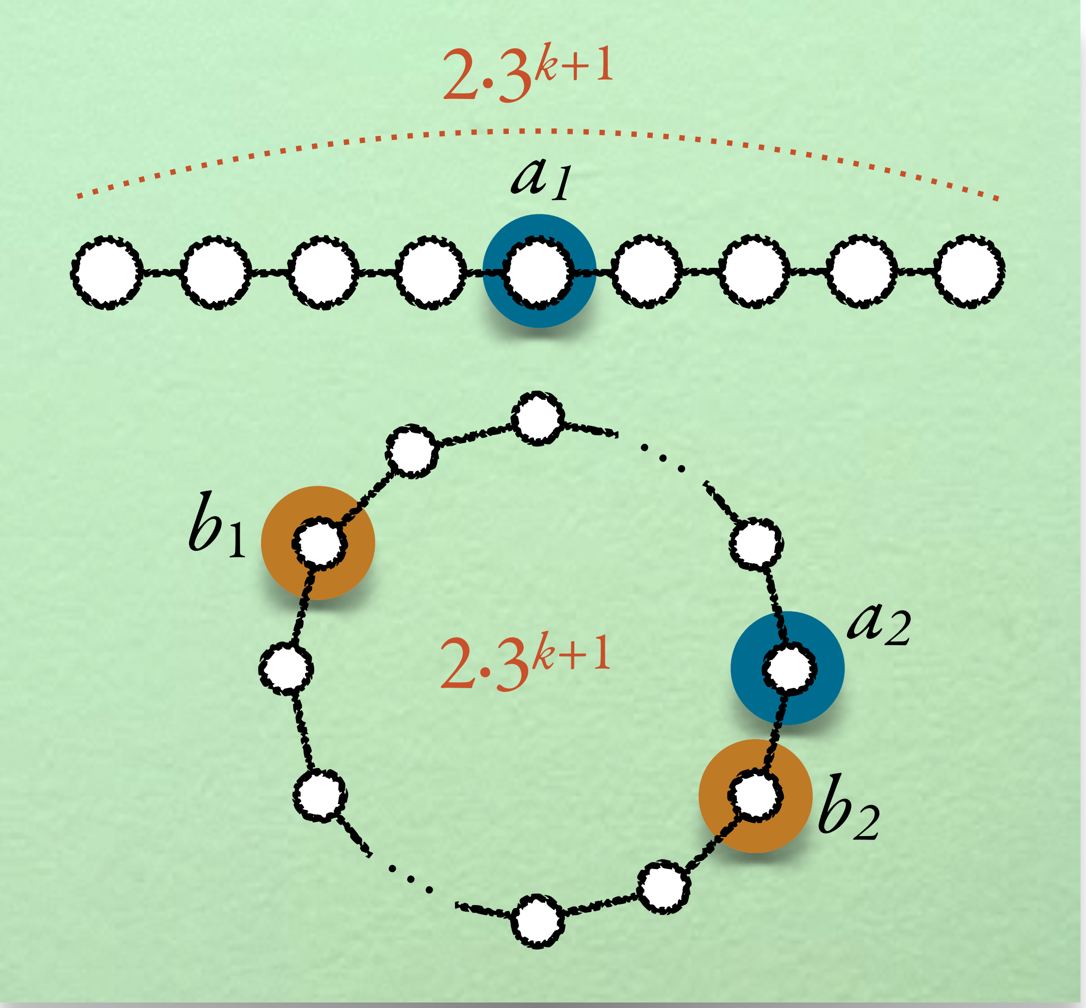

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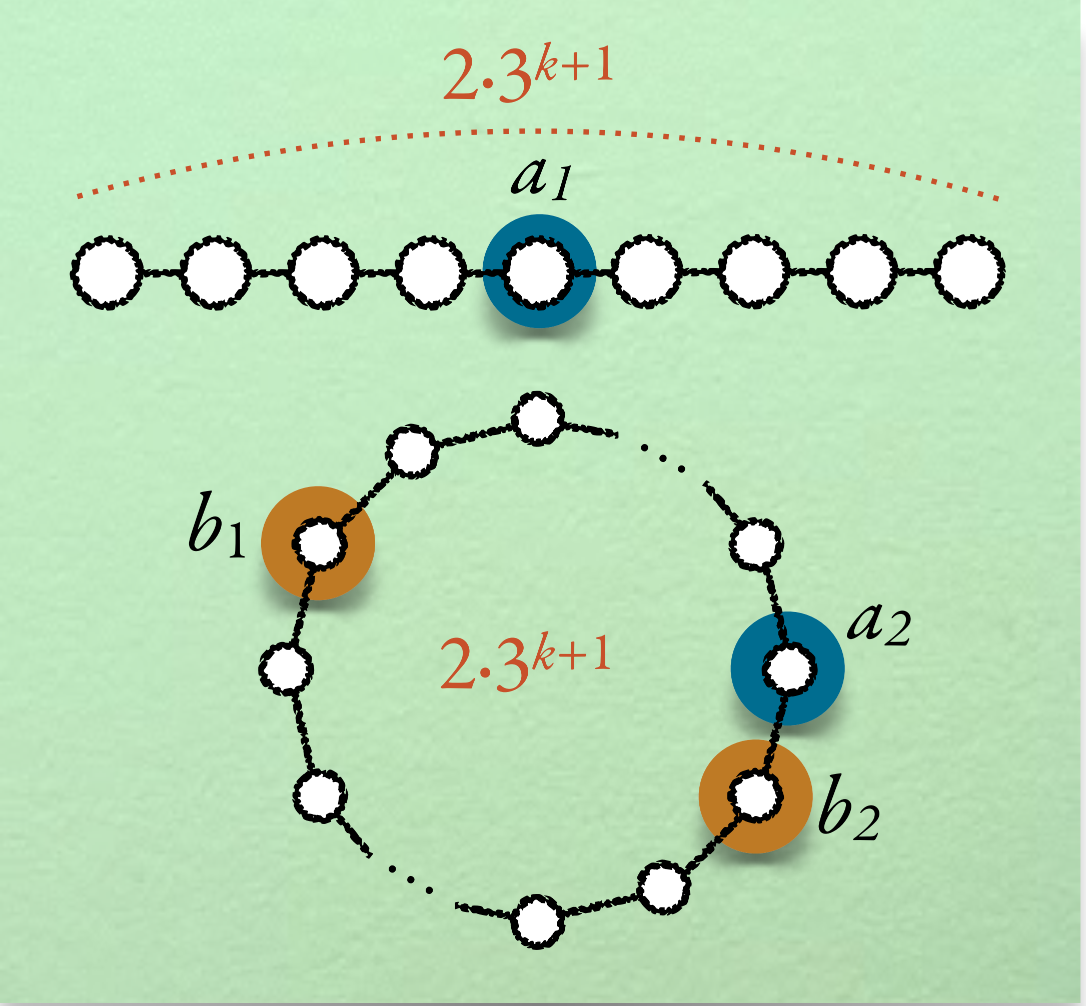
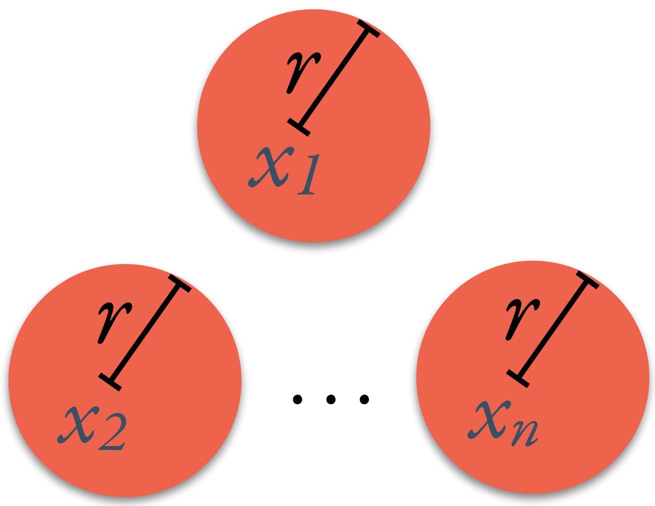


Your turn! Q(x) = "x is a vertex separator"

Gaifman Theorem

Basic local sentence:

$\exists\, x_1, \ldots, x_n$



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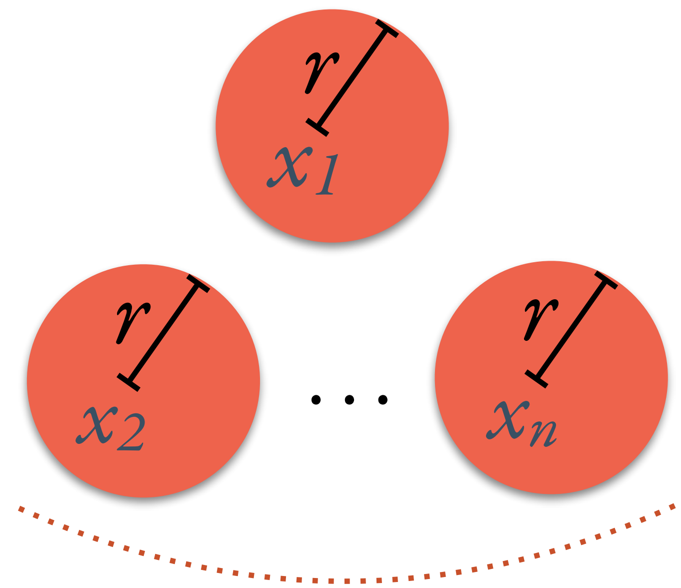

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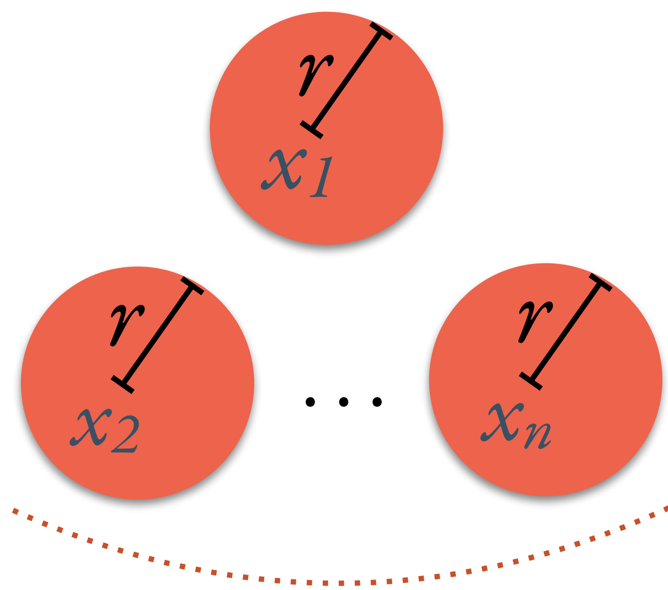

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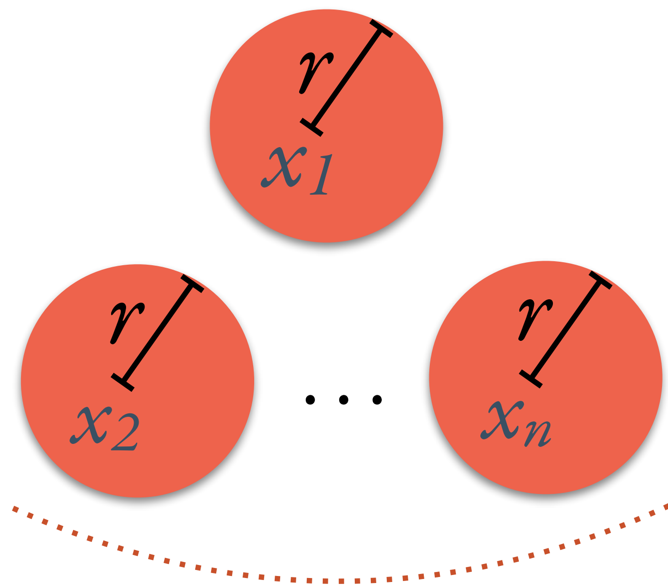

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Inside $\psi_i(x_i)$ we interpret $\exists y \,.\, \phi$ as $\exists y \,.\, \mathrm{d}(x_i, y) \leq r \wedge \phi$

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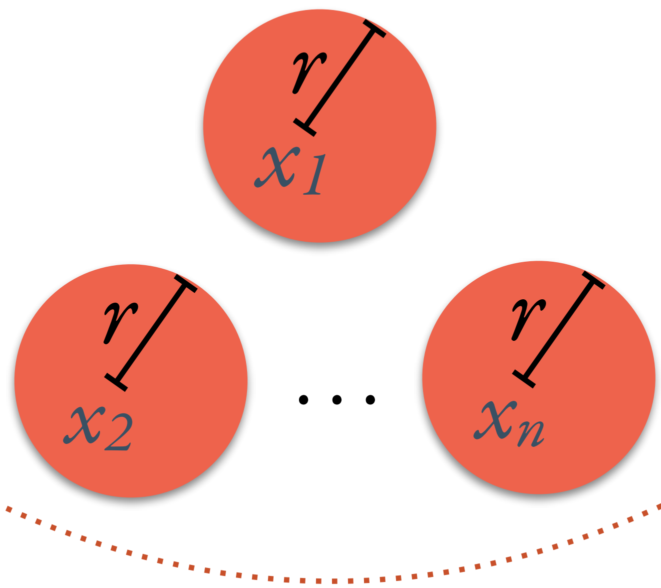

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Gaifman Theorem: Every FO sentence is equivalent to a boolean combination of **basic local sentences**.

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FO sentences with quantifier rank n

=

winning strategies for Spoiler in the n-round EF game

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FO sentences are almost always true or almost always false

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Hanf locality	FO sentences with quantifier rank n = counting 3^n sized balls up to n
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Queries of quantifier rank n output tuples closed under 3^{n+1} balls.

Gaifman Theorem

An FO sentence can only say

"there are some points at distance ≥2r

whose r-balls are isomorphic to certain structures"

or a boolean combination of that.

Some Bibliography

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