Fundamentos lógicos de bases de datos
(Logical foundations of databases)

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Recap

- **Relational model** (tables)
- **Relational Algebra** (union, product, difference, selection, projection)
- **SQL** (SELECT ... FROM ... WHERE ...)
- **First-order logic** (syntax, semantics, active domain)
- **Expressiveness** (FO^{act} = RA = basic SQL)
- **Undecidable problems** (Halting ≤ Domino ≤ Satisfiability ≤ Equivalence)
- **Data complexity / Combined complexity**
- **Complexity of evaluation** (LOGSPACE / PSPACE complexity)

Definability in FO

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Example. $Q(G) = \{ (u,v) \mid G \text{ contains a path from } u \text{ to } v \}$

Is Q expressible as a first-order formula?

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Eg, in $d_0(x, y) = E(x, y)$, and

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Sub-goal: Given a property P and a number *n*,
tell whether P is expressible by a sentence of quantifier rank at most *n*.

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Consider a property (i.e. a set of structures) ***P***.

Suppose that there are $S_1 \in \boldsymbol{P}$, $S_2 \notin \boldsymbol{P}$ *s.t.*

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Example. **P** = { finite structures } seems to be not FO-definable.
One could then aim at proving that

for all n there are $S_1 \in P$ and $S_2 \notin P$ s.t. S_1, S_2 n-equivalent...

Expressive power via games

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Idea: For every two structures (S,S') there is a game where

a player of the game has a **winning strategy**

iff

S,S' are **indistinguishable**

Ehrenfeucht-Fraïssé games

A game between two players

Board: (S_1, S_2)

One player plays in one structure, the other player answers in the other structure.

If after n rounds Duplicator doesn't lose: S_1, S_2 are n-equivalent

Ehrenfeucht-Fraïssé games

Definition. **Partial isomorphism** between S_1 and S_2 = injective partial map

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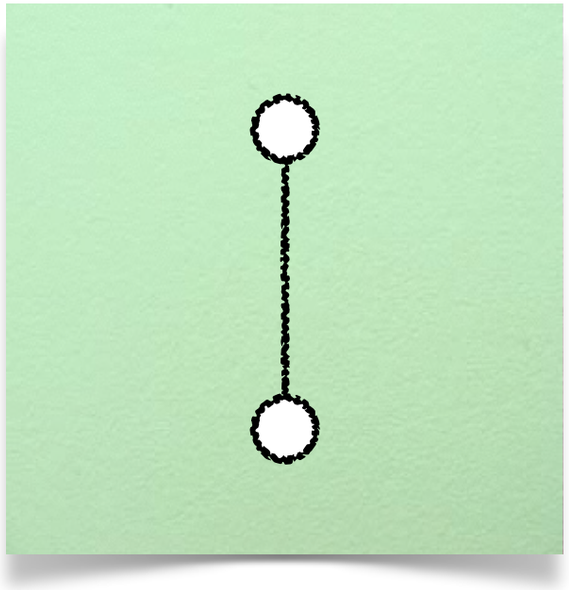
2. **Spoiler** chooses a node y_i from S_2
   and **Duplicator** answers with a node x_i from S_1,

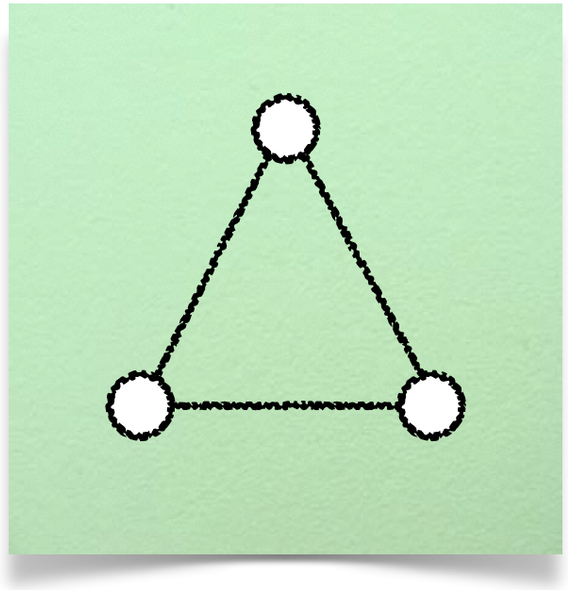
or **Spoiler** wins if $\{ x_i \mapsto y_i \mid 1 \le i \le n \}$ is **not a partial isomorphism** between S_1 and S_2.

Ehrenfeucht-Fraïssé games

S_1



S_2



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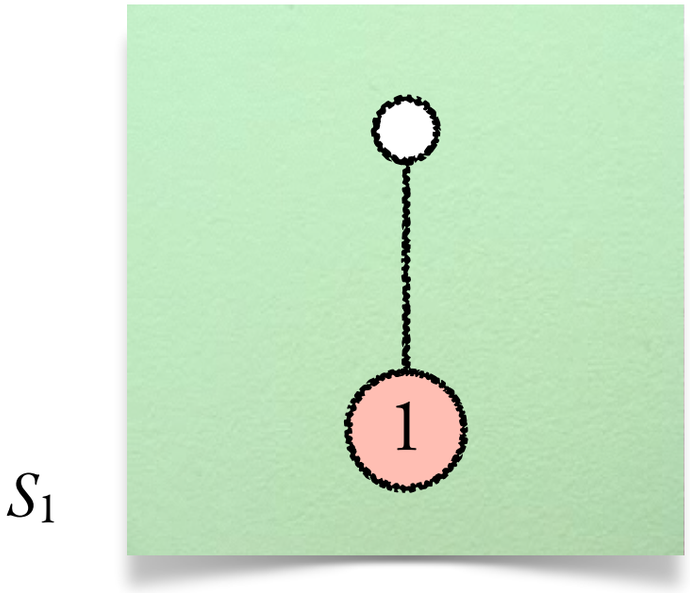
Ehrenfeucht-Fraïssé games



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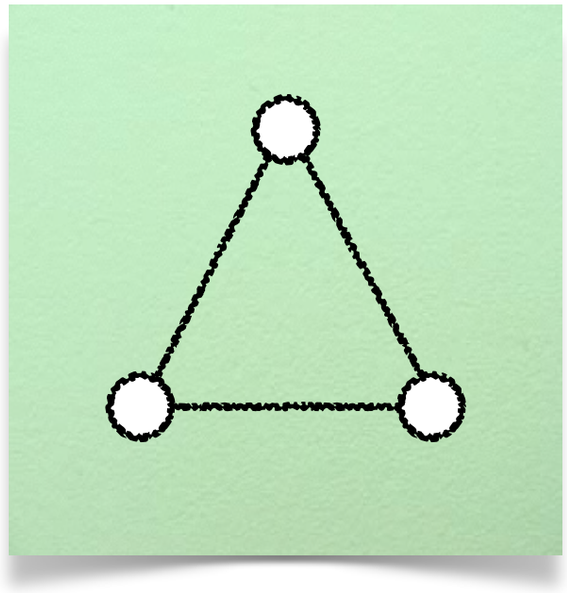

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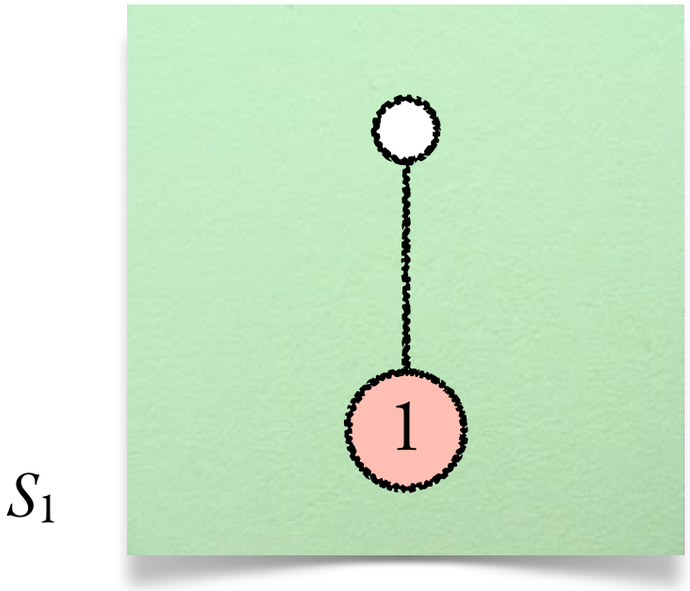

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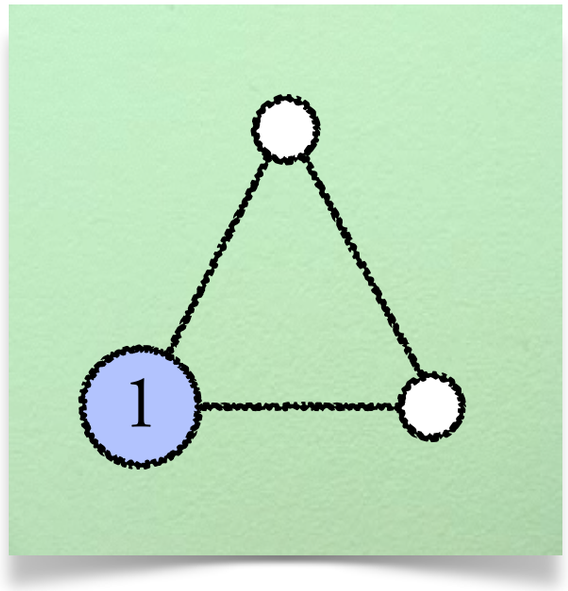

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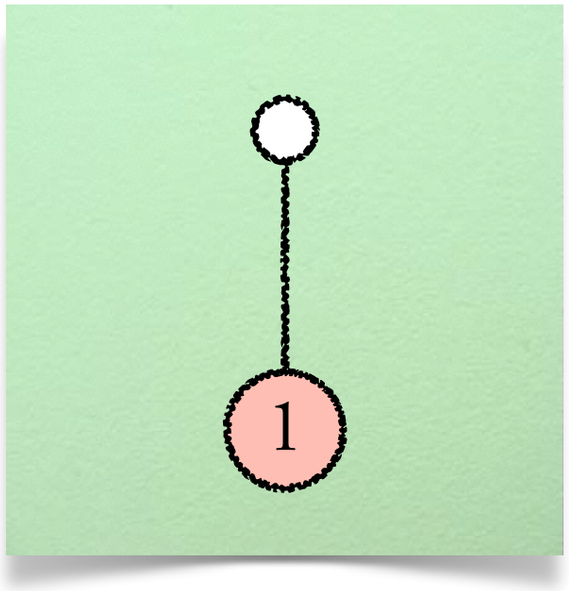

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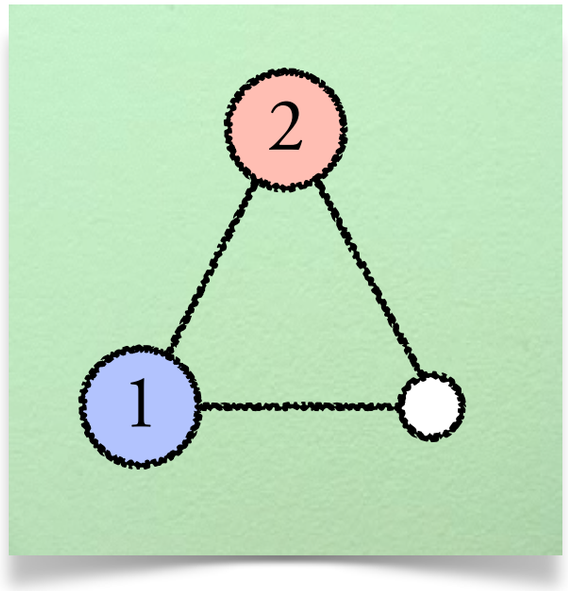

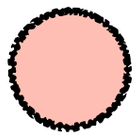

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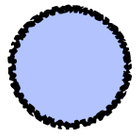

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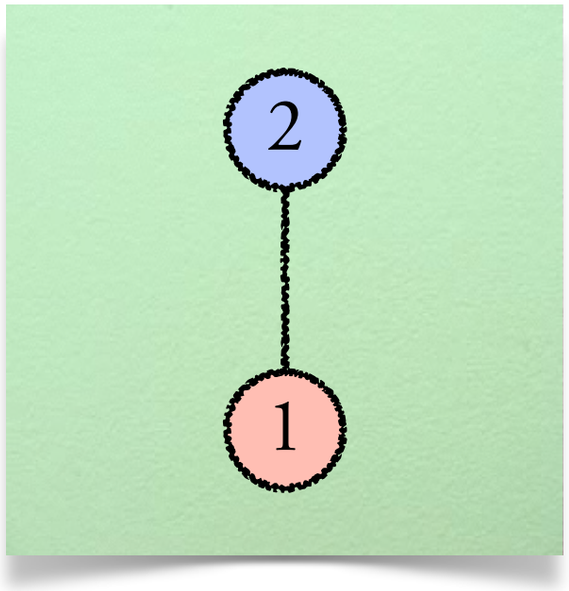

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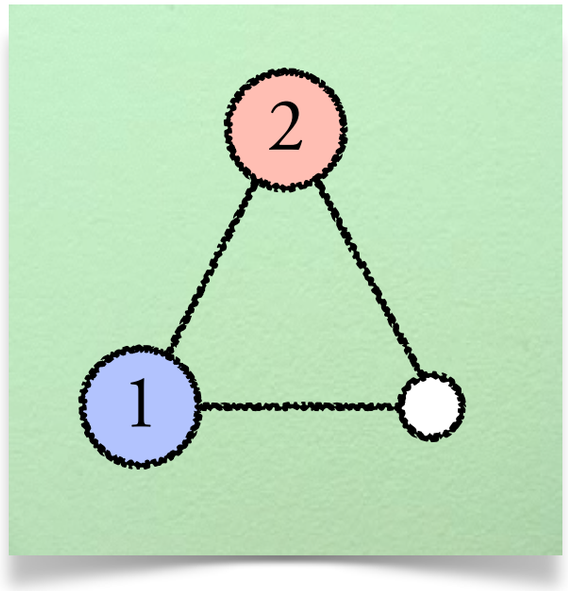

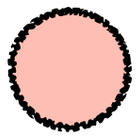
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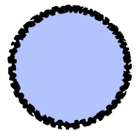
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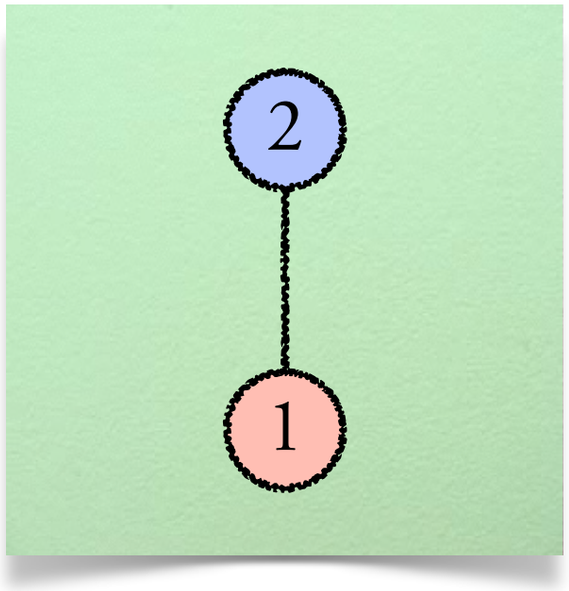

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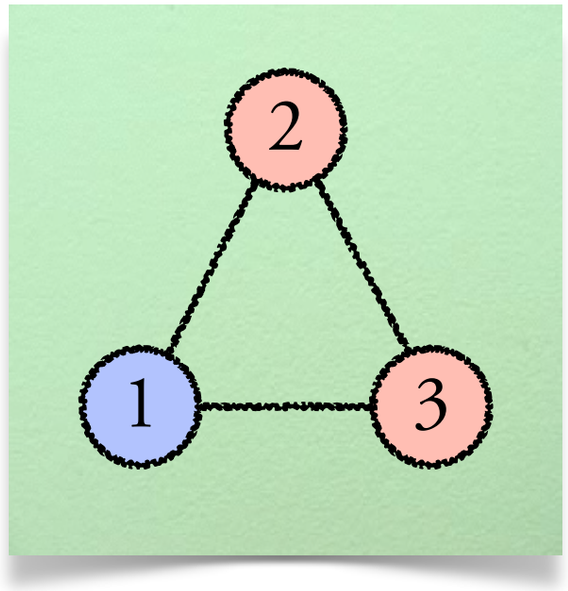

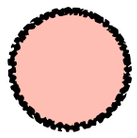
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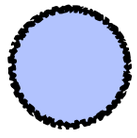
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Question: Can Spoiler win in 3 rounds ?

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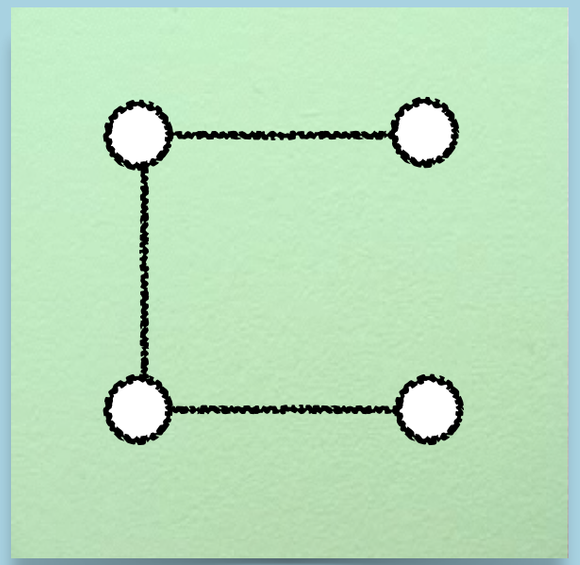

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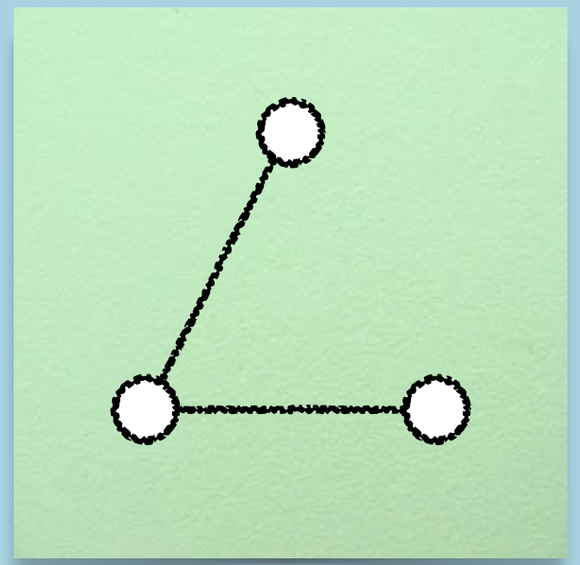

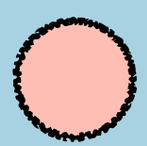 = Spoiler

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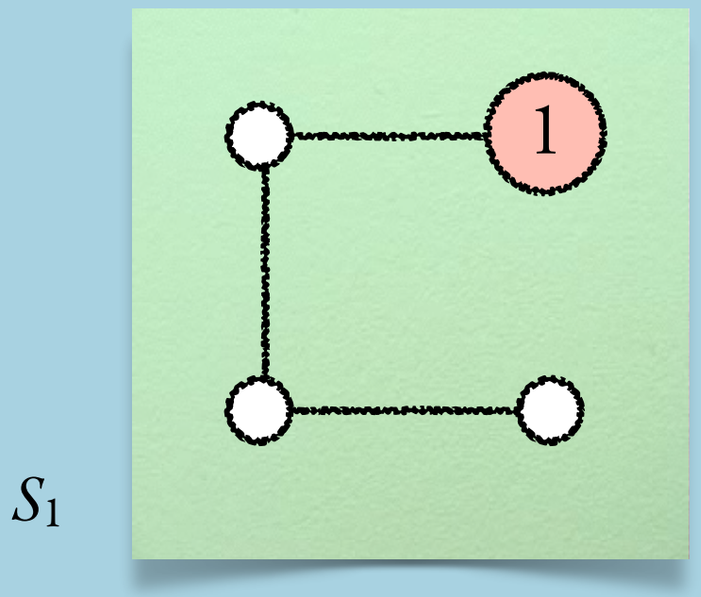

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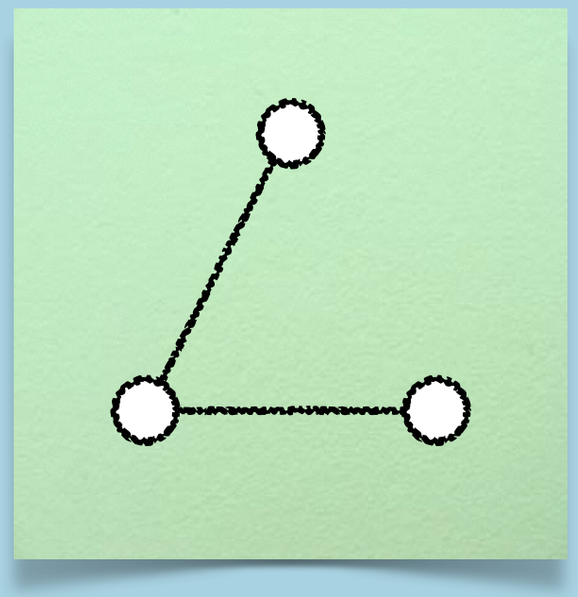

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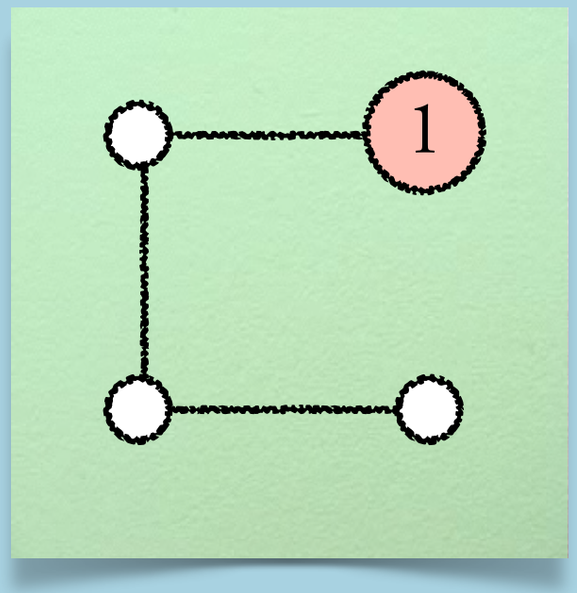

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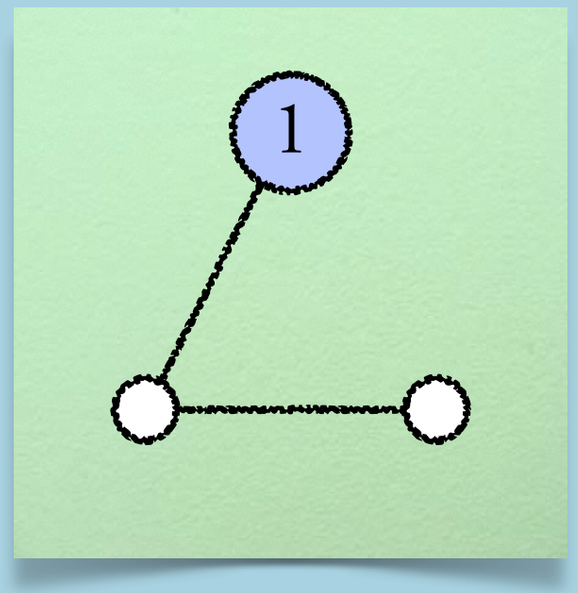

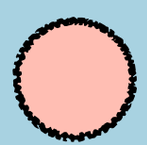

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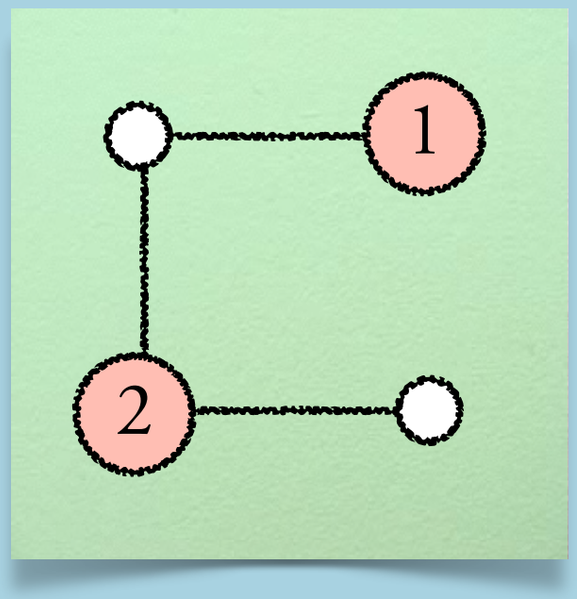

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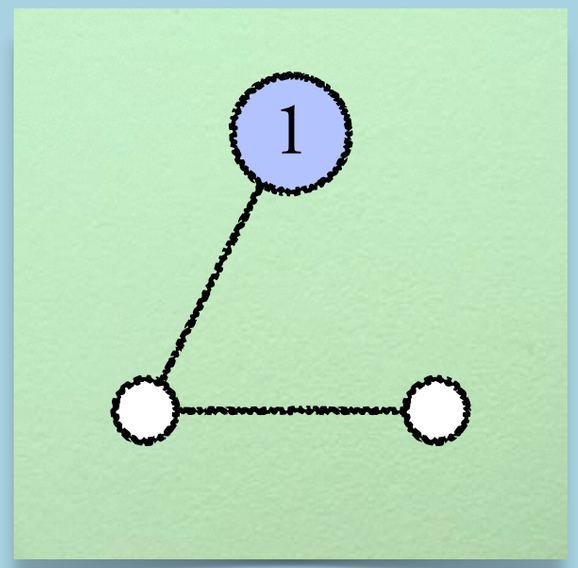

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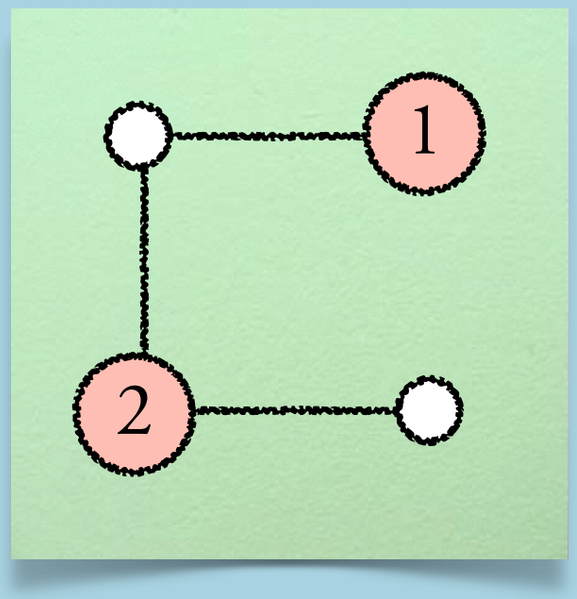

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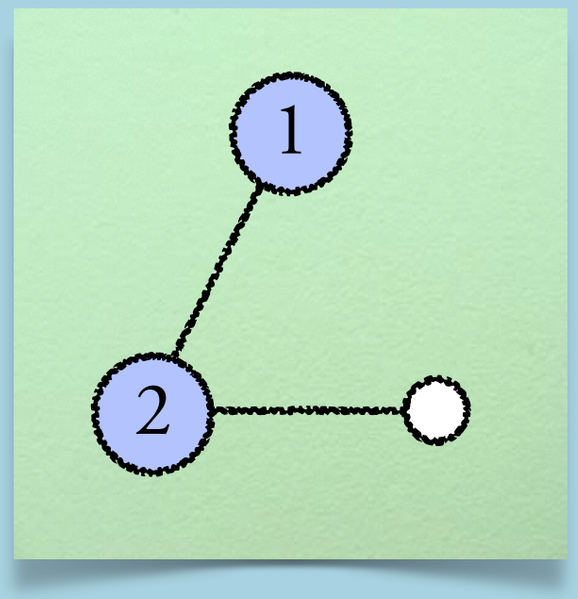

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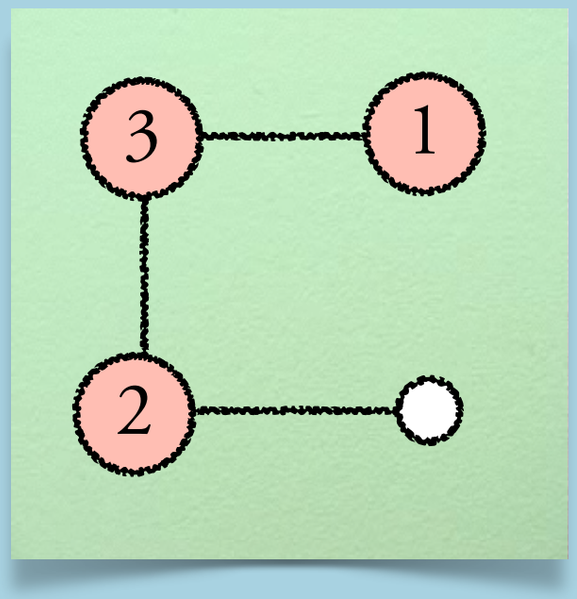

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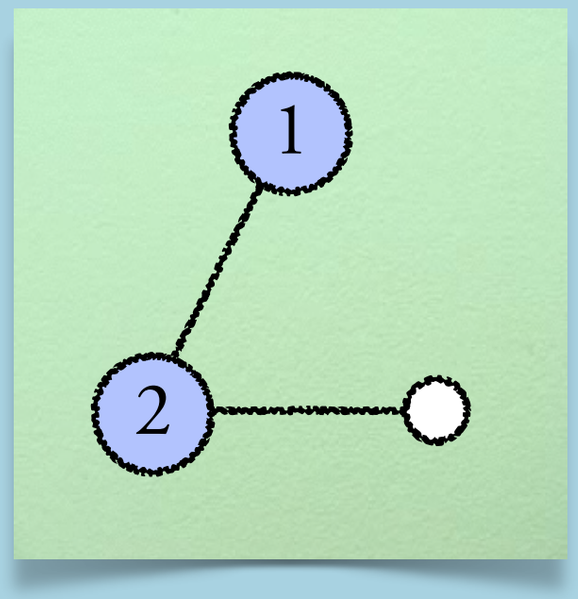

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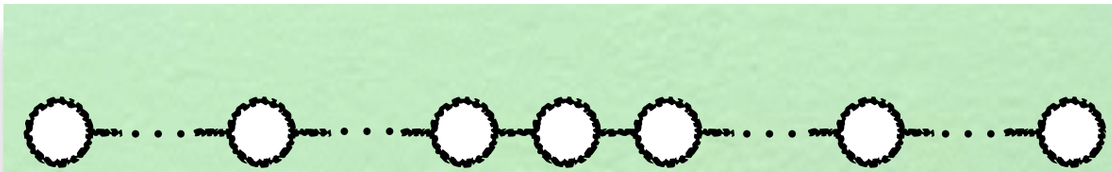

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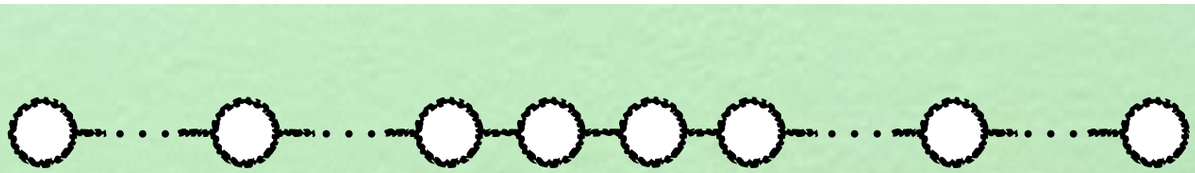

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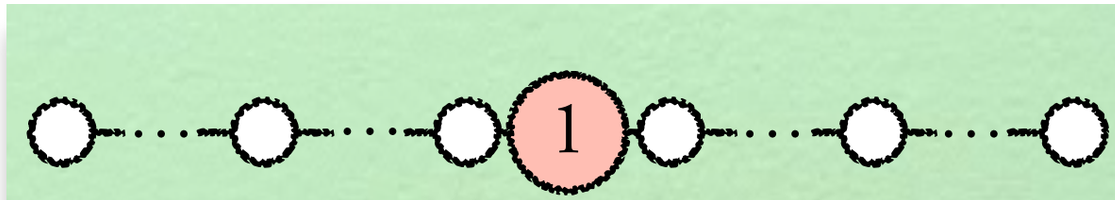

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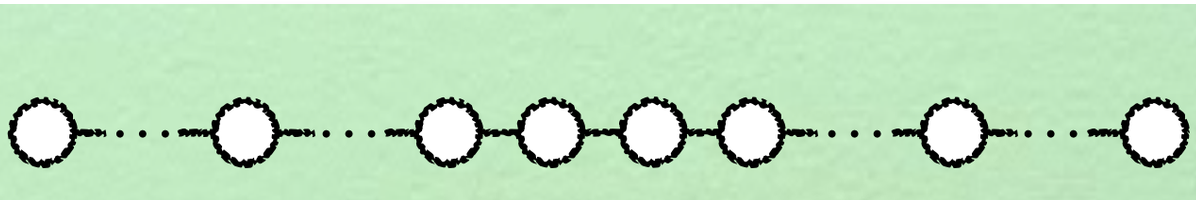

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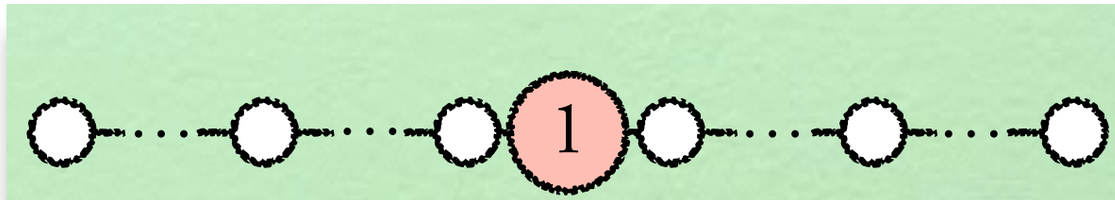

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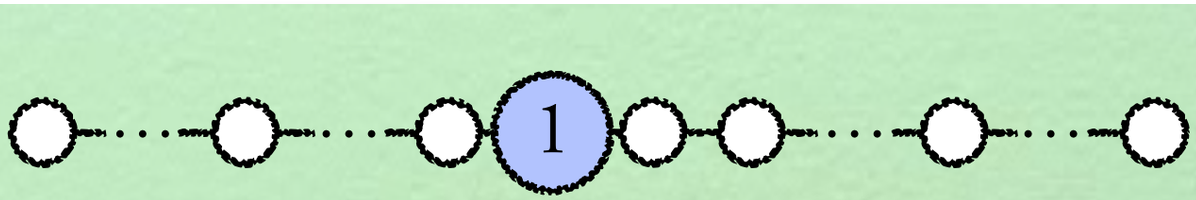

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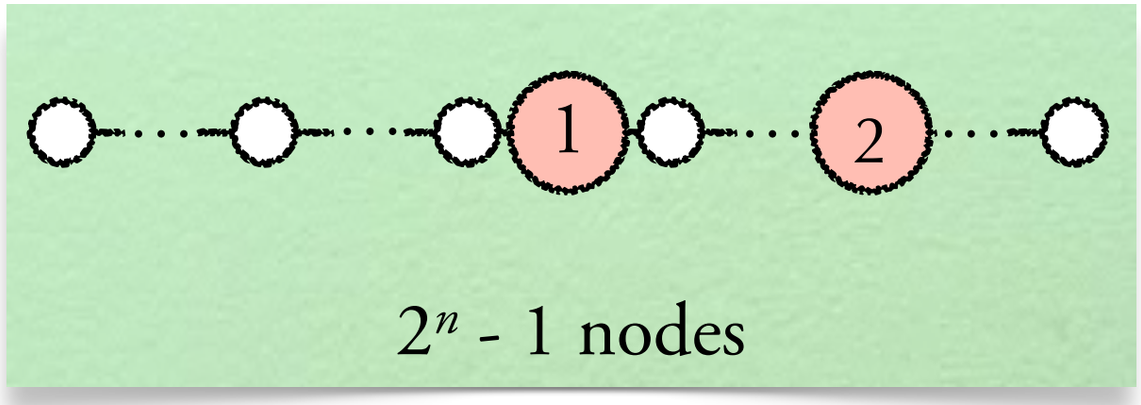

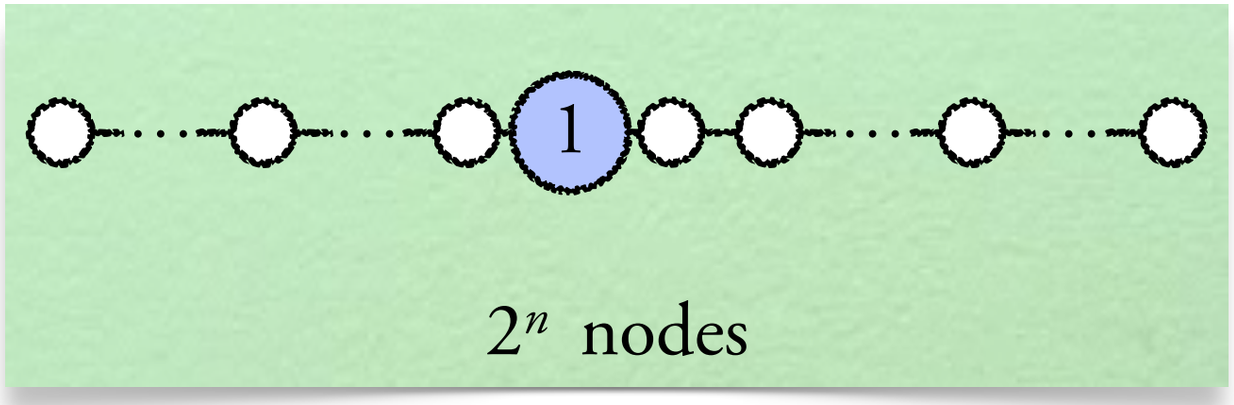

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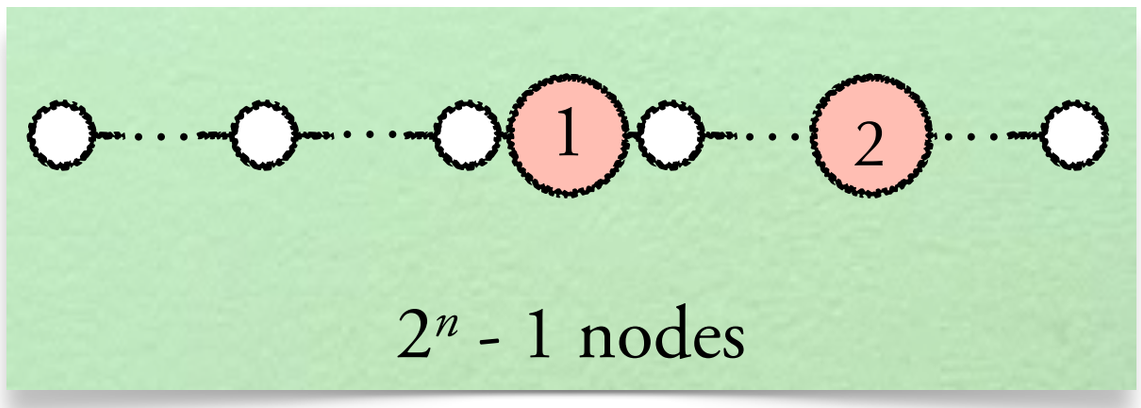

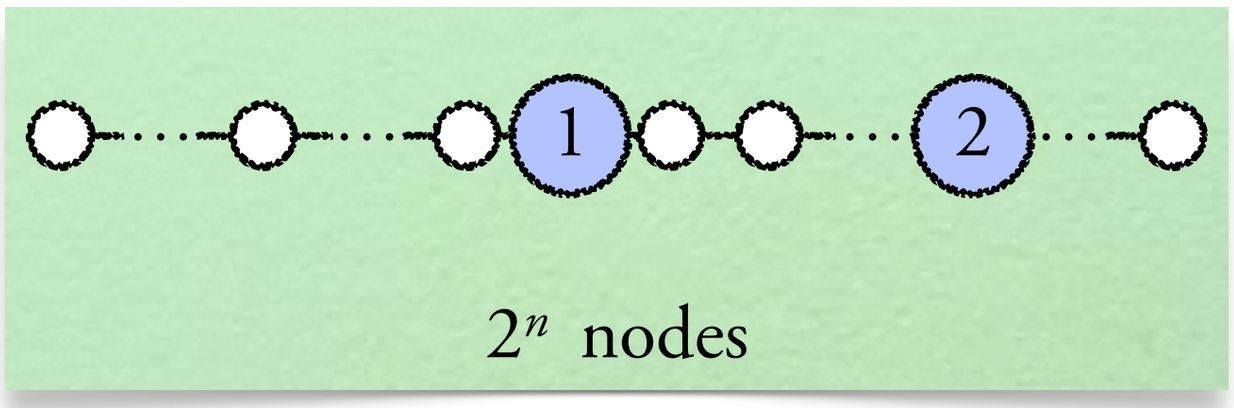

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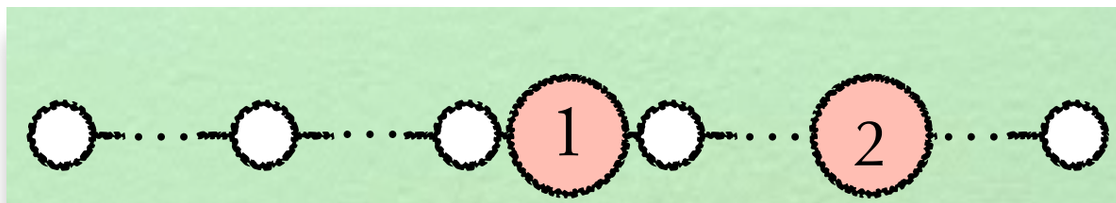

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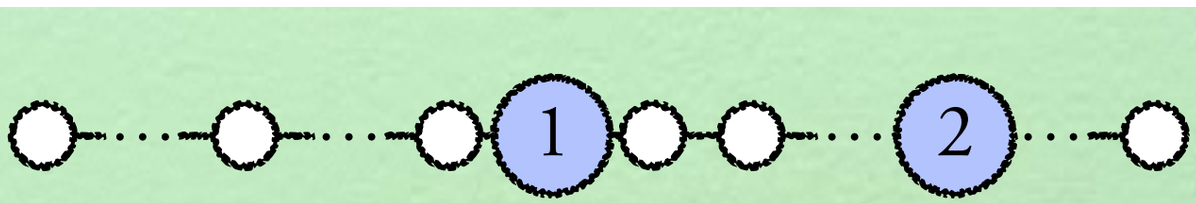

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Any idea?

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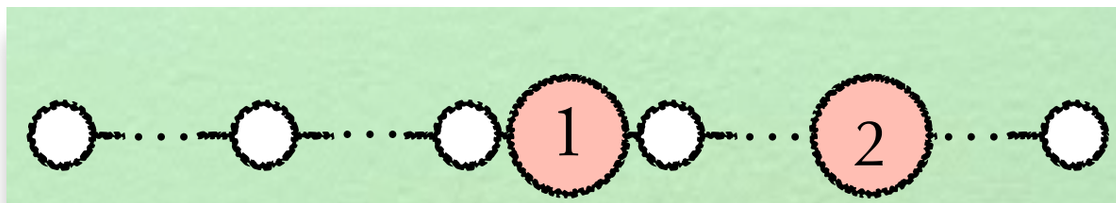

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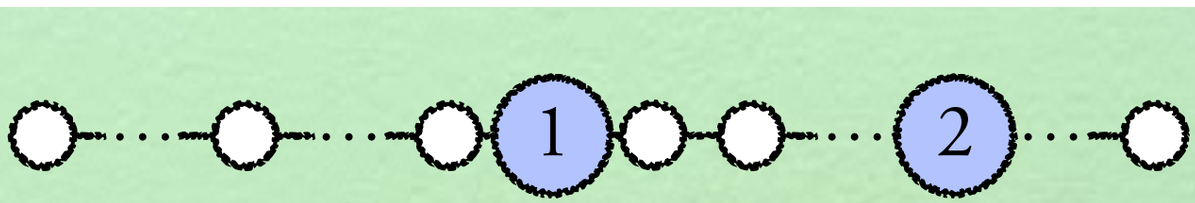

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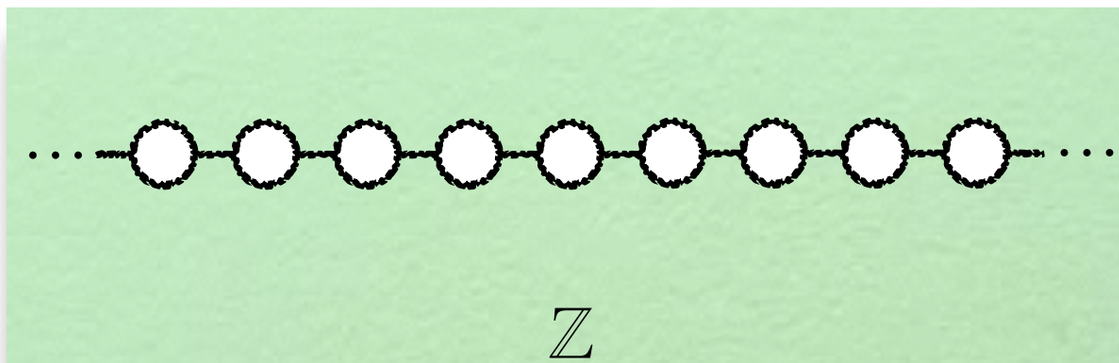

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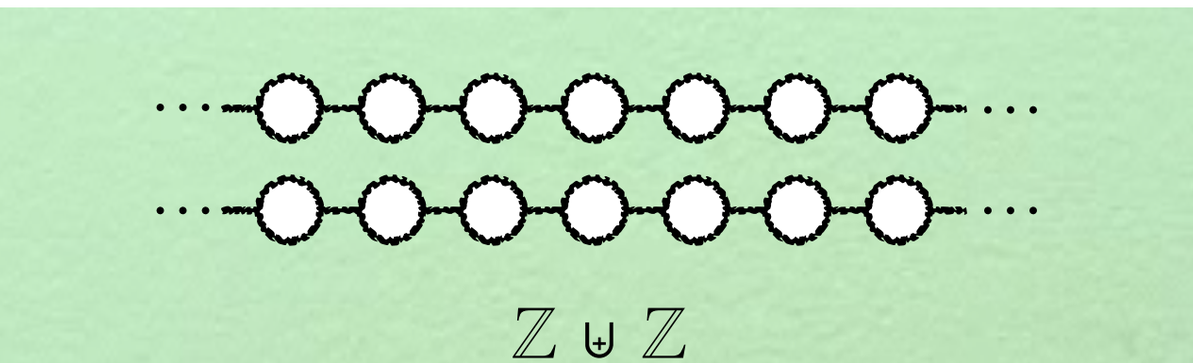

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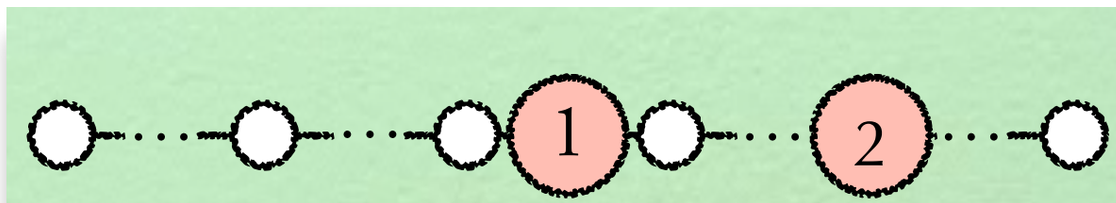

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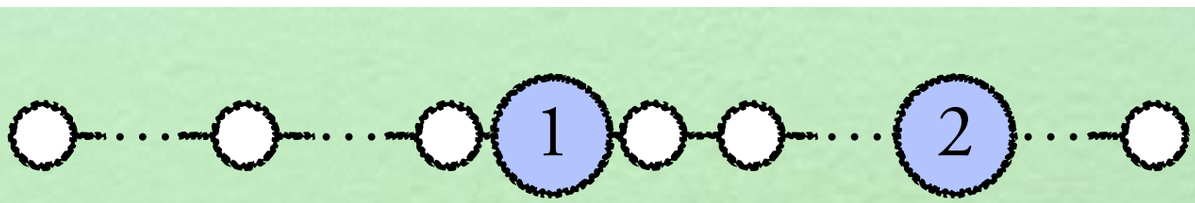

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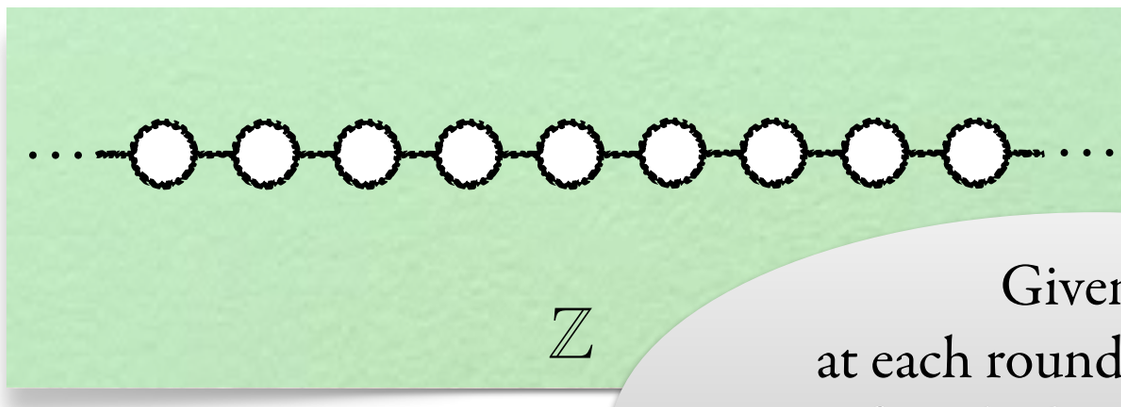

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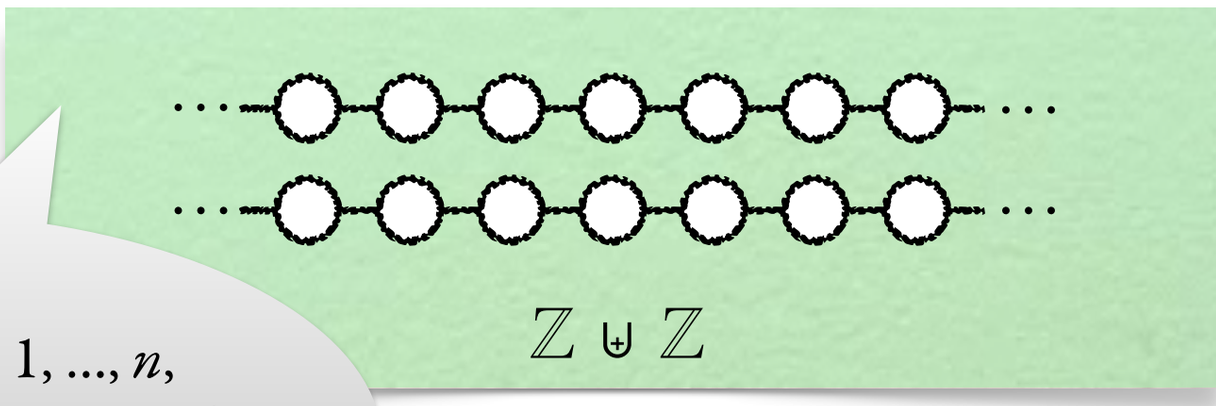

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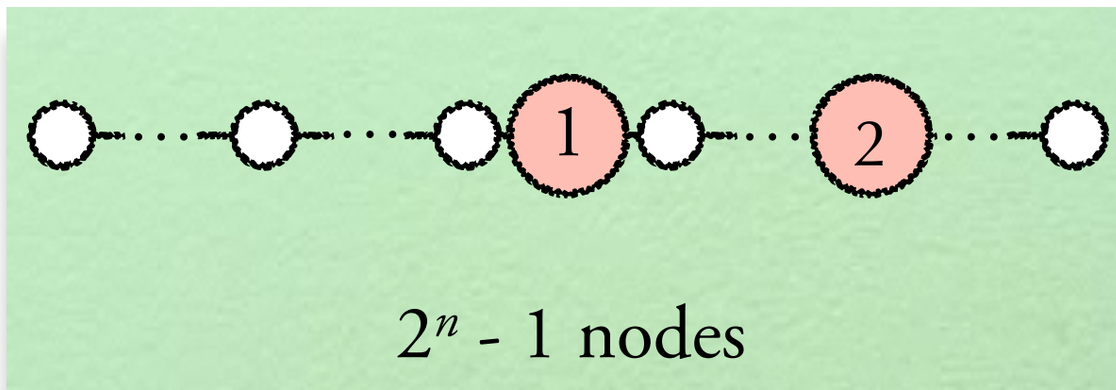

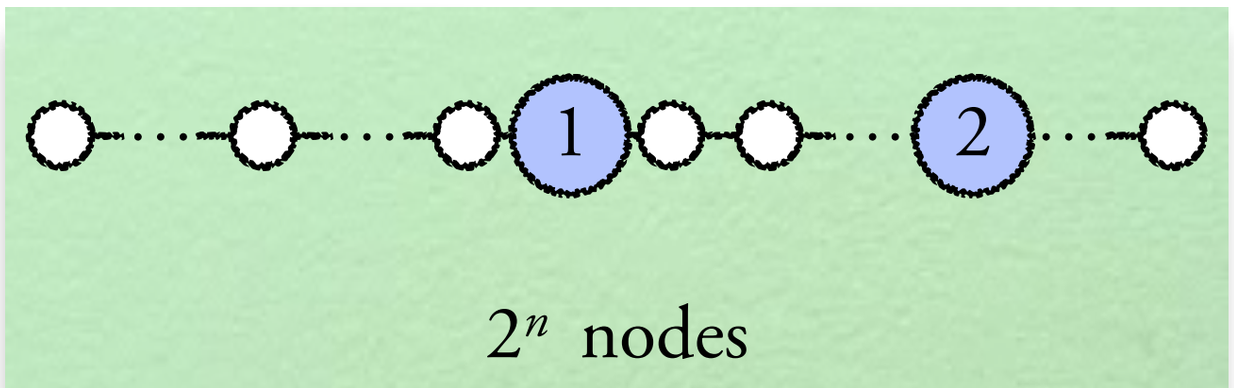

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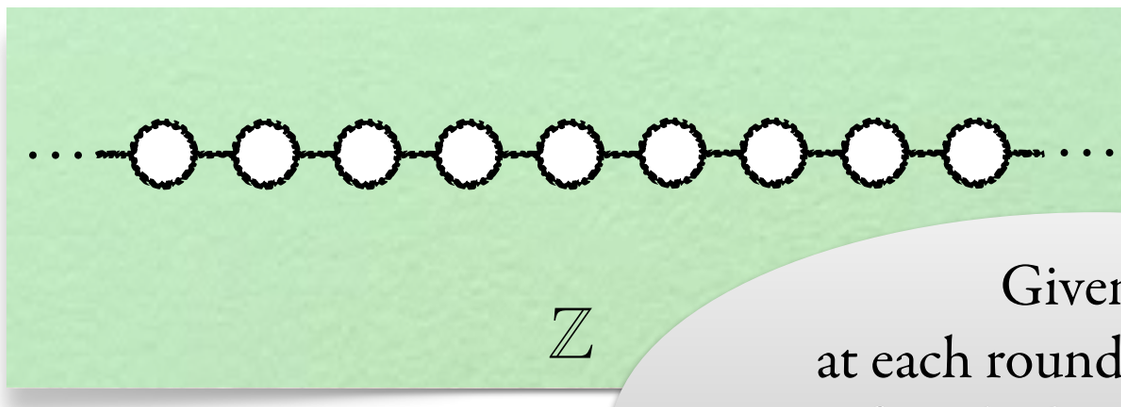

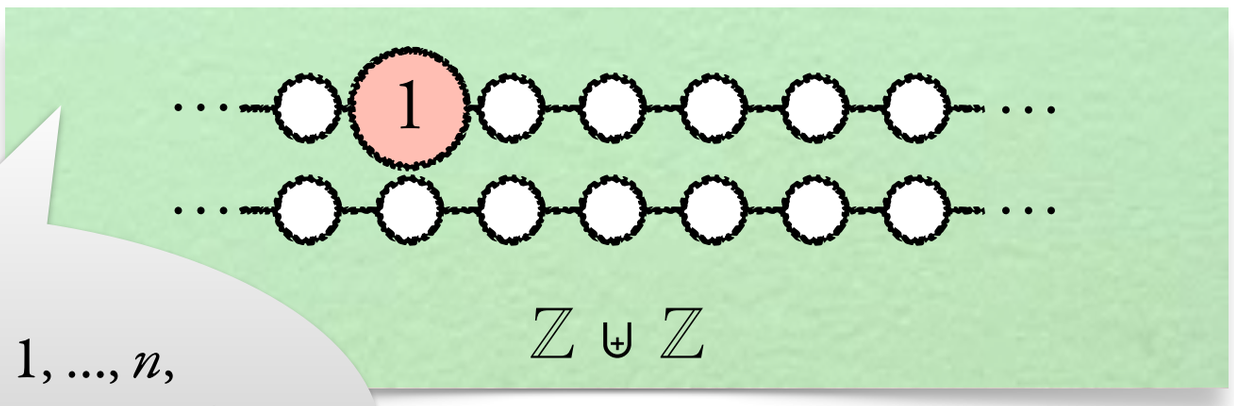

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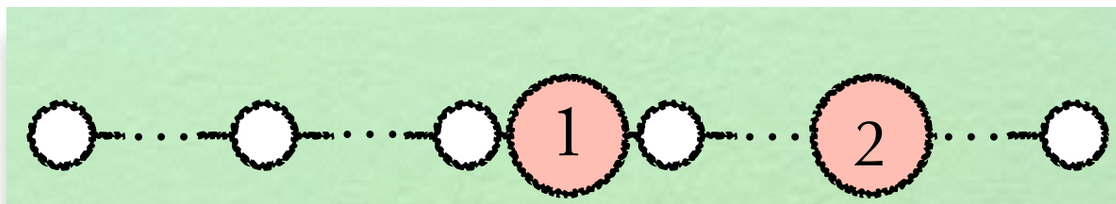

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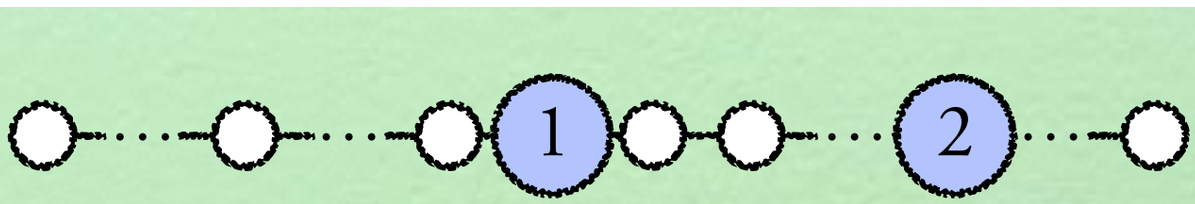

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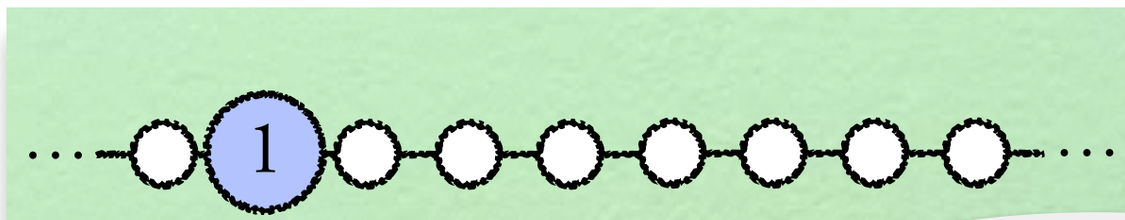

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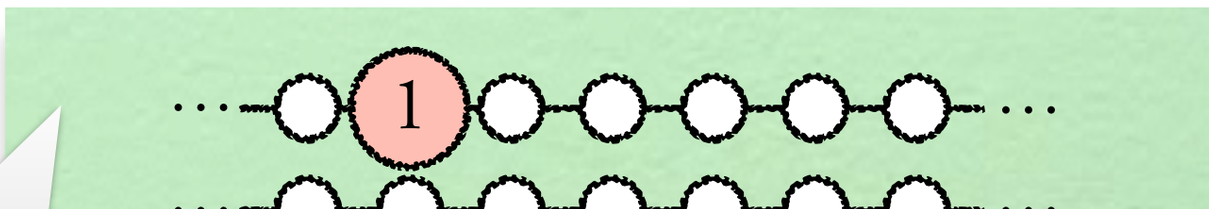

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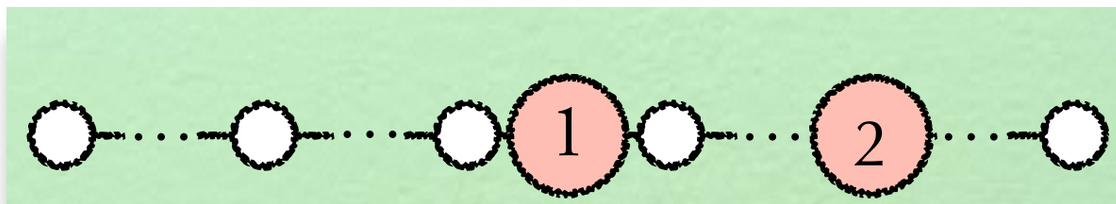

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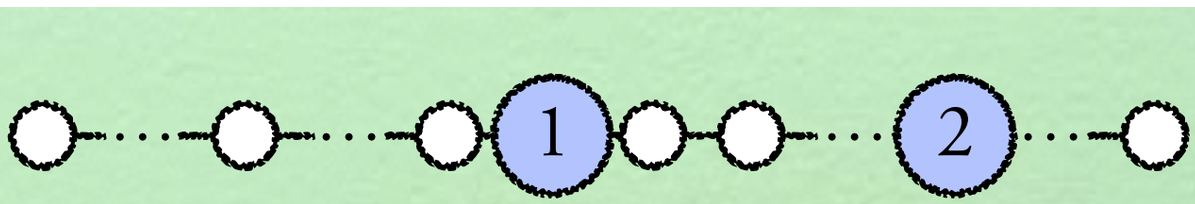

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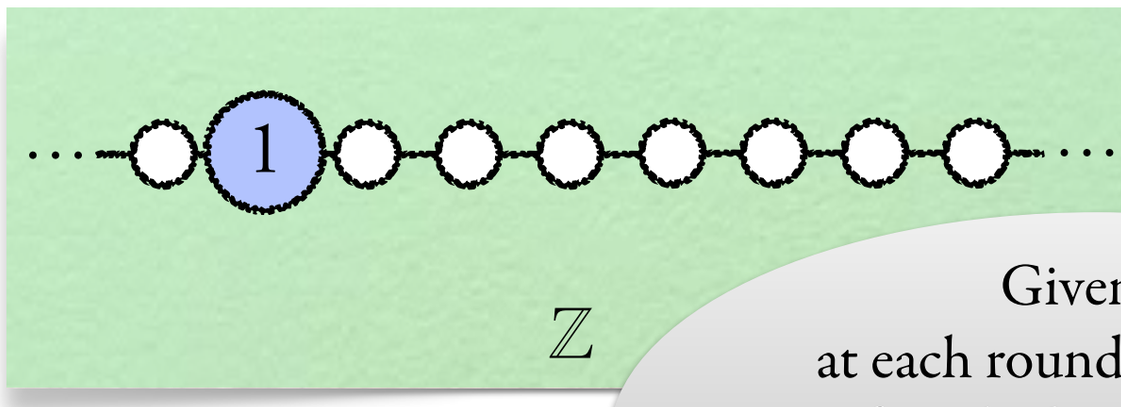

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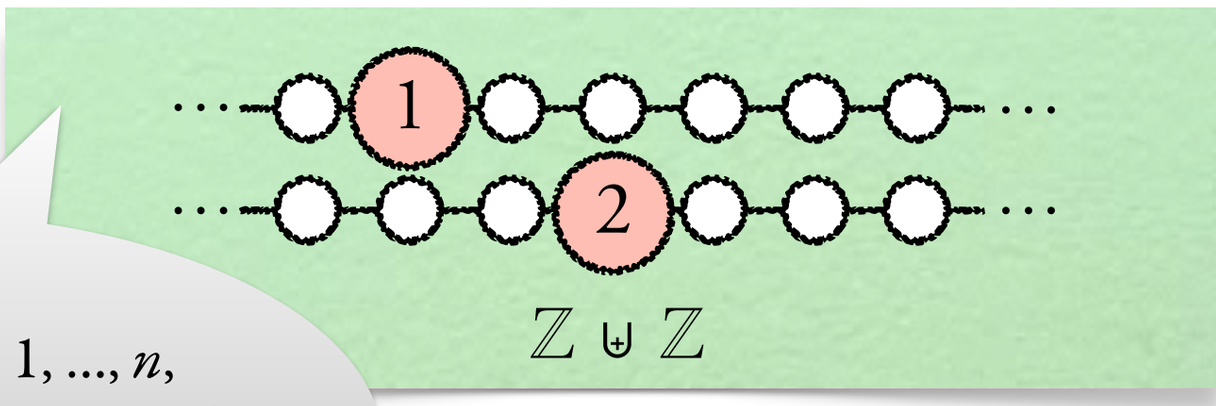

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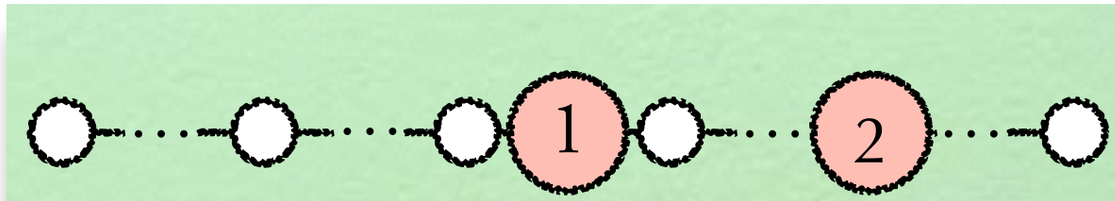

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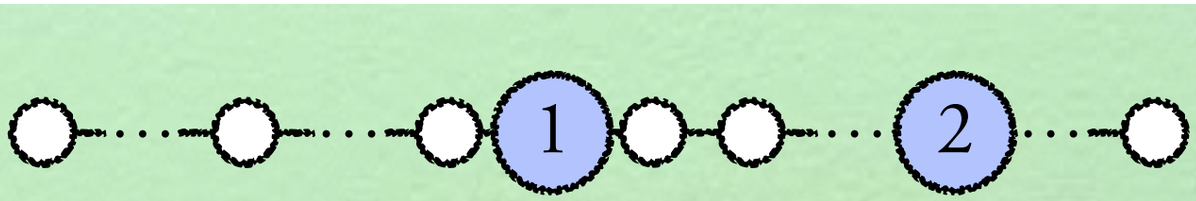

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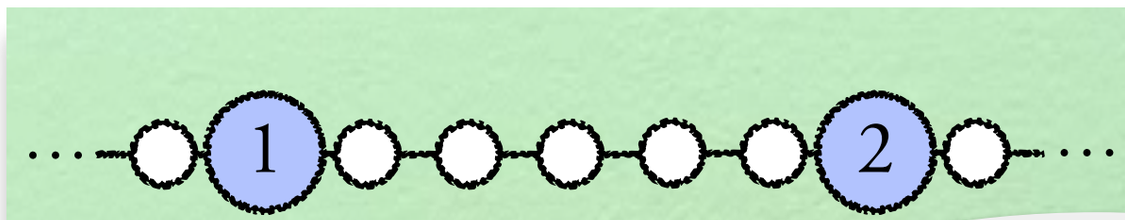

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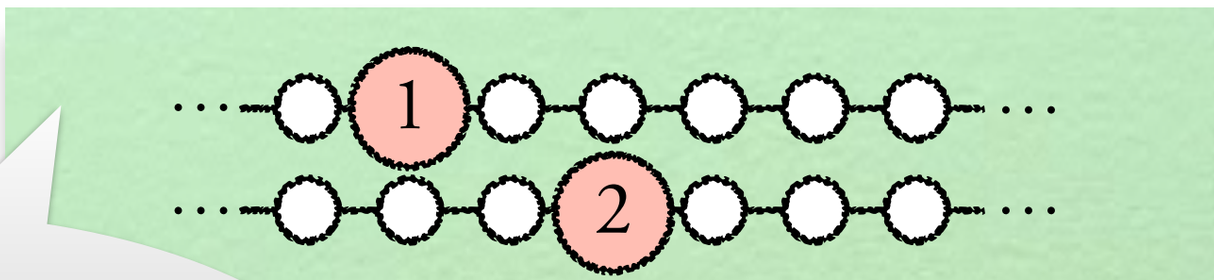

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Proof: Suppose that **P** is defined by an FO sentence ϕ and ❄ holds.

From ❄ and the above theorem: $\forall n \; \exists S_1 \in \mathbf{P} \; \exists S_2 \notin \mathbf{P}$. S_1, S_2 are n-equivalent.

In particular, when n = *quantifier rank of* ϕ, $S_1 \models \phi$ iff $S_2 \models \phi$.

A contradiction!

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A new game to evaluate formulas….

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Lemma. $S \models \phi$ iff **True** wins the semantics game.

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Theorem. S_1 and S_2 are *n-equivalent* [Fraïssé '50, Ehrenfeucht '60]

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Turn winning strategy for True in S_1 into winning strategy for True in S_2 ....

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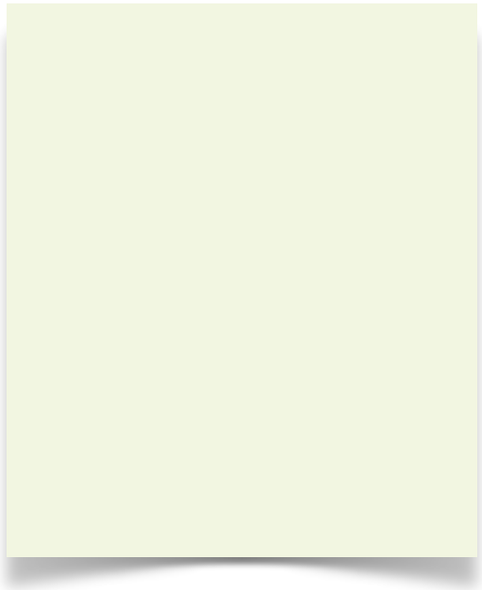

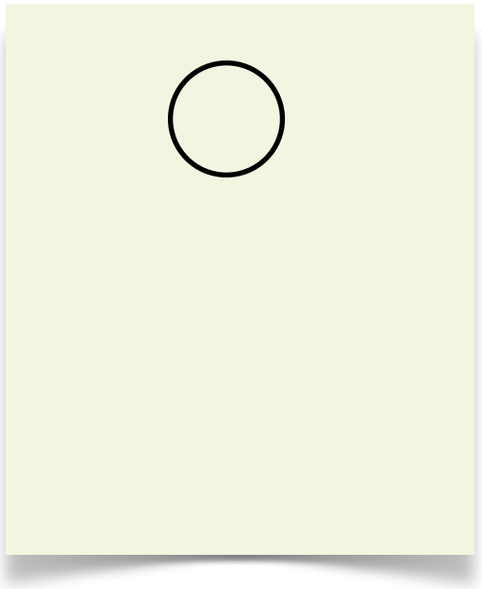

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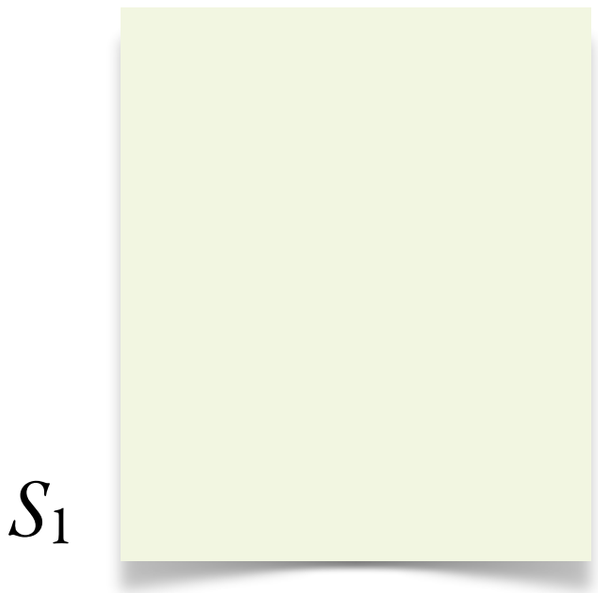

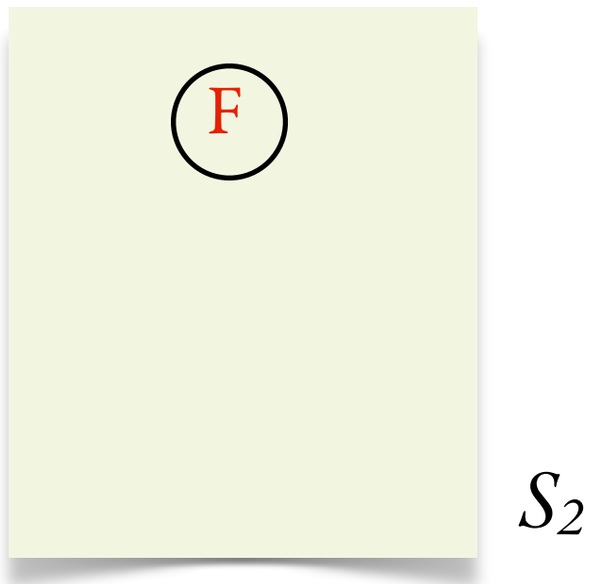

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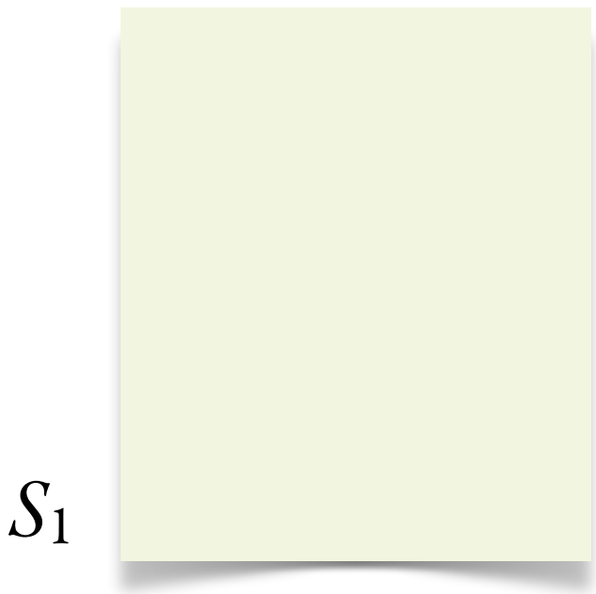

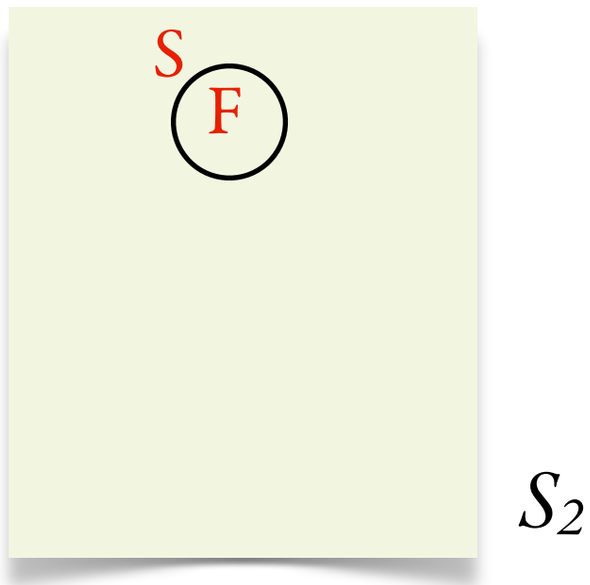

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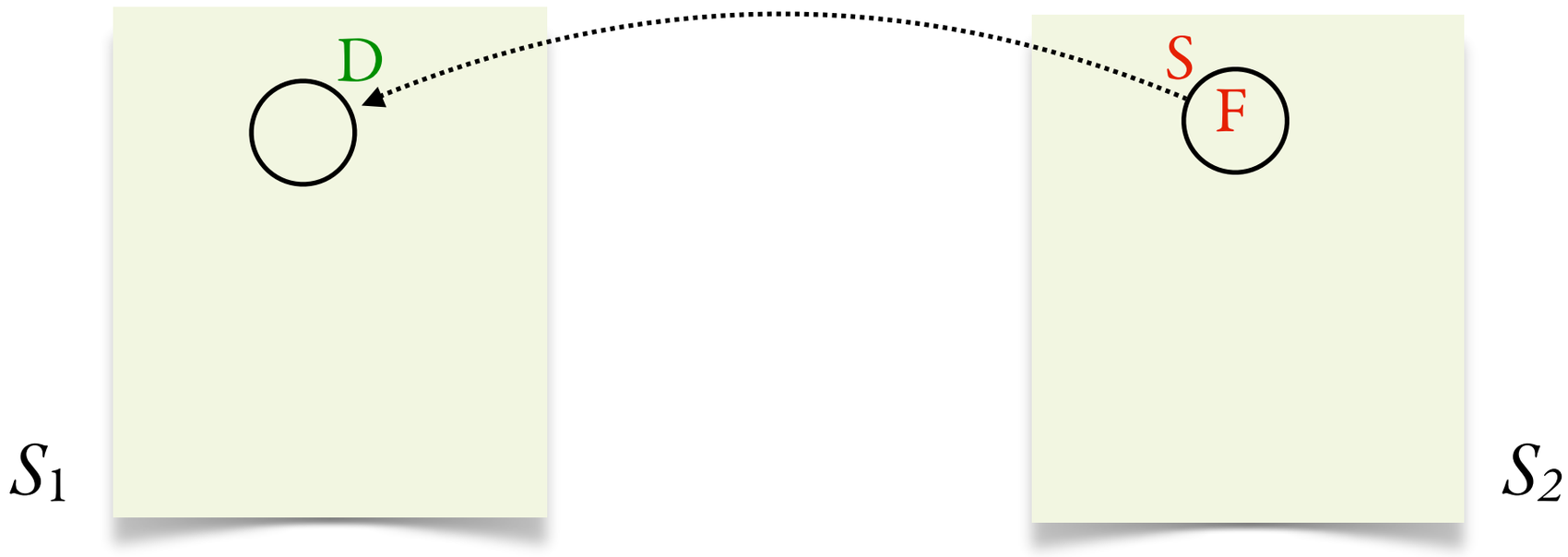

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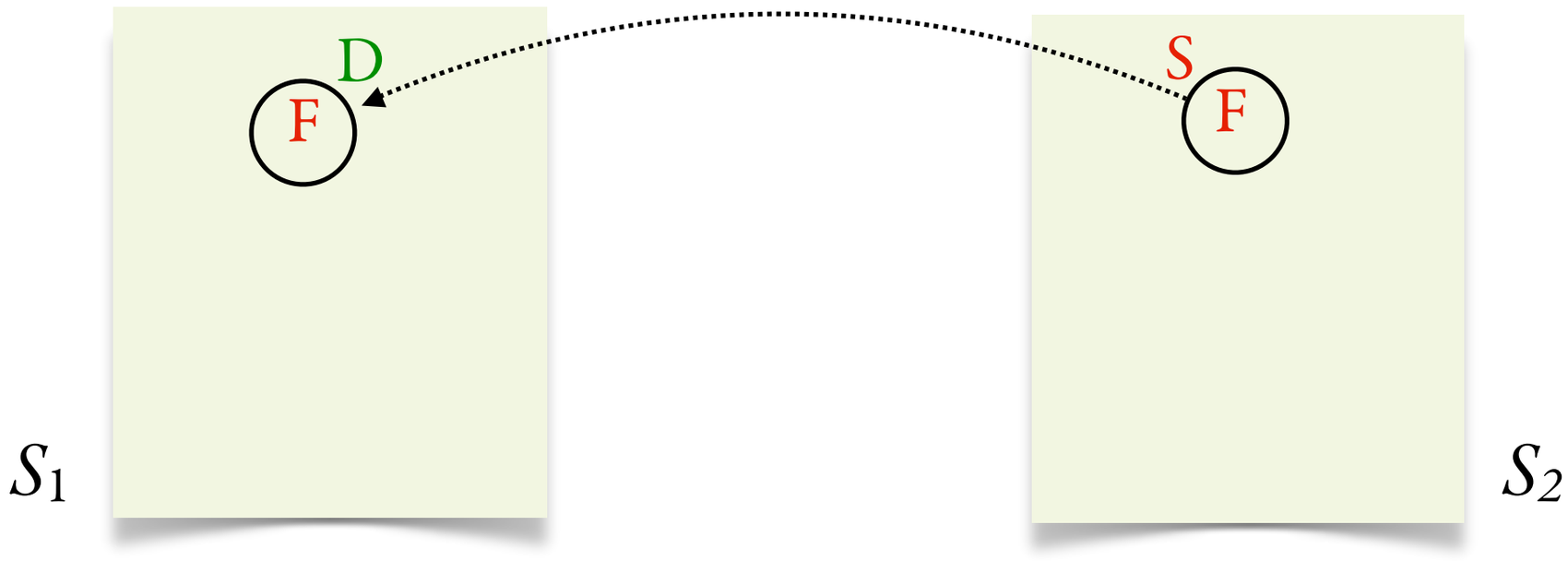

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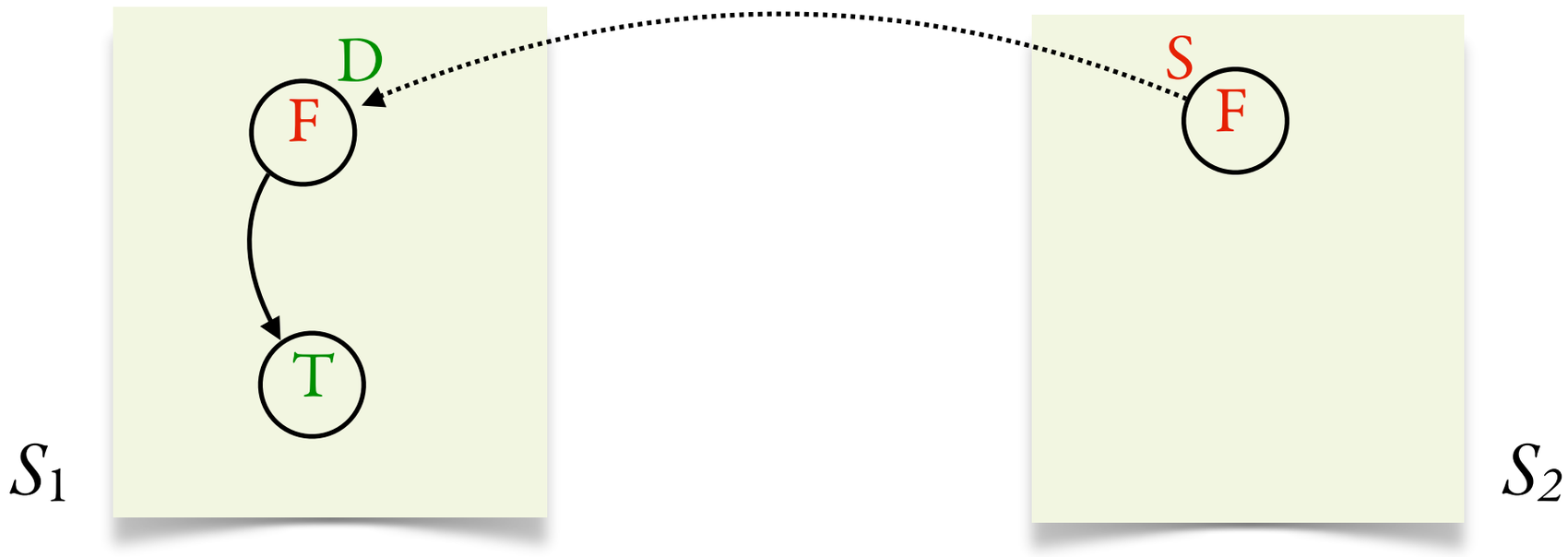

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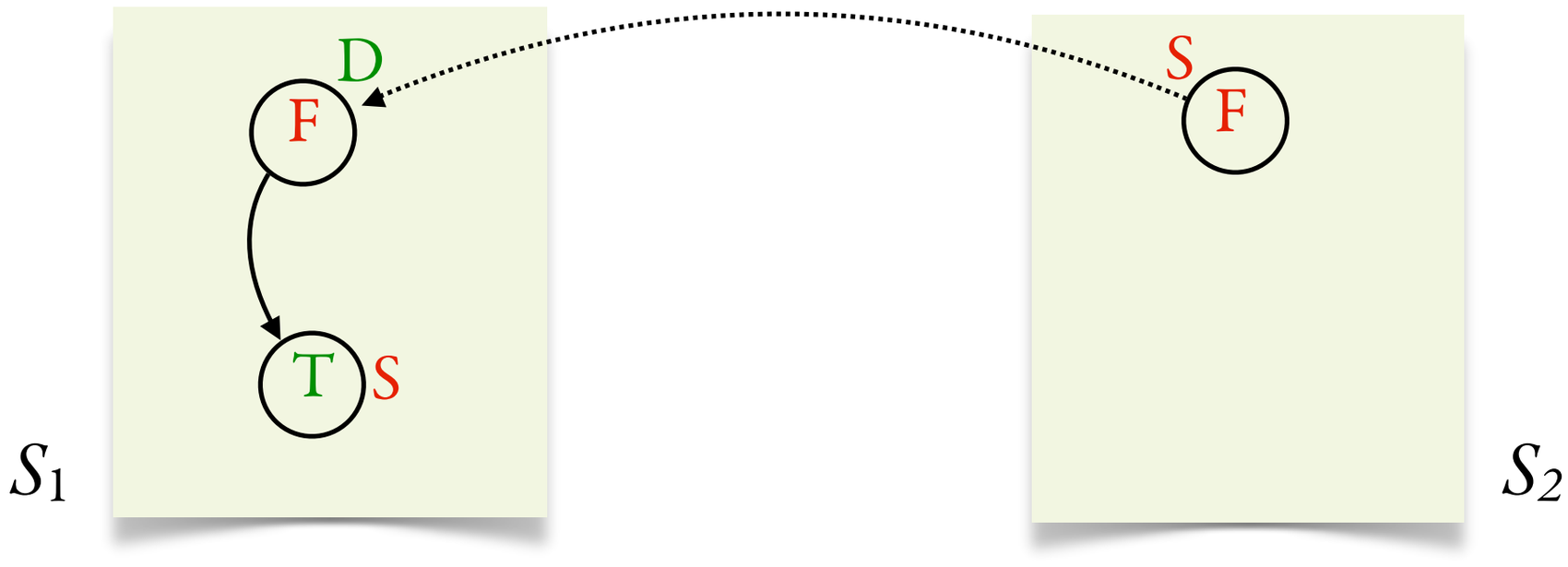

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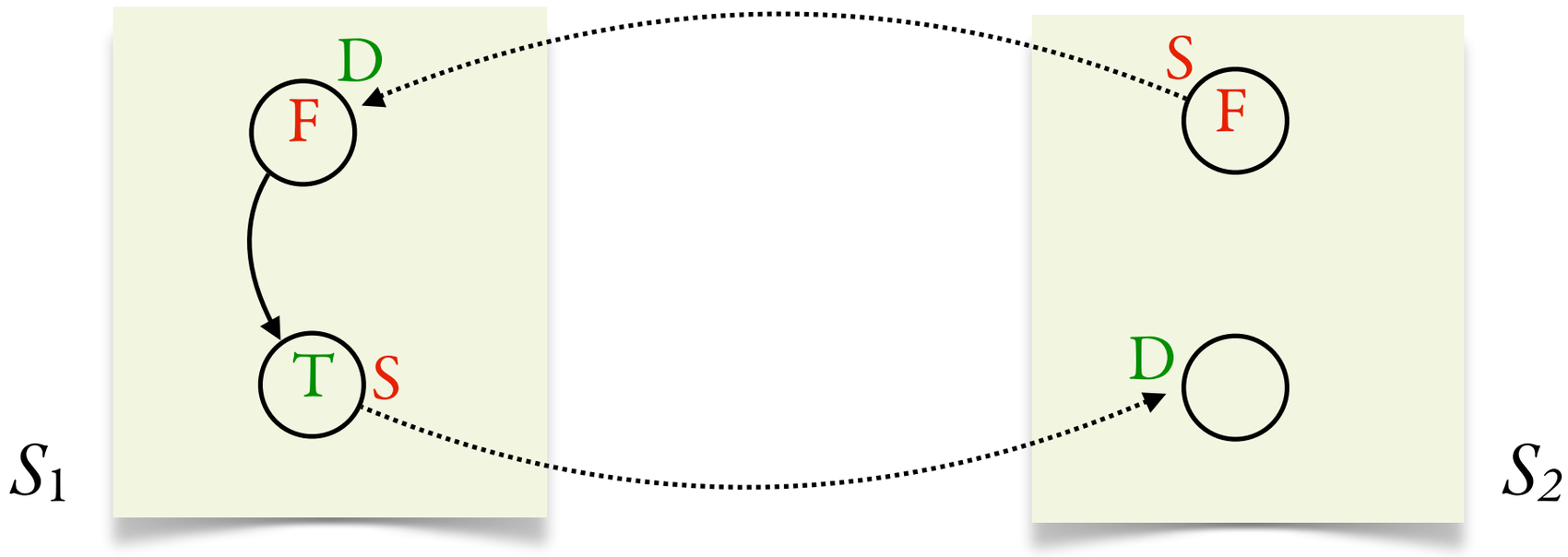

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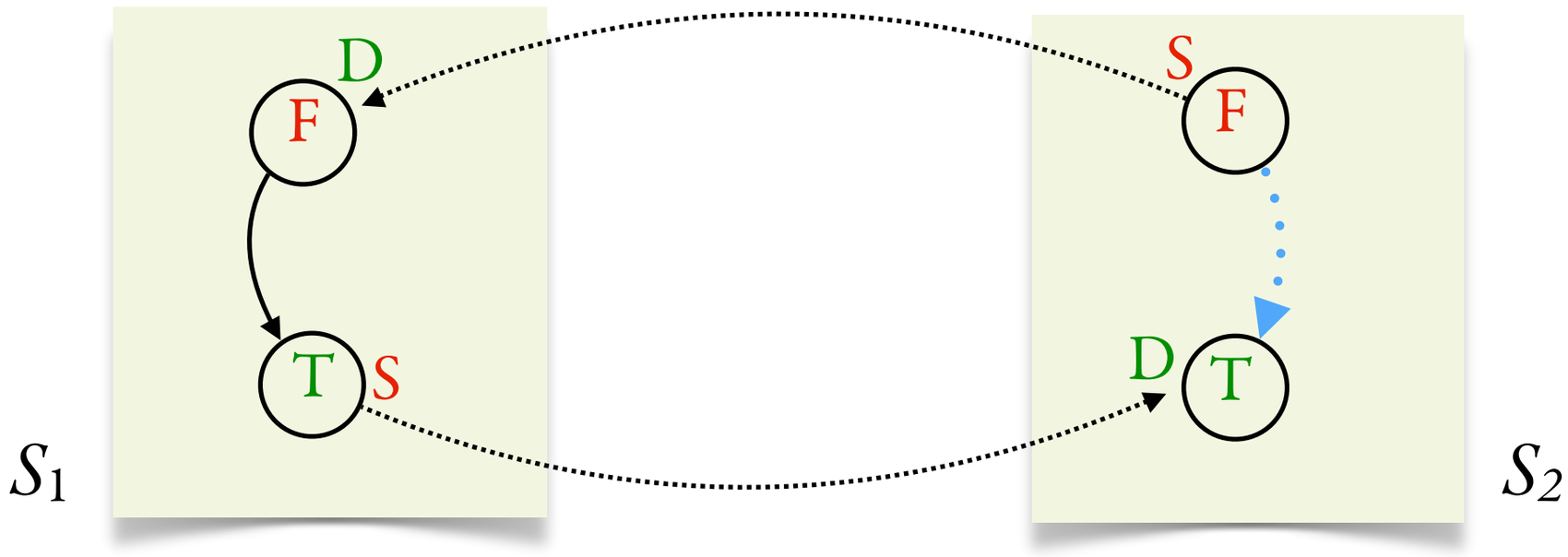

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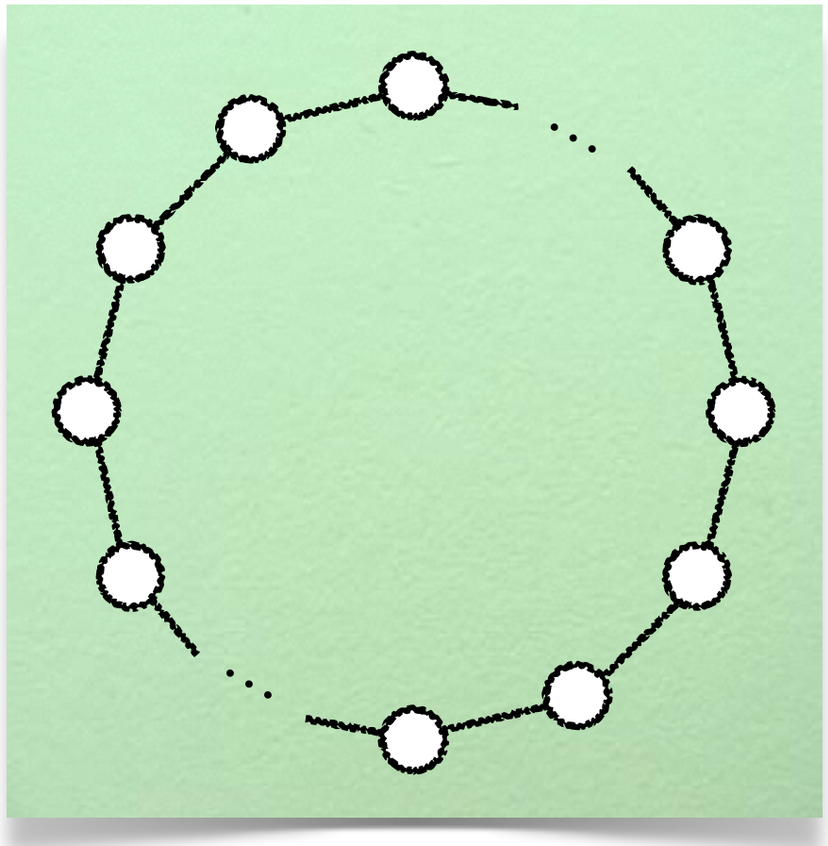

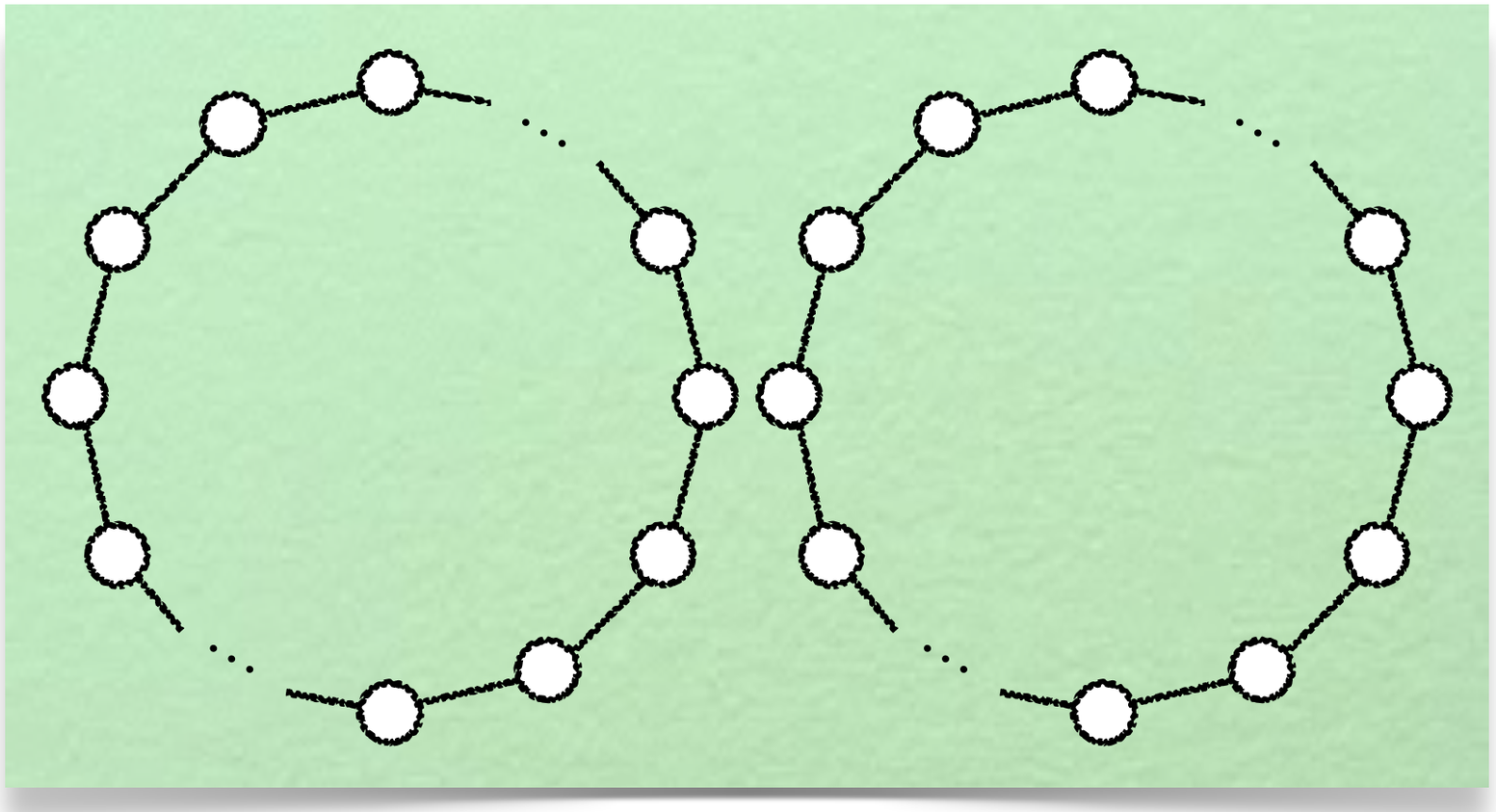

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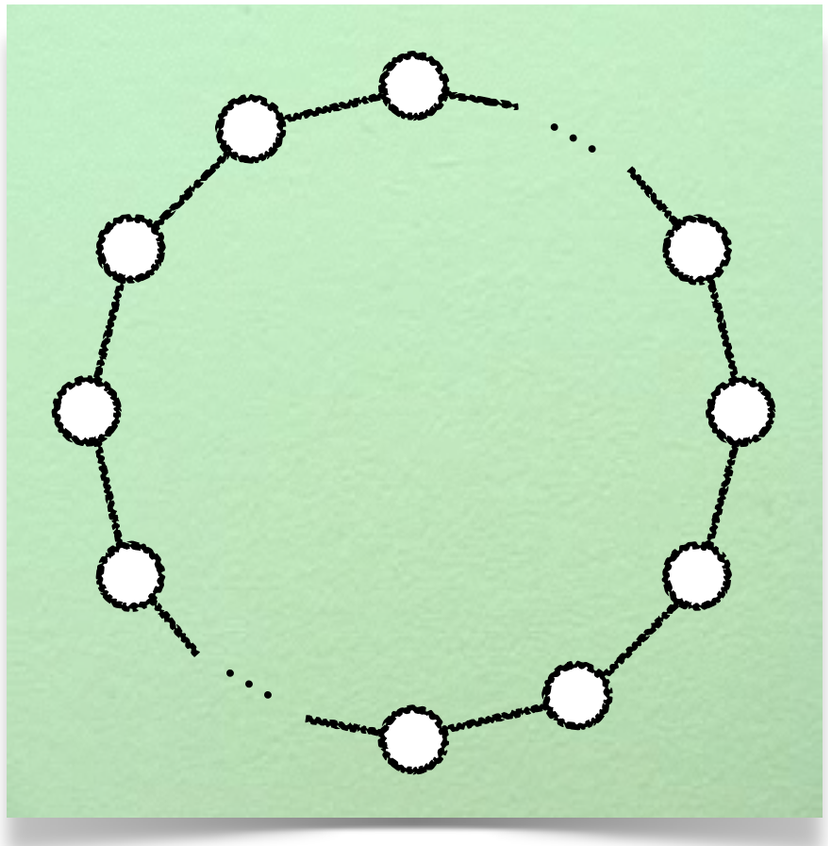

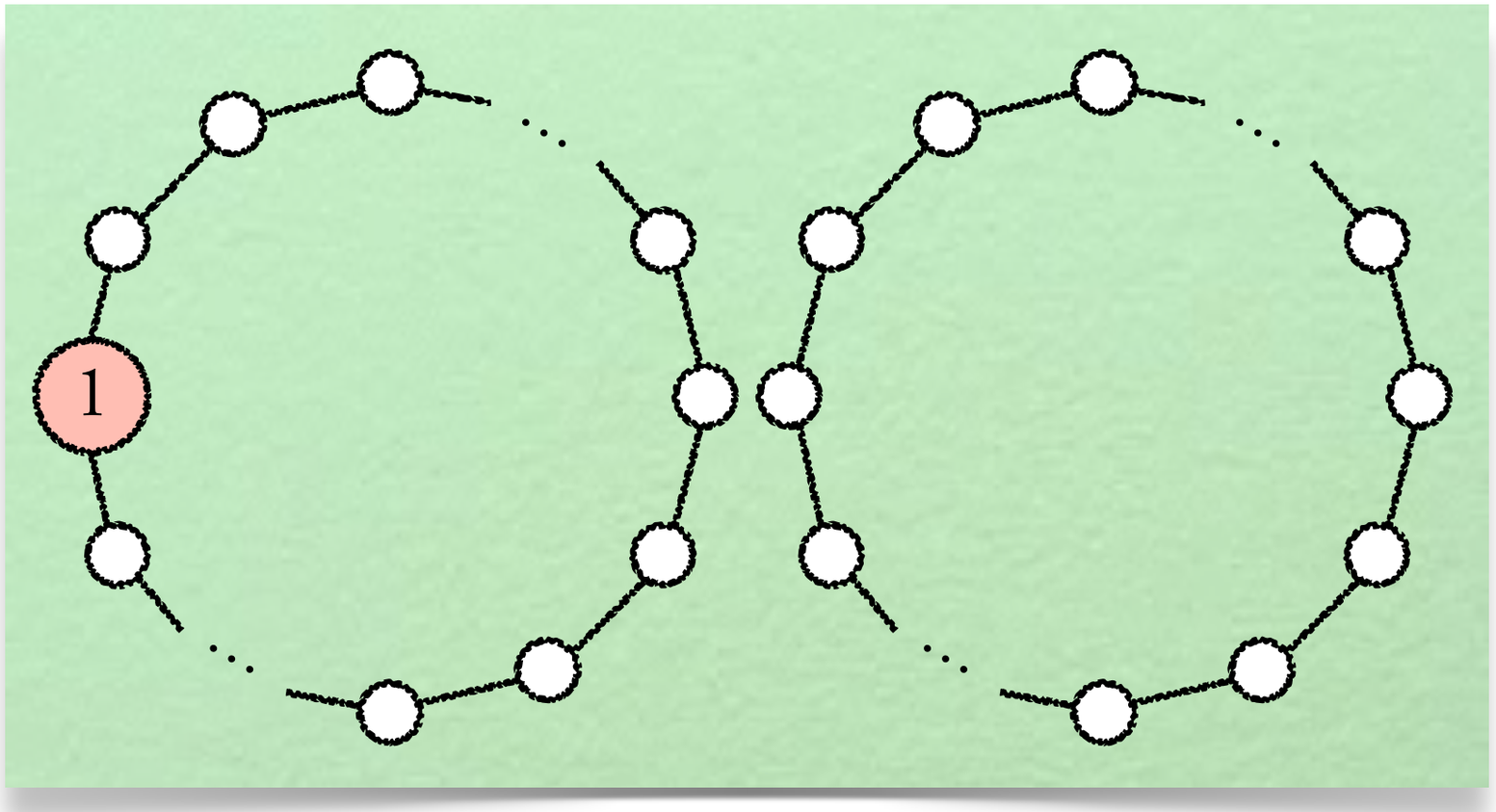

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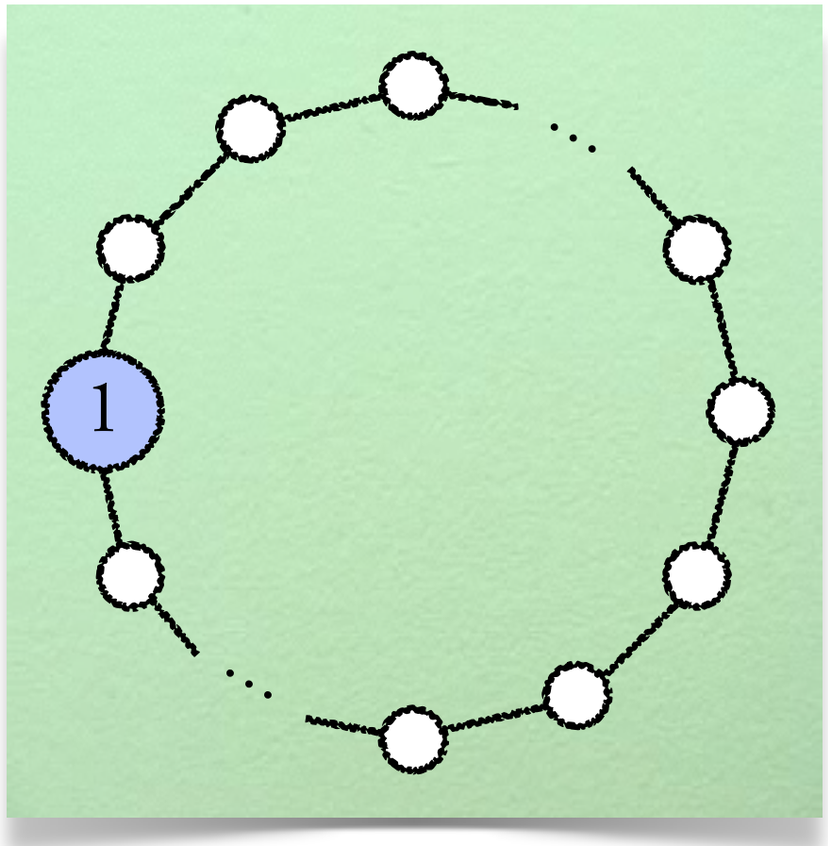

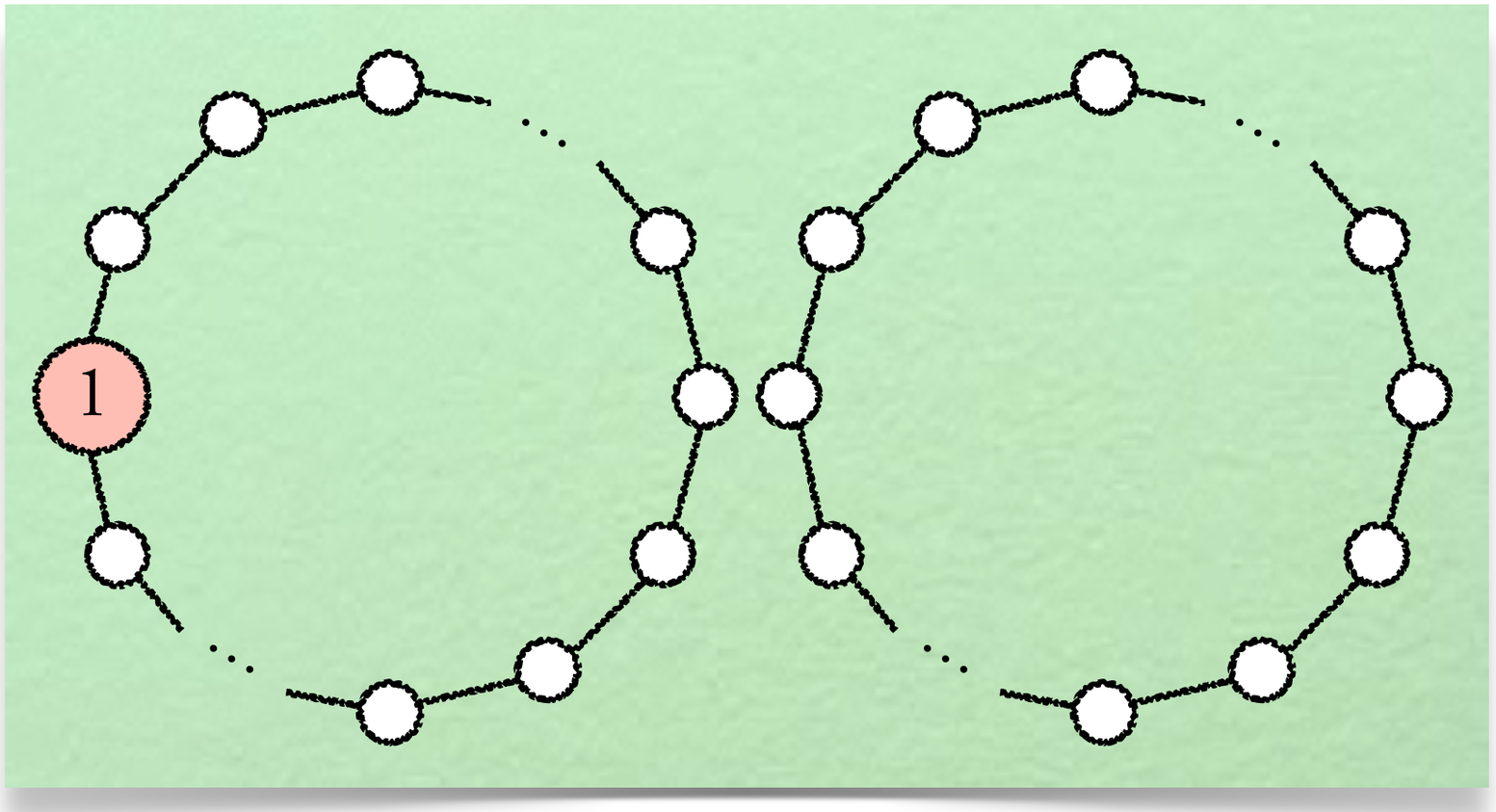

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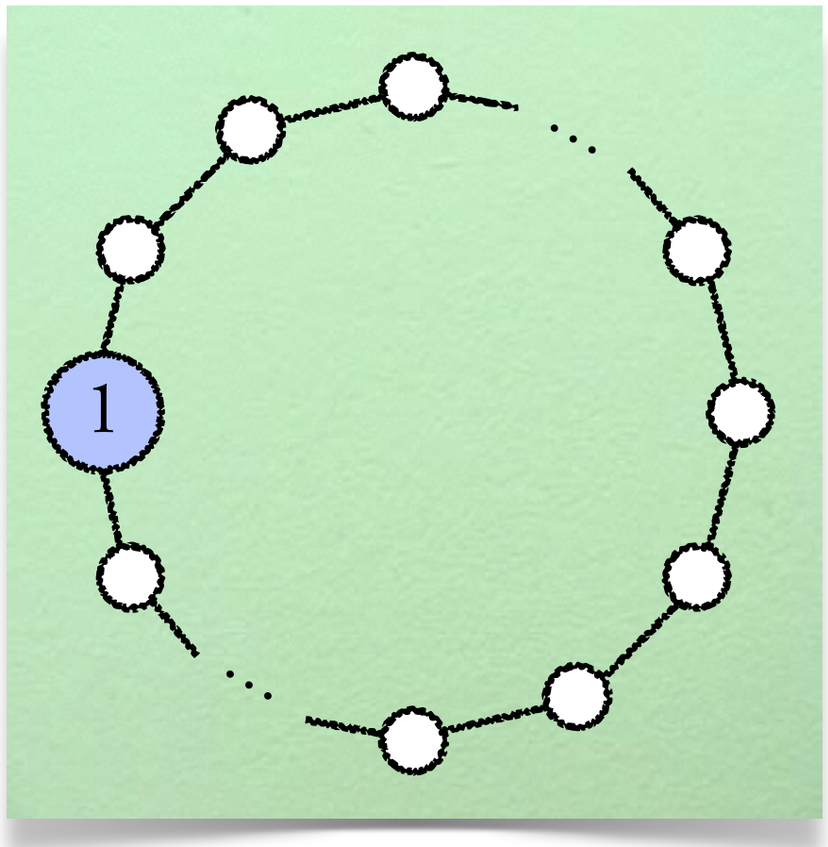

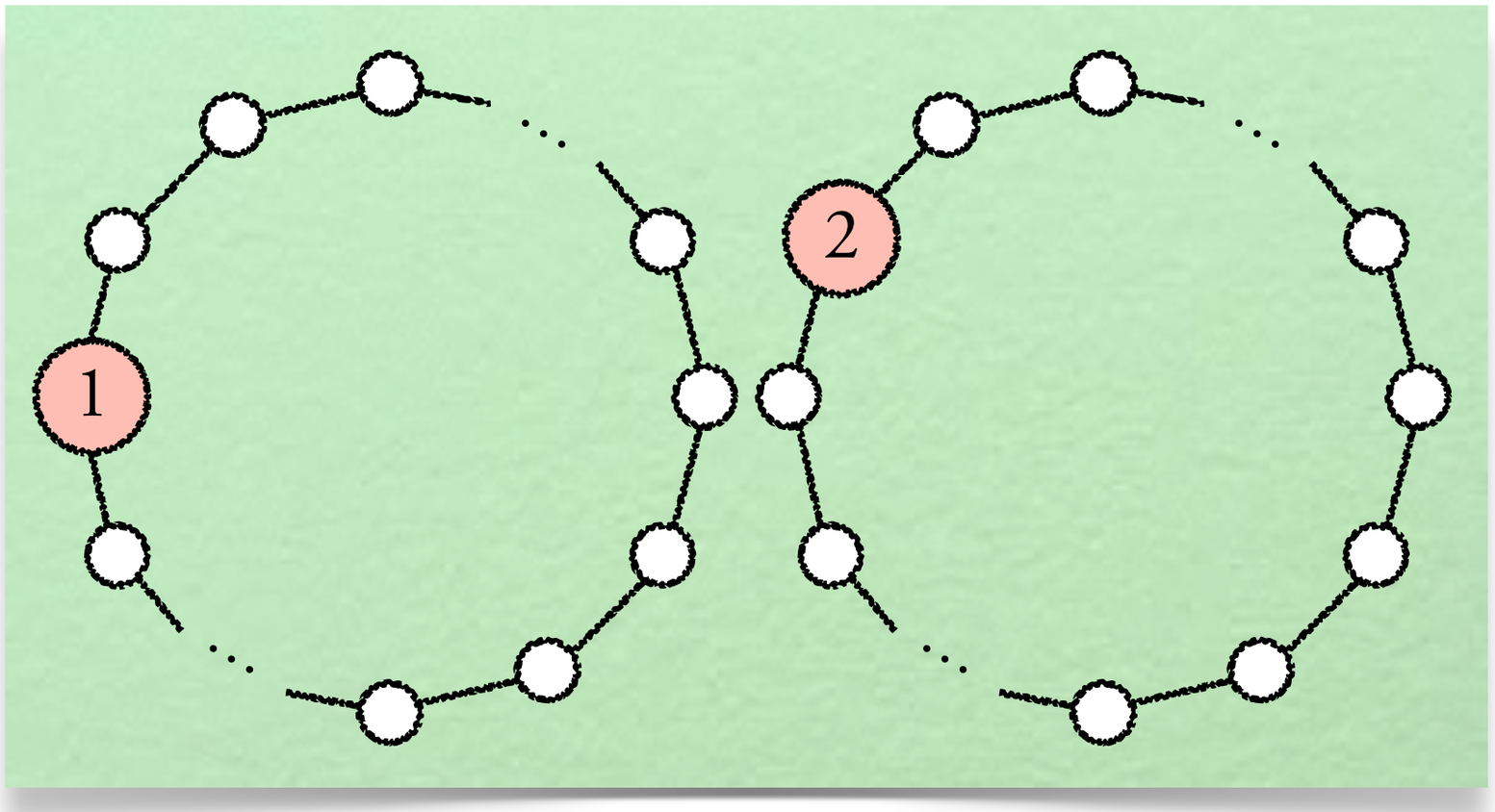

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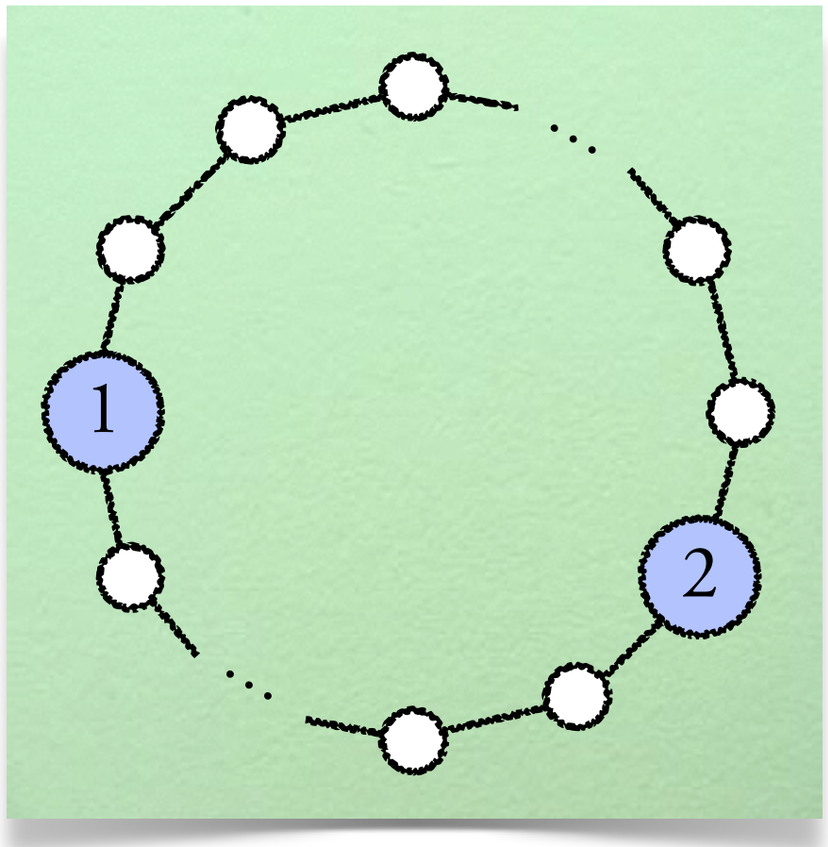

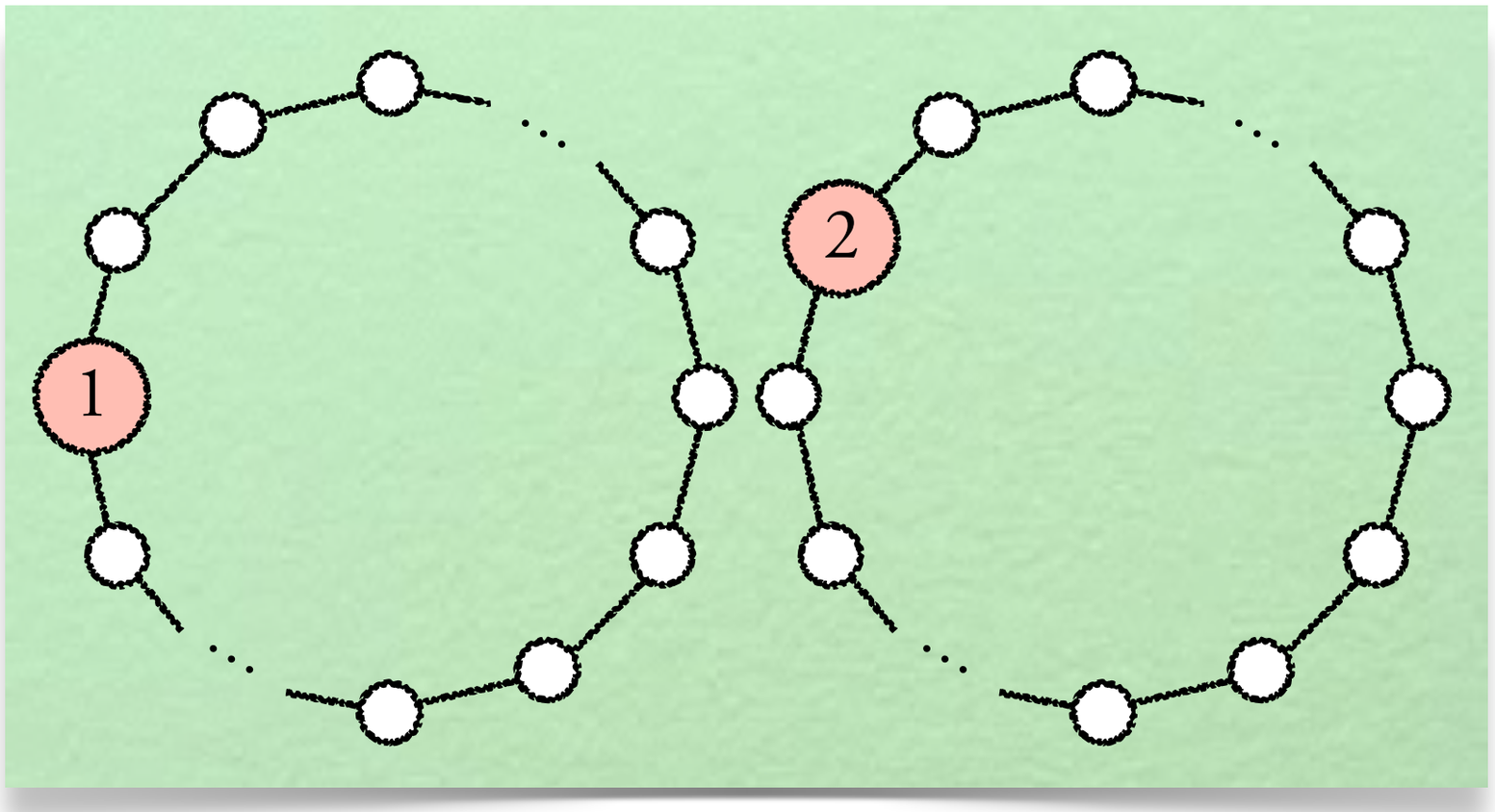

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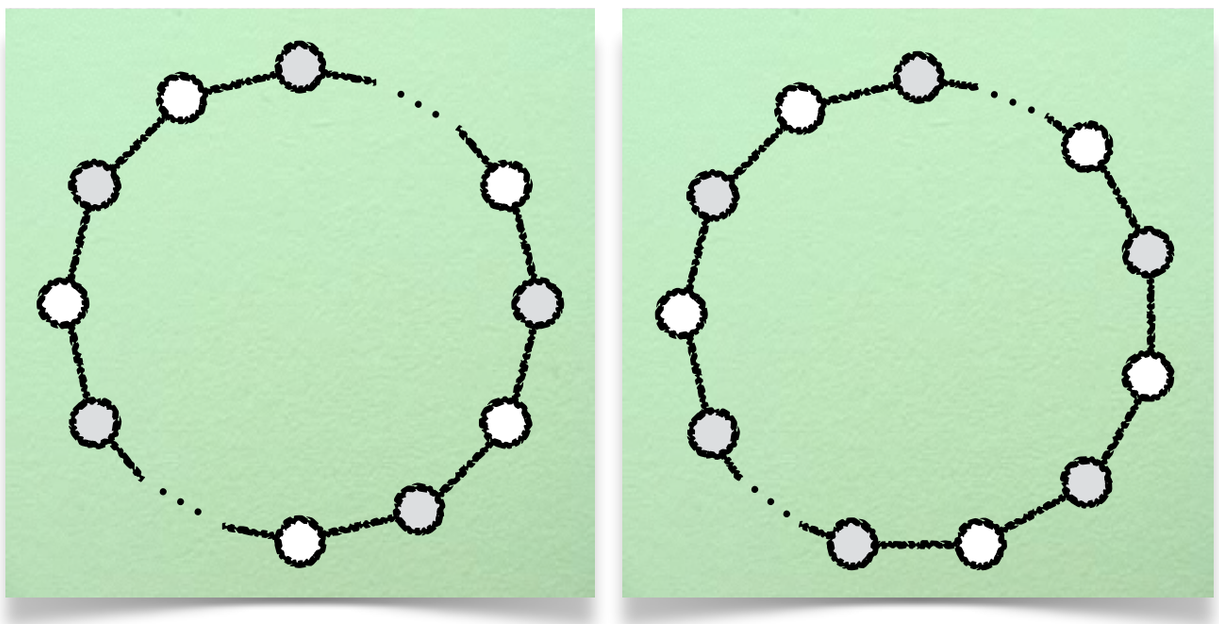

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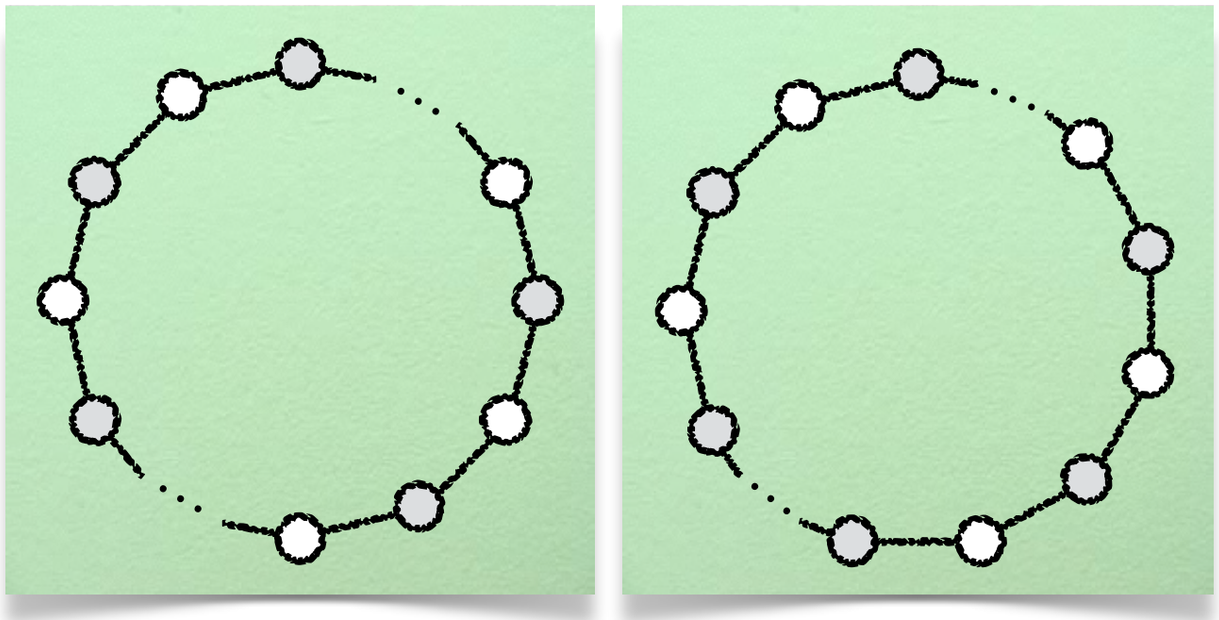

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