20/7/2015

ECI 2015
Buenos Aires

Fundamentos lógicos de bases de datos
(Logical foundations of databases)

Diego Figueira **Gabriele Puppis**

CNRS LaBRI

About the speakers…

Gabriele Puppis
PhD from Udine (Italy)
post-docs in Oxford
Works in LaBRI, Bordeaux
CNRS researcher

Diego Figueira
PhD from ENS Cachan (France),
post-docs in Warsaw, Edinburgh
Works in LaBRI, Bordeaux
CNRS researcher
Alma Mater: UBA !

About the speakers…

Gabriele Puppis
PhD from Udine (Italy)
post-docs in Oxford
Works in LaBRI, Bordeaux
CNRS researcher

Diego Figueira
PhD from ENS Cachan (France),
post-docs in Warsaw, Edinburgh
Works in LaBRI, Bordeaux
CNRS researcher
Alma Mater: UBA !

About the speakers…

Gabriele Puppis
PhD from Udine (Italy)
post-docs in Oxford
Works in LaBRI, Bordeaux
CNRS researcher

hermano del inigualable

!!!

Diego Figueira
PhD from ENS Cachan (France),
post-docs in Warsaw, Edinburgh
Works in LaBRI, Bordeaux
CNRS researcher
Alma Mater: UBA !

First and foremost…

interrupt!

ask! (in any language)

Organization

Format: 45 min slots / 20 min breaks: 45’ + 20’ + 45’ + 20’ + 45’

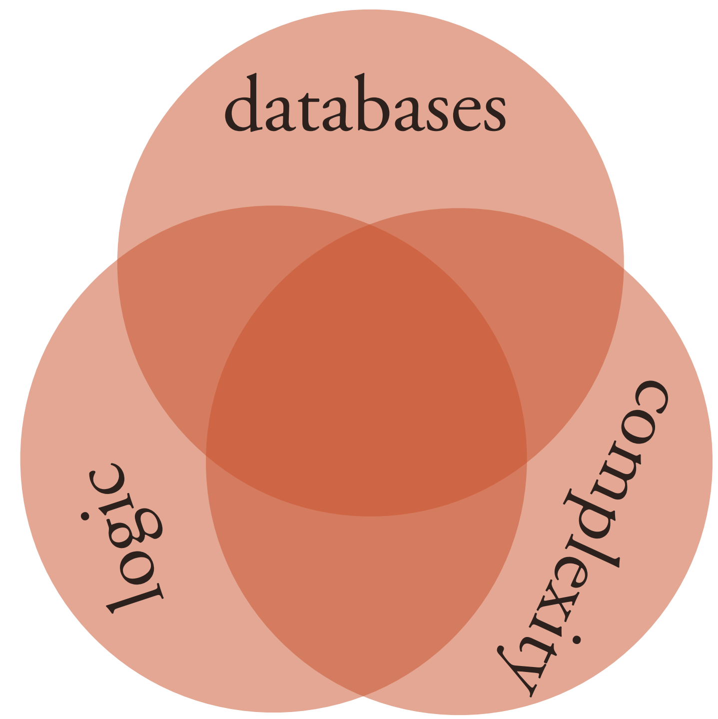
Schedule:

Relational Algebra

First-Order logic

EF games, 0-1 law, Locality

Conjunctive Queries



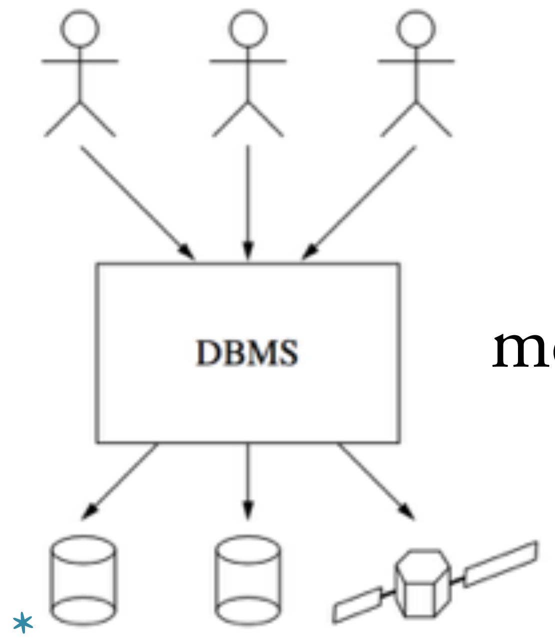
Evaluation: A short test on Saturday

Slides available at

http://www.labri.fr/perso/dfigueir/ECI15/

Databases

database = a collection of data, structured in some way + a way of defining, querying, updating the data inside

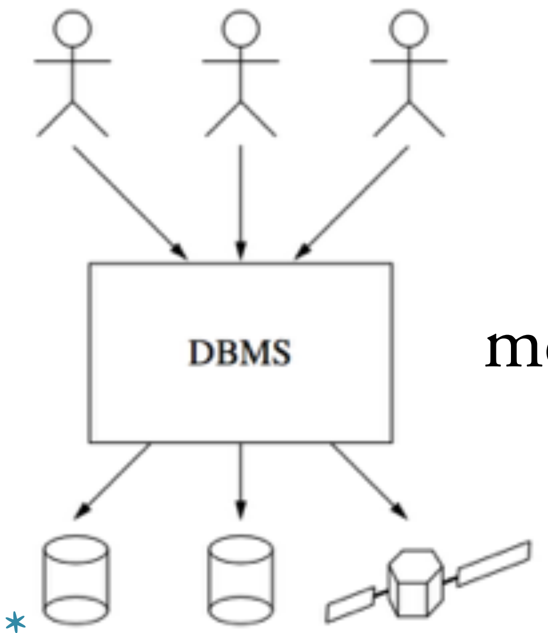


mediate between humans, processes & data

* [Abitebou, Hull, Vianu "Foundations of databases"]

Databases

database = a collection of data, structured in some way + a way of defining, querying, updating the data inside



*

mediate between humans, processes & data

Data model

- how the data is logically organized
- mathematical abstraction for representing data
- independent from physical organisation

DBMS also implement: transactions, concurrency, access control, resiliency…

* [Abitebou, Hull, Vianu "Foundations of databases"]

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a **relation**?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a **relation**?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

like:

$\{ (1,a,2), (2,b,6), (2,a,1) \} \subseteq N \times \{a,b\} \times N$

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a **relation**?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

like:

$\{ (1,a,2), (2,b,6), (2,a,1) \} \subseteq N \times \{a,b\} \times N$

(2,b,6) → a "tuple" (a "3-tuple")

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a **relation**?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

like:

{ (1,a,2), (2,b,6), (2,a,1) } ⊆ N × {a,b} × N

(2,b,6) → a "tuple" (a "3-tuple")

() → 0-tuple

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a **relation**?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

like:

$\{ (1,a,2), (2,b,6), (2,a,1) \} \subseteq N \times \{a,b\} \times N$

(2,b,6) → a "tuple" (a "3-tuple")

() → 0-tuple

like: "

1	a	2
2	b	6
2	a	1

"

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a relation?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

DB = A schema: names of tables and attributes

An instance: data conforming to the schema

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a relation?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

DB = A schema: names of tables and attributes

Films (Title:string, Director:string, Actor:string)

Schedule (Theatre:string, Title:string)

An instance: data conforming to the schema

Relational databases

Relational data model = data logically organised into relations ("tables").

What's a relation?

- a (finite) subset of the cartesian product of sets
- a "table" with rows and columns

DB =

A schema: names of tables and attributes

Films (Title:string, Director:string, Actor:string)

Schedule (Theatre:string, Title:string)

An instance: data conforming to the schema

Films

Title	Director	Actor
8 1/2	Fellini	Mastroianni
Shining	Kubrick	Nicholson
Dr. Strangelove	Kubrick	Sellers
8 femmes	Ozon	Ardant

Schedule

Theatre	Title
Utopia	Dr. Strangelove
Utopia	8 1/2
UGC	Dr. Strangelove
UGC	8 femmes

Relational databases

Relational data model = data logically organised into relations (“tables”).

We assume all elements come from
a fixed set of *constants* or *data values* **U**.

Relational databases: queries

What is a **query** q ?

A mapping that takes a database instance **D**
returns a relation $q(\mathbf{D}) \subseteq \mathbf{U}^{r}$ of fixed arity **r**

Relational databases: queries

computable!

What is a query q ?

A mapping that takes a database instance **D**
returns a relation $q(\mathbf{D}) \subseteq \mathbf{U}^{\mathbf{r}}$ of fixed arity **r**

Relational databases: queries

computable!

What is a **query** *q* ?

A mapping that
- takes a database instance **D**
- returns a relation $q(\mathbf{D}) \subseteq \mathbf{U}^{\mathbf{r}}$ of fixed arity **r**

generic!
(order independent)

Relational databases: queries

computable!

What is a **query** q ?

A mapping that
- takes a database instance **D**
- returns a relation $q(\mathbf{D}) \subseteq \mathbf{U}^r$ of fixed arity **r**

generic!
(order independent)

Boolean query: r=0
Either "yes" { () } or "no" { }

Relational databases: queries

What is a **query** *q* ?

A mapping that takes a database instance **D**
returns a relation $q(\mathbf{D}) \subseteq \mathbf{U}^{r}$ of fixed arity **r**

What do we care about queries?

expressive power

evaluation

static analysis

The fundamental questions:

How to query the relational data model?

How efficient/expressive is it?

The fundamental questions:

How to query the relational data model?

How efficient/expressive is it?

Query Language

Syntax

Expressions for querying the db,
governed by syntactic rules

"Select X from Y"

"y :- $\forall x\ (x \leq y)$"

+

Semantics

Interpretation of symbols
in terms of some structure

Retrieves all strings
in column X of table Y

Returns the maximum element
of the set.

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=, \neq\} \times \mathbb{N}$

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=, \neq\} \times \mathbb{N}$

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=, \neq\} \times \mathbb{N}$

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1=j_1, \ldots, i_n=j_n\}}(R) \coloneqq \{(x_1, \ldots, x_m) \in R \mid (x_{i_1}=x_{j_1}) \wedge \cdots \wedge (x_{i_n}=x_{j_n})\}$: Selection

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=, \neq\} \times \mathbb{N}$

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=,\neq\} \times \mathbb{N}$

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection

$\sigma_{\{1=3, 1\neq 2\}}(\{(1,2,1), (2,2,2)\}) = \{(1,2,1)\}$

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=,\neq\} \times \mathbb{N}$

$\sigma_{\{1=3, 1\neq 2\}}(\{(1,2,1), (2,2,2)\}) = \{(1,2,1)\}$

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) \coloneqq \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection
- $\pi_{\{i_1, \ldots, i_n\}}(R) \coloneqq \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$: Projection

Relational Algebra (RA)

[Codd, 1970]

Syntax: $E := R, S, \ldots \mid E \cup E \mid E \setminus E \mid E \times E \mid \pi_M(E) \mid \sigma_\Theta(E)$

where $M \subseteq \mathbb{N}$

$\Theta \subseteq \mathbb{N} \times \{=,\neq\} \times \mathbb{N}$

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection
- $\pi_{\{i_1, \ldots, i_n\}}(R) := \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$: Projection

$\sigma_{\{1=3, 1\neq 2\}}(\{(1,2,1), (2,2,2)\}) = \{(1,2,1)\}$

$\pi_{\{1,3\}}(\{(1,2,1),(2,2,2)\}) = \{(1,1), (2,2)\}$

Relational Algebra (RA)

[Codd, 1970]

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection
- $\pi_{\{i_1, \ldots, i_n\}}(R) := \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$: Projection

Question 1: What is the RA expression for

$$\{ (v_1,v_2) \mid \text{there are } w_1 \neq w_2 \text{ so that } (v_1,w_1) \in R_1 \text{ and } (v_2,w_2) \in R_2 \} ?$$

Question 2: $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2)) = ?$

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

Relational Algebra (RA)

[Codd, 1970]

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection
- $\pi_{\{i_1, \ldots, i_n\}}(R) := \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$: Projection

Question 1: What is the RA expression for

$\{ (v_1,v_2) \mid$ there are $w_1 \neq w_2$ so that $(v_1,w_1) \in R_1$ and $(v_2,w_2) \in R_2 \}$?

Answer: $\pi_{\{1,3\}}(\sigma_{1\neq 3}(R_1 \times R_2))$

a	b
b	a
c	a
c	b

Question 2: $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2)) = ?$

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

Relational Algebra (RA)

[Codd, 1970]

- $R_1 \cup R_2$: Set union
- $R_1 \times R_2$: Cartesian product
- $R_1 \setminus R_2$: Set difference
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$: Selection
- $\pi_{\{i_1, \ldots, i_n\}}(R) := \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$: Projection

Question 1: What is the RA expression for

$\{ (v_1,v_2) \mid$ there are $w_1 \neq w_2$ so that $(v_1,w_1) \in R_1$ and $(v_2,w_2) \in R_2 \}$?

Answer: $\pi_{\{1,3\}}(\sigma_{1\neq 3}(R_1 \times R_2))$

a	b
b	a
c	a
c	b

Question 2: $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2)) =$?

Answer (only one element):

b

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R1,…,Rn
Where Z
```

$\Leftrightarrow$ $\pi_X(\sigma_Z(R_1 \times \cdots \times R_n))$

`… or …` $\Leftrightarrow$ union

`… not in (…)` $\Leftrightarrow$ difference

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R₁,…,Rₙ
Where Z
```

$\Leftrightarrow$ $\pi_X (\sigma_Z(R_1 \times \cdots \times R_n))$

`… or …` $\Leftrightarrow$ union

`… not in (…)` $\Leftrightarrow$ difference

$\pi_2 (\sigma_{1 \neq 3}(R_1 \times R_2)) \rightsquigarrow$

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R₁,…,Rₙ
Where Z
```

$\Leftrightarrow$ $\pi_X (\sigma_Z(R_1 \times \cdots \times R_n))$

… or … $\Leftrightarrow$ union

… not in (…) $\Leftrightarrow$ difference

$\pi_2 (\sigma_{1\neq 3}(R_1 \times R_2))$ $\rightsquigarrow$

```
Select R₁.2 as foo
From R₁, R₂
Where R₁.1 ≠ R₂.1
```

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R₁,…,Rₙ
Where Z
```

$\Leftrightarrow \pi_X(\sigma_Z(R_1 \times \cdots \times R_n))$

… or …	$\Leftrightarrow$	union
… not in (…)	$\Leftrightarrow$	difference

$\underbrace{\pi_2(\sigma_{1\neq 3}(R_1 \times R_2))}_{❄} \rightsquigarrow$

```
Select R₁.2 as foo
From R₁, R₂
Where R₁.1 ≠ R₂.1
```

R₁

a	3
b	2
c	4
b	3
a	2

R₂

a	4
b	1
b	2
a	1
b	3

$\pi_2(\sigma_{1=3}(❄ \times R_2)) \rightsquigarrow$

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R₁,…,Rₙ
Where Z
```

$\Leftrightarrow$ $\pi_X(\sigma_Z(R_1 \times \cdots \times R_n))$

… or … $\Leftrightarrow$ union

… not in (…) $\Leftrightarrow$ difference

$\pi_2(\sigma_{1 \neq 3}(R_1 \times R_2))$ ⤳

```
Select R₁.2 as foo
From R₁, R₂
Where R₁.1 ≠ R₂.1
```

★

❄

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

$\pi_2(\sigma_{1=3}($ ❄ $\times R_2))$ ⤳

RA = Basic SQL

no domain-specific features, aggregation, etc

```
Select X
From R₁,…,Rₙ
Where Z
```

$\Leftrightarrow$ $\pi_X(\sigma_Z(R_1 \times \cdots \times R_n))$

… or … $\Leftrightarrow$ union

… not in (…) $\Leftrightarrow$ difference

$\pi_2(\sigma_{1\neq 3}(R_1 \times R_2))$ ⇝ ❄

```
Select R₁.2 as foo
From R₁, R₂
Where R₁.1 ≠ R₂.1
```

★

$\pi_2(\sigma_{1=3}($ ❄ $\times R_2))$ ⇝

```
Select foo
From ★, R₂
Where foo = R₂.2
```

R_1

a	3
b	2
c	4
b	3
a	2

R_2

a	4
b	1
b	2
a	1
b	3

Denotational languages

Procedural

Algebra $\rightsquigarrow$ *How* to obtain the result

Logics $\rightsquigarrow$ *What* is the property of the result

Declarative

Denotational languages

Relational Algebra

operations on tables

Algebra $\rightsquigarrow$ *How* to obtain the result

Procedural

Logics $\rightsquigarrow$ *What* is the property of the result

Declarative

Denotational languages

Relational Algebra
operations on tables

Procedural

Algebra $\rightsquigarrow$ *How* to obtain the result

Logics $\rightsquigarrow$ *What* is the property of the result

Declarative

First Order logic
properties on mathematical structures

Denotational languages

Relational Algebra
operations on tables

Procedural

Algebra $\rightsquigarrow$ *How* to obtain the result

Logics $\rightsquigarrow$ *What* is the property of the result

Declarative

First Order logic
properties on mathematical structures

FO = First-Order logic

Denotational languages

A **structure** is:

$$A = (D, R_1, \ldots, R_n, f_1, \ldots f_n)$$

D is a non-empty set, the domain

R_i is an m-ary relation for some m (ie, $R_i \subseteq D^m$)

f_i is an n-ary function for some n (ie, $f_i : D^n \longrightarrow D$)

Denotational languages

A **structure** is:

$$A = (D, R_1, \ldots, R_n, f_1, \ldots f_n)$$

D is a non-empty set, the domain

R_i is an m-ary relation for some m (ie, $R_i \subseteq D^m$)

f_i is an n-ary function for some n (ie, $f_i : D^n \longrightarrow D$)

A **graph** G = (V,E)

- V: nodes
- $E \subseteq V^2$: edges (binary relation)
- (no functions)

A **group**, like $(\mathbb{N},+)$

- $\mathbb{N}$: natural numbers
- (no relations)
- $+: \mathbb{N}^2 \longrightarrow \mathbb{N}$ addition (binary function)

First-order logic

First-order logic

FO

variables x, y, z, …

quantifiers: $\exists, \forall$

Boolean connectives: $\neg$, $\wedge$, $\vee$

A language to talk about **structures**

Variables range over the **domain**

Atomic formulas: $R(x_1, \ldots, x_m)$, $x=y$

First-order logic

FO

variables x, y, z, …

quantifiers: ∃,∀

Boolean connectives: ¬, ∧, ∨

A language to talk about **structures**

Variables range over the **domain**

Atomic formulas: $R(x_1, \ldots, x_m)$, $x=y$

A **graph** G = (V,E)

- V: nodes
- $E \subseteq V^2$: edges (binary relation)
- (no functions)

Language to talk about **graphs**

Variables range over **nodes**

Atomic formulas: E(x,y), x = y

First-order logic

FO

variables x, y, z, …

quantifiers: ∃,∀

Boolean connectives: ¬, ∧, ∨

A **graph** G = (V,E)

- V: nodes
- $E \subseteq V^2$: edges (binary relation)
- (no functions)

A language to talk about **structures**

Variables range over the **domain**

Atomic formulas: $R(x_1, \ldots, x_m)$, $x=y$

Language to talk about **graphs**

Variables range over **nodes**

Atomic formulas: E(x,y), x = y

Formulas: Atomic formulas + connectives + quantifiers

“The node x has at least two neighbours”

$$\exists y\ \exists z\ (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$$

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

free

x is free = not quantified

(a property of a node in the graph)

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

↳ free

x is free = not quantified

(a property of a node in the graph)

"Each node has at least two neighbours"

$\forall x\, \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

“The node x has at least two neighbours”

$\varphi(x) = \exists y\ \exists z\ (\neg(y=z) \land E(x,y) \land E(x,z))$

↳ free

x is free = not quantified
(a property of a node in the graph)

“Each node has at least two neighbours”

$\psi = \forall x\ \exists y\ \exists z\ (\neg(y=z) \land E(x,y) \land E(x,z))$

the formula is a sentence
= no free variables
(a property of the graph)

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

x → free

x is free = not quantified
(a property of a node in the graph)

"Each node has at least two neighbours"

$\psi = \forall x\, \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

the formula is a sentence
= no free variables
(a property of the graph)

Question:
- How to express in FO
  "*Every two neighbours have a common neighbour*" ?
- Does it have free variables? Is it a sentence?

"The node x has at least two neighbours"

$\varphi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

x → free

x is free = not quantified
(a property of a node in the graph)

"Each node has at least two neighbours"

$\psi = \forall x\, \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

the formula is a sentence
= no free variables
(a property of the graph)

Question: • How to express in FO
"*Every two neighbours have a common neighbour*" ?

• Does it have free variables? Is it a sentence?

Answer: $\forall x \forall y \left(\neg E(x,y) \vee \exists z \left(\left(E(x,z) \vee E(z,x)\right) \wedge \left(E(y,z) \vee E(z,y)\right) \right) \right)$

Binding

To evaluate a formula ϕ we need a graph G=(V,E) and a **binding** α that maps free variables of ϕ to nodes of G.

$G \models_\alpha \phi(x_1,\ldots,x_n)$ $\alpha : \{x_1,\ldots,x_n\} \longrightarrow V$ assigns nodes to free variables

Binding

To evaluate a formula ϕ we need a graph $G=(V,E)$ and a **binding** α that maps free variables of ϕ to nodes of G.

"G,α **satisfy** ϕ" → "ϕ is **satisfiable**"

$G \models_\alpha \phi(x_1,\ldots,x_n)$ $\alpha : \{x_1,\ldots,x_n\} \longrightarrow V$ assigns nodes to free variables

Binding

To evaluate a formula ϕ we need a graph G=(V,E) and a **binding** α that maps free variables of ϕ to nodes of G.

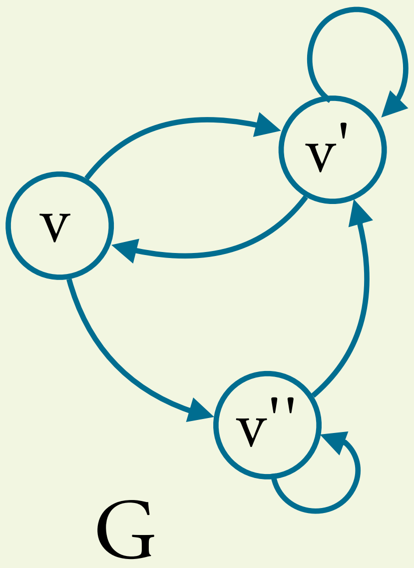
"**G,α satisfy ϕ**" → "**ϕ is satisfiable**"

$G \models_\alpha \phi(x_1,\ldots,x_n)$ $\alpha : \{x_1,\ldots,x_n\} \longrightarrow V$ assigns nodes to free variables

"The node x has at least two neighbours"

$\phi(x) = \exists y\ \exists z\ (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

$G \models_\alpha \phi$ if $\alpha = \{x \mapsto v\}$



Binding

To evaluate a formula ϕ we need a graph G=(V,E) and a **binding** α that maps free variables of ϕ to nodes of G.

"G,α **satisfy** ϕ" ⋯▸ "ϕ is **satisfiable**"

$G \vDash_{\alpha} \phi(x_1,\ldots,x_n)$ $\alpha : \{x_1,\ldots,x_n\} \longrightarrow V$ assigns nodes to free variables

"The node x has at least two neighbours"

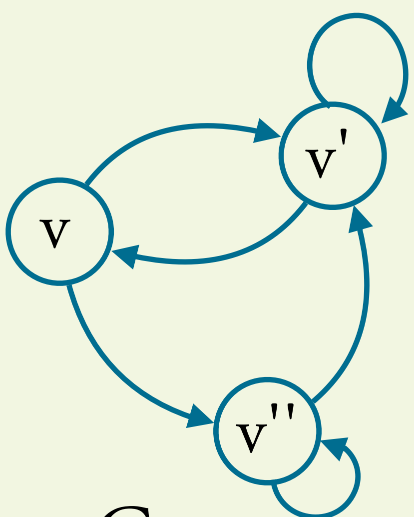
$\phi(x) = \exists y\, \exists z\, (\neg(y{=}z) \wedge E(x,y) \wedge E(x,z))$

$G \vDash_{\alpha} \phi$ if $\alpha = \{x \mapsto v\}$

"Every node has at least two neighbours"

$\psi = \forall x\, \exists y\, \exists z\, (\neg(y{=}z) \wedge E(x,y) \wedge E(x,z))$

$G \vDash_{\varnothing} \psi$



G

Formal Semantics of FO

$G \models_\alpha \exists x\, \phi$ iff for **some** $v \in V$ and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

$G \models_\alpha \forall x\, \phi$ iff for **every** $v \in V$ and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

$G \models_\alpha \phi \wedge \psi$ iff $G \models_\alpha \phi$ and $G \models_\alpha \psi$

$G \models_\alpha \neg\phi$ iff it is not true that $G \models_\alpha \phi$

$G \models_\alpha x=y$ iff $\alpha(x)=\alpha(y)$

$G \models_\alpha E(x,y)$ iff $(\alpha(x),\alpha(y)) \in E$

Formulas as queries

$\phi(x_1, \ldots, x_n)$ evaluated on $G=(V,E)$ yields all the bindings that satisfy ϕ:

$$\phi(G) = \{ \, (\alpha(x_1),\ldots,\alpha(x_n)) \mid G \models_\alpha \phi, \ \alpha\colon \{x_1,\ldots,x_n\} \longrightarrow V \, \}$$

Formulas as queries

$\phi(x_1, \ldots, x_n)$ evaluated on $G=(V,E)$ yields all the bindings that satisfy ϕ:

$$\phi(G) = \{ \, (\alpha(x_1),\ldots,\alpha(x_n)) \mid G \models_\alpha \phi, \; \alpha\colon \{x_1,\ldots,x_n\} \longrightarrow V \, \}$$

"The node x has at least two neighbours"

$\phi(x) = \exists y \, \exists z \, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

$\rightsquigarrow$ "Return all nodes with at least two neighbours"

Formulas as queries

$\phi(x_1, \ldots, x_n)$ evaluated on $G=(V,E)$ yields all the bindings that satisfy ϕ:

$$\phi(G) = \{ \, (\alpha(x_1),\ldots,\alpha(x_n)) \mid G \models_\alpha \phi,\ \alpha\colon \{x_1,\ldots,x_n\} \longrightarrow V \, \}$$

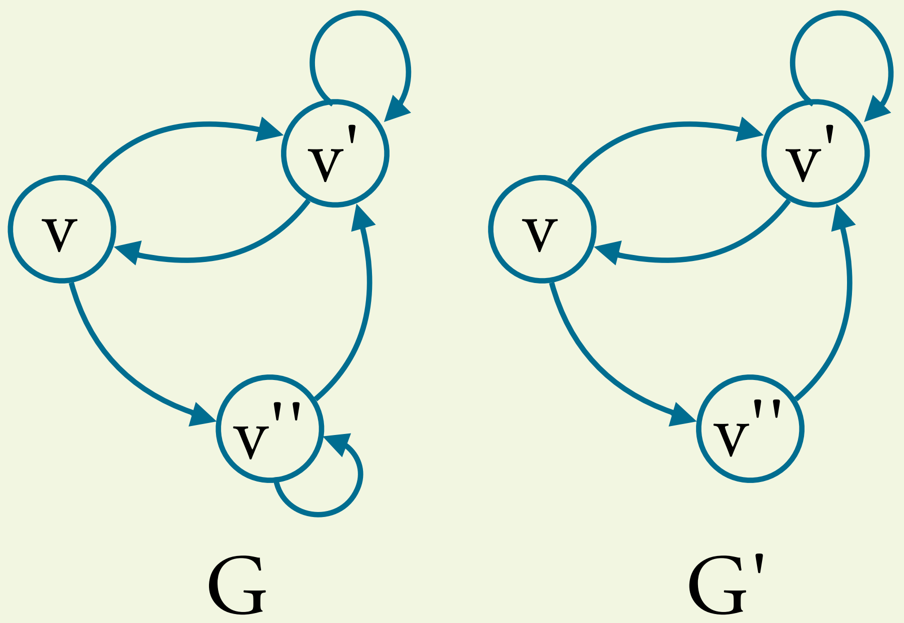
"The node x has at least two neighbours"

$\phi(x) = \exists y\, \exists z\, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

$\rightsquigarrow$ "Return all nodes with at least two neighbours"

$\phi(G) = \{v, v', v''\}$

$\phi(G') = \{v, v'\}$



Formulas as queries

$\phi(x_1, \ldots, x_n)$ evaluated on $G=(V,E)$ yields all the bindings that satisfy ϕ:

$$\phi(G) = \{ \, (\, \alpha(x_1),\ldots,\alpha(x_n) \,) \mid G \models_\alpha \phi, \; \alpha\colon \{x_1,\ldots,x_n\} \longrightarrow V \, \}$$

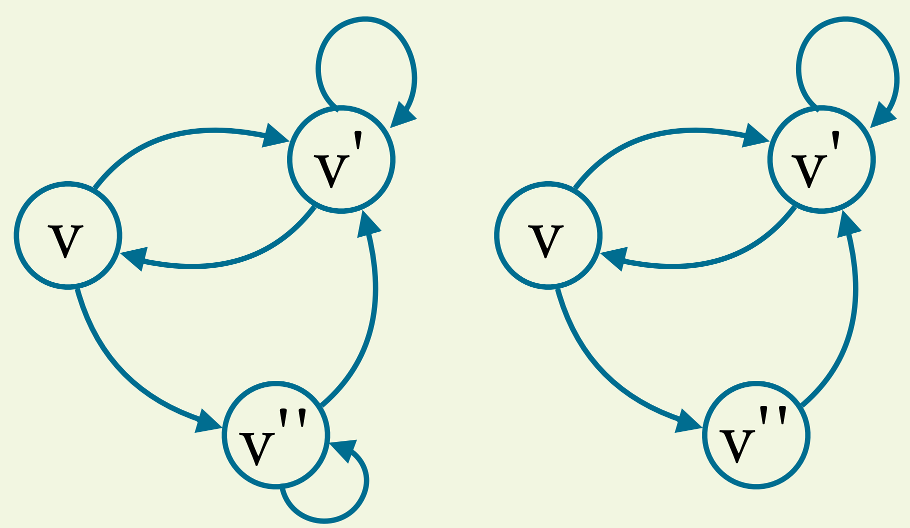
"The node x has at least two neighbours"

$\phi(x) = \exists y \, \exists z \, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

⤳ "Return all nodes with at least two neighbours"

$\phi(G) = \{v, v', v''\}$

$\phi(G') = \{v, v'\}$



G   G'

"Every node has two neighbours"

$\psi = \forall x \, \exists y \, \exists z \, (\neg(y=z) \wedge E(x,y) \wedge E(x,z))$

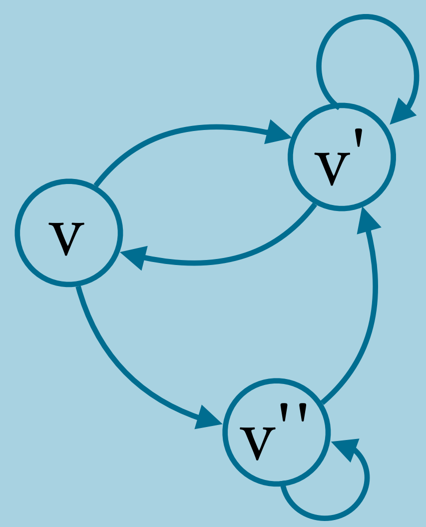
$\psi(G) = \{()\}$ ⤳ set with one element: the 0-tuple

$\psi(G') = \{\}$ ⤳ empty set

Question: Which bindings α verify $G \models_\alpha \phi$ for

$$\phi(x,y) = \exists z\,(E(x,z) \wedge E(z,y))$$

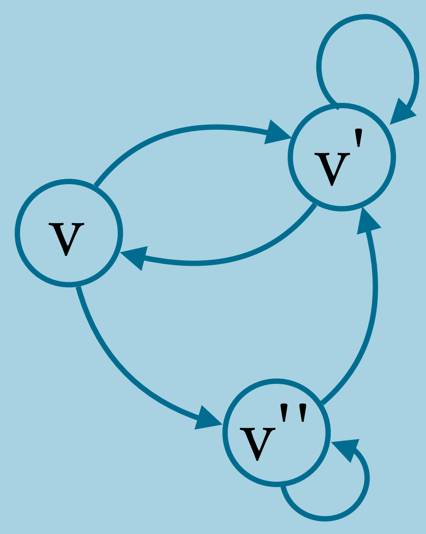
and $G =$



?

Question: Which bindings α verify $G \models_\alpha \phi$ for

$$\phi(x,y) = \exists z\,(E(x,z) \wedge E(z,y))$$

and G =  ?

Answer:

- $\alpha = \{\, x \mapsto v,\, y \mapsto v' \,\}$,
- $\alpha = \{\, x \mapsto v,\, y \mapsto v \,\}$,
- $\alpha = \{\, x \mapsto v',\, y \mapsto v' \,\}$,
- ... and all the rest

$$\phi(G) = \{v, v', v''\} \times \{v, v', v''\}$$

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

Particular to databases:

- Use of constants
- No functions
- Finite structure
- Quantification over active domain

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

Particular to databases:

- Use of constants
- No functions
- Finite structure
- Quantification over active domain

$\models_{\text{finite}}$ is different from $\models$

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

Particular to databases:

- Use of constants
- No functions
- Finite structure
- Quantification over active domain

$\models_{\text{finite}}$ is different from $\models$

There are formulas ϕ that are satisfiable only on infinite structures.

Like which?

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

Particular to databases:

- Use of constants
- No functions
- Finite structure
- Quantification over active domain

$\models_{finite}$ is different from $\models$

There are formulas ϕ that are satisfiable only on infinite structures.

Like which?

ϕ = " R(x,y) *is an infinite linear order*"

Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables = Relations

Queries = Formulas

Rows = Tuples

Particular to databases:

- Use of constants
- No functions
- Finite structure
- Quantification over active domain

$\models_{finite}$ is different from $\models$

There are formulas ϕ that are satisfiable only on infinite structures.

Like which?

ϕ = " R(x,y) *is an infinite linear order* "

Finite model theory

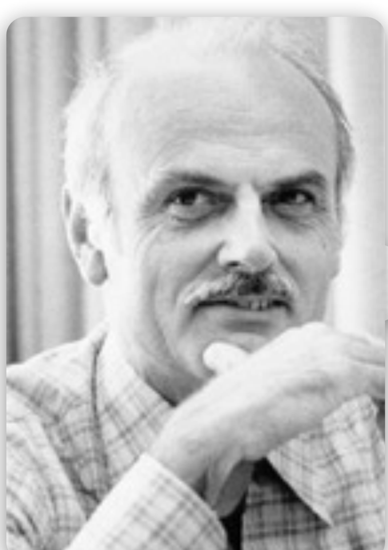
Formulas as queries

FO can serve as a **declarative** query language on relational databases : we express the properties of the answer

Tables	=	Relations
Queries	=	Formulas
Rows	=	Tuples

RA $=^*$ FO

How = *What*



RA and FO logic have roughly* the same expressive power!
[E.F. Codd 1972]

*FO without functions, with equality, on finite domains, …

Formulas as queries

$RA \subseteq FO$

- $R_1 \times R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \wedge R_2(x_{n+1}, \ldots, x_m)$
- $R_1 \cup R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \vee R_2(x_1, \ldots, x_n)$
- $\sigma_{\{i_1=j_1,\ldots,i_n=j_n\}}(R) \quad \rightsquigarrow \quad R(x_1, \ldots, x_m) \wedge (x_{i_1}=x_{j_1}) \wedge \cdots \wedge (x_{i_n}=x_{j_n})$
- $\pi_{\{i_1,\ldots,i_n\}}(R) \quad \rightsquigarrow \quad \exists(\{x_1,\ldots,x_m\} \setminus \{x_{i_1},\ldots,x_{i_n}\}).\ R(x_1, \ldots, x_m)$
- $R_1 \setminus R_2 \quad \rightsquigarrow \quad R_1(x_1, \ldots, x_n) \wedge \neg R_2(x_1, \ldots, x_n)$
- …

Formulas as queries

$\boxed{\mathrm{FO} \subseteq \mathrm{RA}}$ does not hold in general!

Formulas as queries

$FO \subseteq RA$ does not hold in general!

"the complement of R" $\notin$ RA
$\in$ FO : $\neg R(x)$

Formulas as queries

FO $\not\subseteq$ RA

"the complement of R" $\notin$ RA
$\in$ FO : $\neg R(x)$

Formulas as queries

$\text{FO} \not\subseteq \text{RA}$

"the complement of R" $\notin$ RA
$\in$ FO : $\neg R(x)$

⇝ We restrict variables to range over **active domain**

Formulas as queries

FO $\not\subseteq$ RA

"the complement of R" $\notin$ RA
$\in$ FO : $\neg R(x)$

⇝ We restrict variables to range over **active domain** ▸ elements in the relations

FO^{act}

=

FO restricted
to active domain

Formulas as queries

FO $\not\subseteq$ RA

"the complement of R" $\notin$ RA
$\in$ FO : $\neg R(x)$

⇝ We restrict variables to range over **active domain** → elements in the relations

FO^act

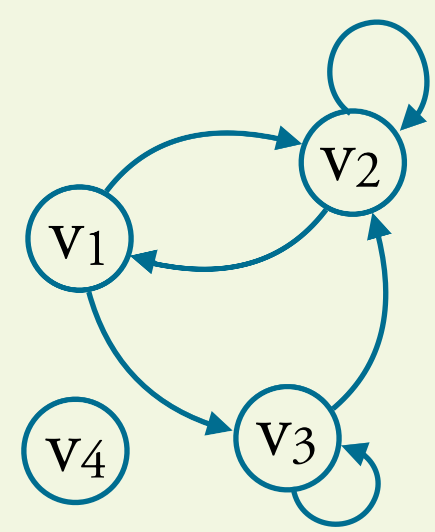
=

FO restricted to active domain

$$\phi_1(x,y) = \neg E(x,y)$$
$$\phi_1(G) = \{(v_1,v_1),(v_3,v_1)\}$$

$$\phi_2(x) = \forall y\, E(y,x)$$
$$\phi_2(G) = \{v_2\}$$

G =



First-order logic restricted to active domain

Formal Semantics of FO^{act}

$G \models_\alpha \exists x\, \phi$ iff for **some** $v \in$ ACT(G) and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

$G \models_\alpha \forall x\, \phi$ iff for **every** $v \in$ ACT(G) and $\alpha' = \alpha \cup \{x \mapsto v\}$ we have $G \models_{\alpha'} \phi$

$G \models_\alpha \phi \wedge \psi$ iff $G \models_\alpha \phi$ and $G \models_\alpha \psi$

$G \models_\alpha \neg\phi$ iff it is not true that $G \models_\alpha \phi$

$G \models_\alpha x=y$ iff $\alpha(x)=\alpha(y)$

$G \models_\alpha E(x,y)$ iff $(\alpha(x),\alpha(y)) \in E$

ACT(G) = {v | **for some** v': (v,v') ∈ E or (v',v) ∈ E}

$\mathbf{FO}^{\mathbf{act}} \subseteq \mathrm{RA}$

$FO^{act} \subseteq RA$

Assume:

1. ϕ has variables $x_1,\ldots,x_n$,
2. ϕ in normal form: $(\exists^* (\neg\exists)^*)^*$ + quantifier-free $\psi(x_1,\ldots,x_n)$

$$\exists x_1,x_2 \neg\exists x_3 \exists x_4 . (E(x_1,x_3) \wedge \neg E(x_4,x_2)) \vee (x_1=x_3)$$

$FO^{act} \subseteq RA$

Assume:

1. ϕ has variables $x_1,\ldots,x_n$,
2. ϕ in normal form: $(\exists^* (\neg\exists)^*)^*$ + quantifier-free $\psi(x_1,\ldots,x_n)$

$$\exists x_1,x_2\ \neg\exists x_3\ \exists x_4\ .\ (\ E(x_1,x_3) \wedge \neg E(x_4,x_2)\ ) \vee (x_1=x_3)$$

Translation

Adom = RA expression for active domain = "$\pi_1(E) \cup \pi_2(E)$"

- $(\ \exists x_i\ \phi(x_{i_1},\ldots,x_{i_n})\ )^+ \rightsquigarrow \pi_{\{i_1,\ldots,i_n\}\setminus\{i\}}(\ \phi^+\ )$
- $(x_i = x_j)^+ \rightsquigarrow \sigma_{\{i=j\}}(\ \mathbf{Adom} \times \ldots \times \mathbf{Adom}\ )$
- $(\psi_1(x_1,\ldots,x_n) \wedge \psi_2(x_1,\ldots,x_n))^+ \rightsquigarrow \psi_1^+ \cap \psi_2^+$
- $(\neg\phi(x_{i_1},\ldots,x_{i_n}))^+ \rightsquigarrow \mathbf{Adom} \times \ldots \times \mathbf{Adom}\ \setminus\ \phi^+$

$FO^{act} \subseteq RA$

Assume:

1. ϕ has variables $x_1,\dots,x_n$,
2. ϕ in normal form: $(\exists^* (\neg\exists)^*)^*$ + quantifier-free $\psi(x_1,\dots,x_n)$

$$\exists x_1,x_2\ \neg\exists x_3\ \exists x_4\ .\ (\ E(x_1,x_3) \wedge \neg E(x_4,x_2)\ ) \vee (x_1=x_3)$$

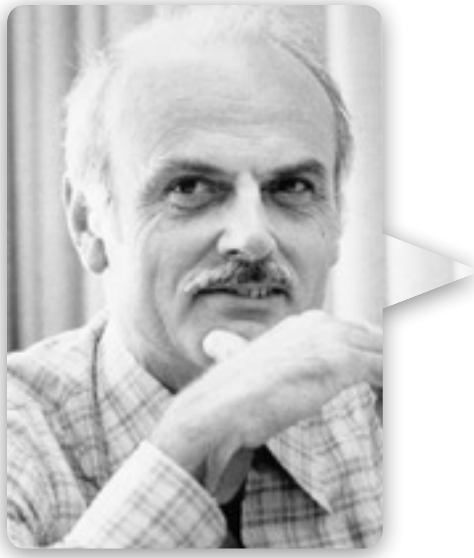
Translation

Adom = RA expression for active domain = "$\pi_1(E) \cup \pi_2(E)$"

- $(\ \exists x_i\ \phi(x_{i_1},\dots,x_{i_n})\ )^+ \rightsquigarrow \pi_{\{i_1,\dots,i_n\}\setminus\{i\}}(\ \phi^+\ )$
- $(x_i = x_j)^+ \rightsquigarrow\ \sigma_{\{i=j\}}(\ \mathbf{Adom} \times \dots \times \mathbf{Adom}\ )$
- $(\psi_1(x_1,\dots,x_n) \wedge \psi_2(x_1,\dots,x_n))^+ \rightsquigarrow \psi_1^+ \cap \psi_2^+$
- $(\neg\phi(x_{i_1},\dots,x_{i_n}))^+ \rightsquigarrow \mathbf{Adom} \times \dots \times \mathbf{Adom}\ \setminus\ \phi^+$

$A \cap B = (A \cup B) \setminus A \setminus B$

Corollary



FO^{act} is equivalent to RA

Question 1: How is $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2))$ expressed in FO?
Remember: R_1,R_2 are binary

Question 2: How is $\exists y,z \,.\, (R_1(x,y) \land R_1(y,z) \land x \neq z)$ expressed in RA?
Remember: The signature is the same as before (R_1,R_2 binary)

- $R_1 \cup R_2$
- $R_1 \times R_2$
- $R_1 \setminus R_2$
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \land \cdots \land (x_{i_n} \overset{\neq}{=} x_{j_n})\}$
- $\pi_{\{i_1, \ldots, i_n\}}(R) := \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$

Question 1: How is $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2))$ expressed in FO?
Remember: R_1,R_2 are binary

Answer: $\exists x_2 \,.\, \big(\exists x_1,x_4 \,.\, (R_1(x_1,x_2) \wedge R_2(x_1,x_4)) \wedge R_2(x_2,x_5) \big)$

Question 2: How is $\exists y,z \,.\, (R_1(x,y) \wedge R_1(y,z) \wedge x \neq z)$ expressed in RA?
Remember: The signature is the same as before (R_1,R_2 binary)

- $R_1 \cup R_2$
- $R_1 \times R_2$
- $R_1 \setminus R_2$
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$
- $\pi_{\{i_1,\ldots,i_n\}}(R) := \{(x_{i_1},\ldots,x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$

Question 1: How is $\pi_2(\sigma_{1=3}(\pi_2(\sigma_{1=3}(R_1 \times R_2)) \times R_2))$ expressed in FO?
Remember: R_1, R_2 are binary

Answer: $\exists x_2 . \Big(\exists x_1, x_4 . \big(R_1(x_1,x_2) \wedge R_2(x_1,x_4)\big) \wedge R_2(x_2,x_5) \Big)$

Question 2: How is $\exists y,z . (R_1(x,y) \wedge R_1(y,z) \wedge x \neq z)$ expressed in RA?
Remember: The signature is the same as before (R_1, R_2 binary)

- $R_1 \cup R_2$
- $R_1 \times R_2$
- $R_1 \setminus R_2$
- $\sigma_{\{i_1 \overset{\neq}{=} j_1, \ldots, i_n \overset{\neq}{=} j_n\}}(R) := \{(x_1, \ldots, x_m) \in R \mid (x_{i_1} \overset{\neq}{=} x_{j_1}) \wedge \cdots \wedge (x_{i_n} \overset{\neq}{=} x_{j_n})\}$
- $\pi_{\{i_1, \ldots, i_n\}}(R) := \{(x_{i_1}, \ldots, x_{i_n}) \mid (x_1, \ldots, x_m) \in R\}$

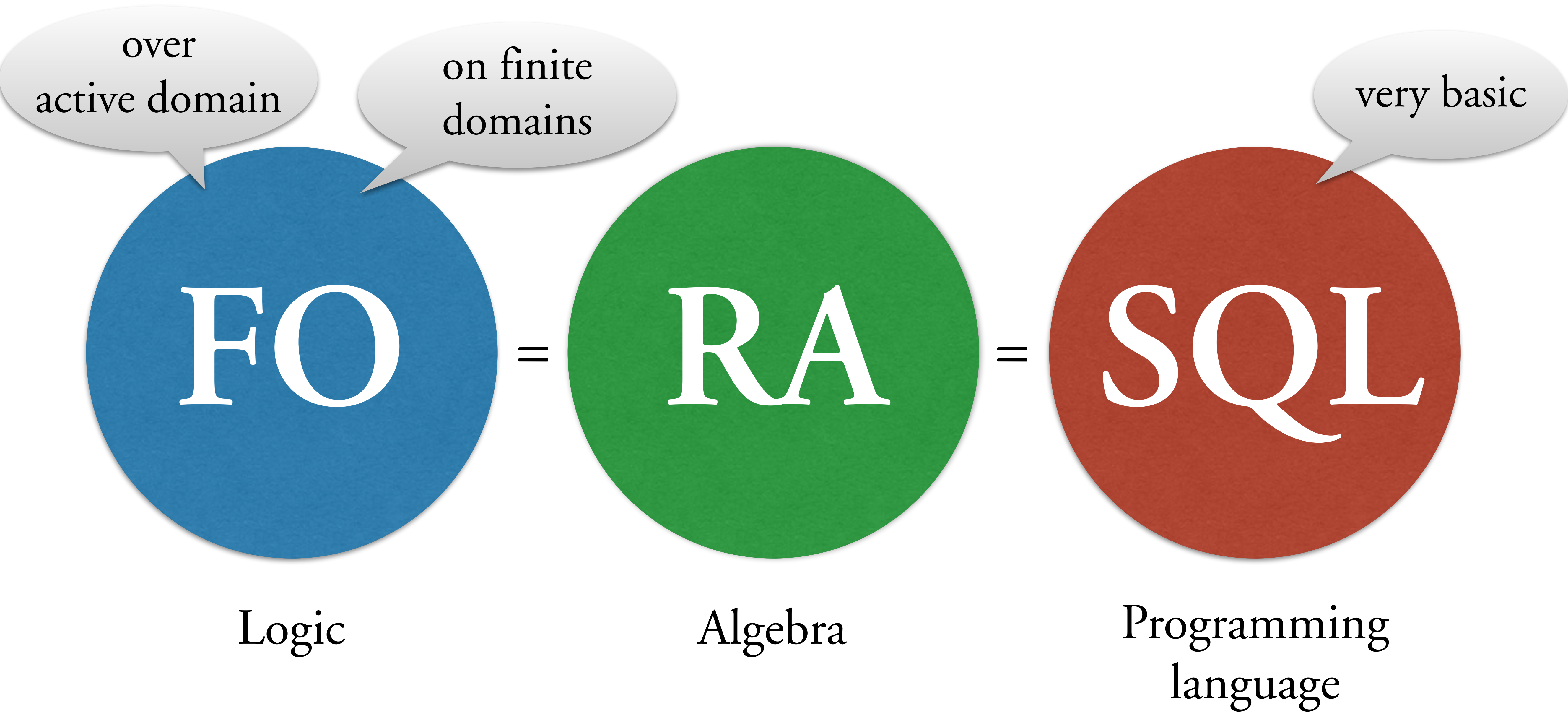
Answer: $\pi_1(\sigma_{\{2=3, 1\neq 4\}}(R_1 \times R_1))$

FO = RA = SQL

Logic

Algebra

Programming language


over
active domain
on finite
domains
very basic
FO
=
RA
=
SQL
Logic
Algebra
Programming
language

Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is **t** ∈ **Q**(**db**) ?

⇝ How hard is it to retrieve data?

Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is **t** ∈ **Q**(**db**) ?

⇝ How hard is it to retrieve data?

Emptiness problem: Given a query **Q**, is there a database instance **db** so that **Q**(**db**) ≠ ∅ ?

⇝ Does Q make sense? Is it a contradiction? (Query optimization)

Algorithmic problems for query languages

Evaluation problem: Given a query **Q**, a database instance **db**, and a tuple **t**, is **t** ∈ **Q**(**db**) ?

⇝ How hard is it to retrieve data?

Emptiness problem: Given a query **Q**, is there a database instance **db** so that **Q**(**db**) ≠ ∅ ?

⇝ Does Q make sense? Is it a contradiction? (Query optimization)

Equivalence problem: Given queries $\mathbf{Q_1}$, $\mathbf{Q_2}$, is

$$\mathbf{Q_1}(\mathbf{db}) = \mathbf{Q_2}(\mathbf{db})$$

for all database instances **db**?

⇝ Can we safely replace a query with another? (Query optimization)

Complexity theory

What can be **mechanized**? $\rightsquigarrow$ decidable/undecidable

How **hard** is it to mechanise? $\rightsquigarrow$ complexity classes

Complexity theory

What can be **mechanized**? ⤳ decidable/undecidable

- Domino
- H's 10th
- PCP
- K
- ⋮

How **hard** is it to mechanise? ⤳ complexity classes

Complexity theory

Domino H's 10th PCP K ⋮

What can be **mechanized**? ⇝ decidable/undecidable

How **hard** is it to mechanise? ⇝ complexity classes

usage of resources:
- time
- memory

Complexity theory

Domino, H's 10th, PCP, K, ⋮

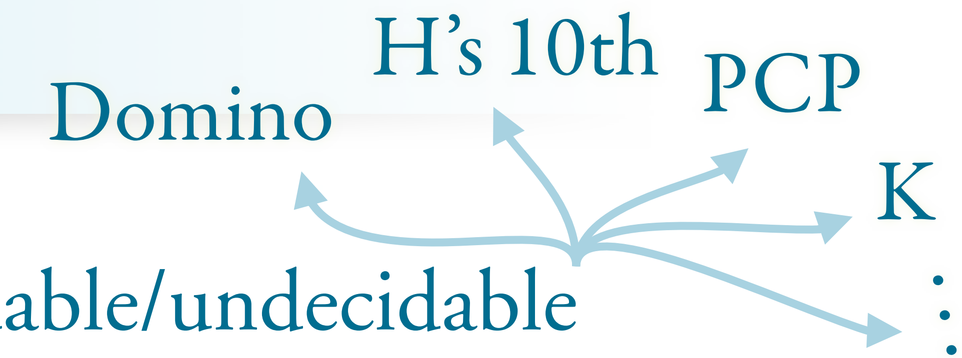
What can be **mechanized**? ⇝ decidable/undecidable

How **hard** is it to mechanise? ⇝ complexity classes

usage of resources:
- time
- memory

Algorithm Alg is TIME-bounded
by a function $f: \mathrm{N} \longrightarrow \mathrm{N}$ if
Alg(*input*) uses less than $f(|input|)$ units of TIME.

Complexity theory

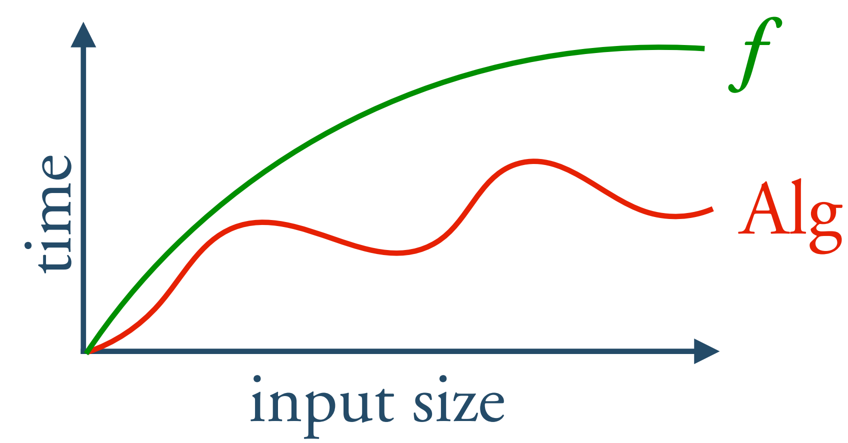


What can be **mechanized**? ⇝ decidable/undecidable

How **hard** is it to mechanise? ⇝ complexity classes

⇢ usage of resources:
- time
- memory

Algorithm **Alg** is TIME-bounded by a function f: N ⟶ N if **Alg**(*input*) uses less than f(|*input*|) units of TIME.



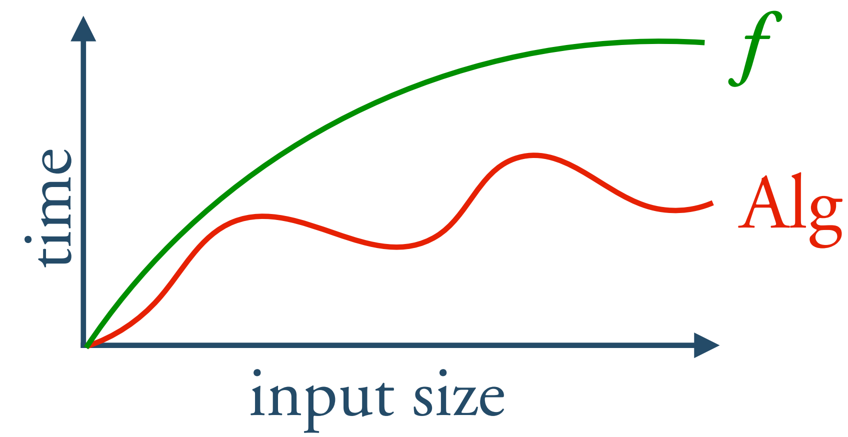
Complexity theory

What can be **mechanized**? ⇝ decidable/undecidable

How **hard** is it to mechanise? ⇝ complexity classes

⇢ usage of resources:
- time
- memory

Algorithm **Alg** is ~~TIME~~ SPACE-bounded
by a function f: N ⟶ N if
Alg(*input*) uses less than f(|*input*|) units of ~~TIME~~ SPACE.



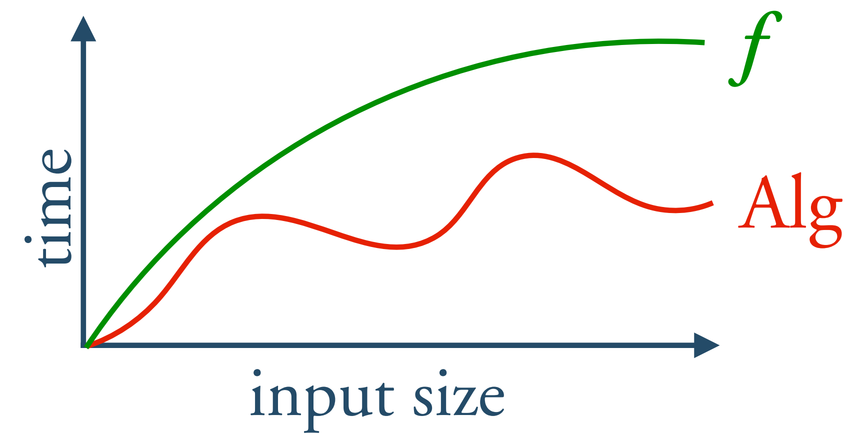
Complexity theory

What can be **mechanized**? ⇝ decidable/undecidable → Domino, H's 10th, PCP, K, ⋮

How **hard** is it to mechanise? ⇝ complexity classes

⇢ usage of resources:
- time
- memory

Algorithm **Alg** is ~~TIME~~ SPACE-bounded
by a function f: N ⟶ N if
Alg(*input*) uses less than $f(|input|)$ units of ~~TIME~~ SPACE.



LOGSPACE $\subsetneq$ PTIME $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq \cdots$

Complexity theory

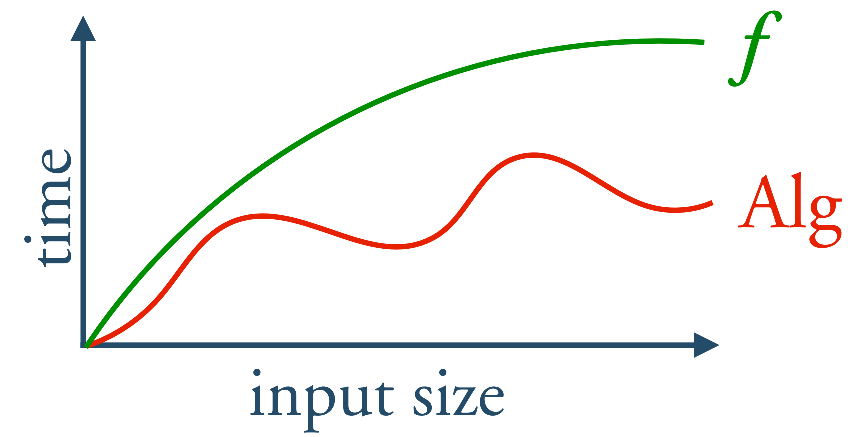
Domino H's 10th PCP K ⋮

What can be **mechanized**? ⇝ decidable/undecidable

How **hard** is it to mechanise? ⇝ complexity classes

→ usage of resources:
- time
- memory

Algorithm Alg is ~~TIME~~ SPACE-bounded by a function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ if Alg(*input*) uses less than $f(|input|)$ units of ~~TIME~~ SPACE.



LOGSPACE $\subsetneq$ PTIME $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq \cdots$

- PTIME: TIME-bounded by a polynomial
- PSPACE: SPACE-bounded by a polynomial
- LOGSPACE: SPACE-bounded by log(n)

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, ..., x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

Equivalence problem: Given FO formulae ϕ,ψ, is

$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

for all graphs G and bindings α?

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, ..., x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

Equivalence problem: Given FO formulae ϕ,ψ, is

$G \models_\alpha \phi$ iff $G \models_\alpha \psi$

for all graphs G and bindings α?

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{\text{finite}}$

Equivalence problem: Given FO formulae ϕ,ψ, is

$G \models_\alpha \phi$ iff $G \models_\alpha \psi$

for all graphs G and bindings α?

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{finite}$

Equivalence problem: Given FO formulae ϕ,ψ, is

$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

for all graphs G and bindings α?

💀 UNDECIDABLE ⇝ by reduction to the satisfiability problem

Algorithmic problems for FO

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{\text{finite}}$ [Trakhtenbrot '50]

Algorithmic problems for FO

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE $\rightsquigarrow$ both for $\models$ and $\models_{finite}$ [Trakhtenbrot '50]

Proof: By reduction from the Domino (aka Tiling) problem.

Algorithmic problems for FO

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_{\alpha} \phi$?

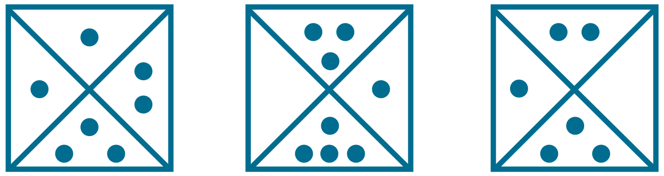
💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{\text{finite}}$ [Trakhtenbrot '50]

Proof: By reduction from the Domino (aka Tiling) problem.

Reduction from P to P': Algorithm that solves P using a O(1) procedure
" P'(x) "
that returns the truth value of P'(x).

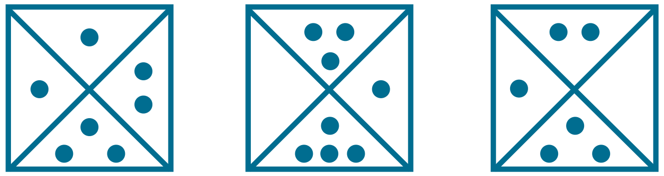
The (undecidable) Domino problem

Domino

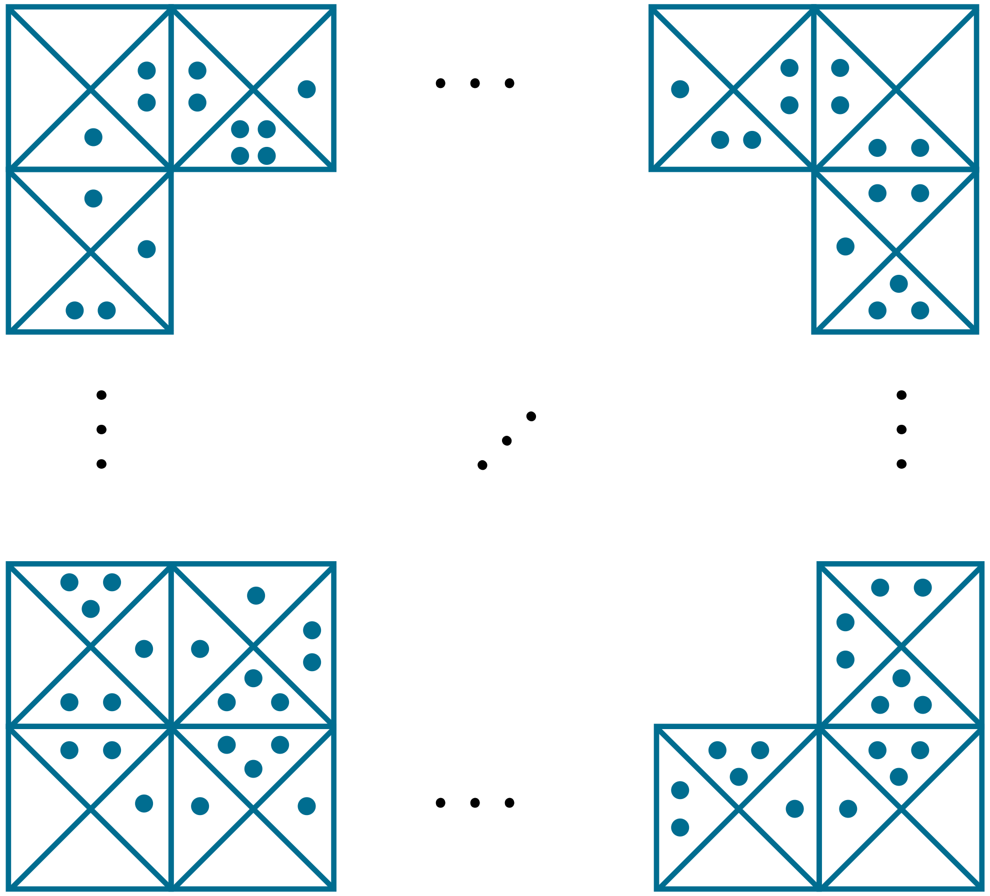
Input: 4-sided dominos: 

The (undecidable) Domino problem

Domino

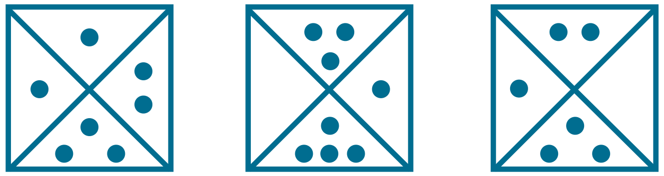
Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)

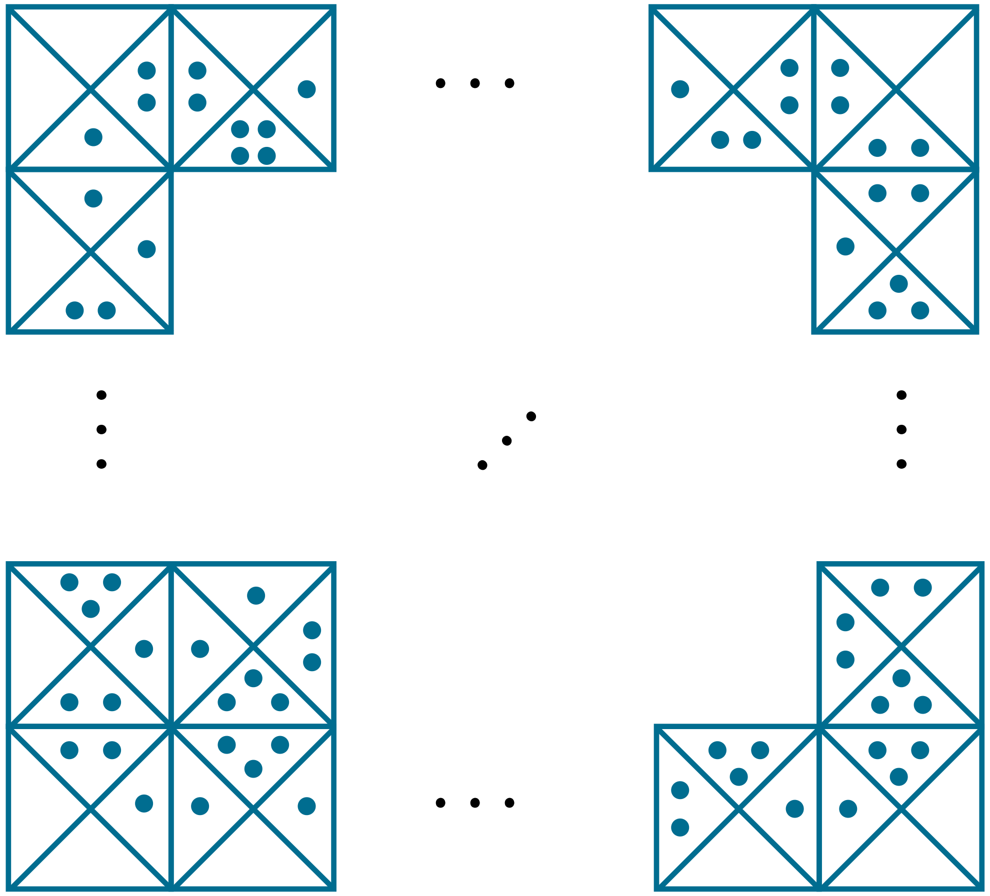


The (undecidable) Domino problem

Domino

Input: 4-sided dominos: 

Output: Is it possible to form a white-bordered rectangle? (of any size)

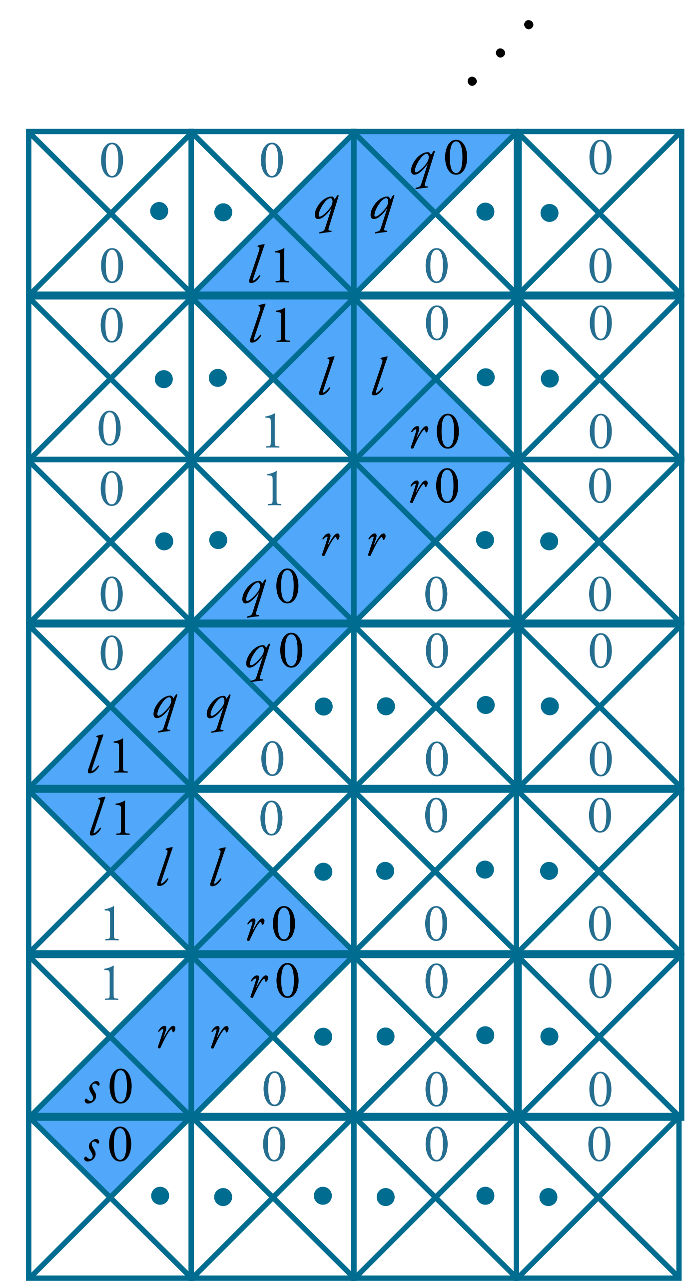


Rules: sides must match,
you can't rotate the dominos, but you can 'clone' them.

The (undecidable) Domino problem

Domino - Why is it undecidable?

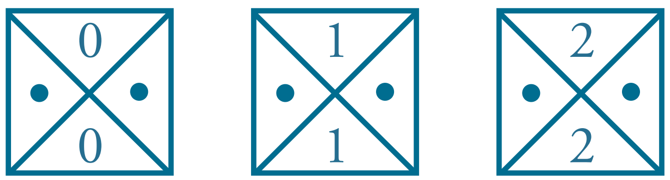
It can easily encode *halting* computations of Turing machines:



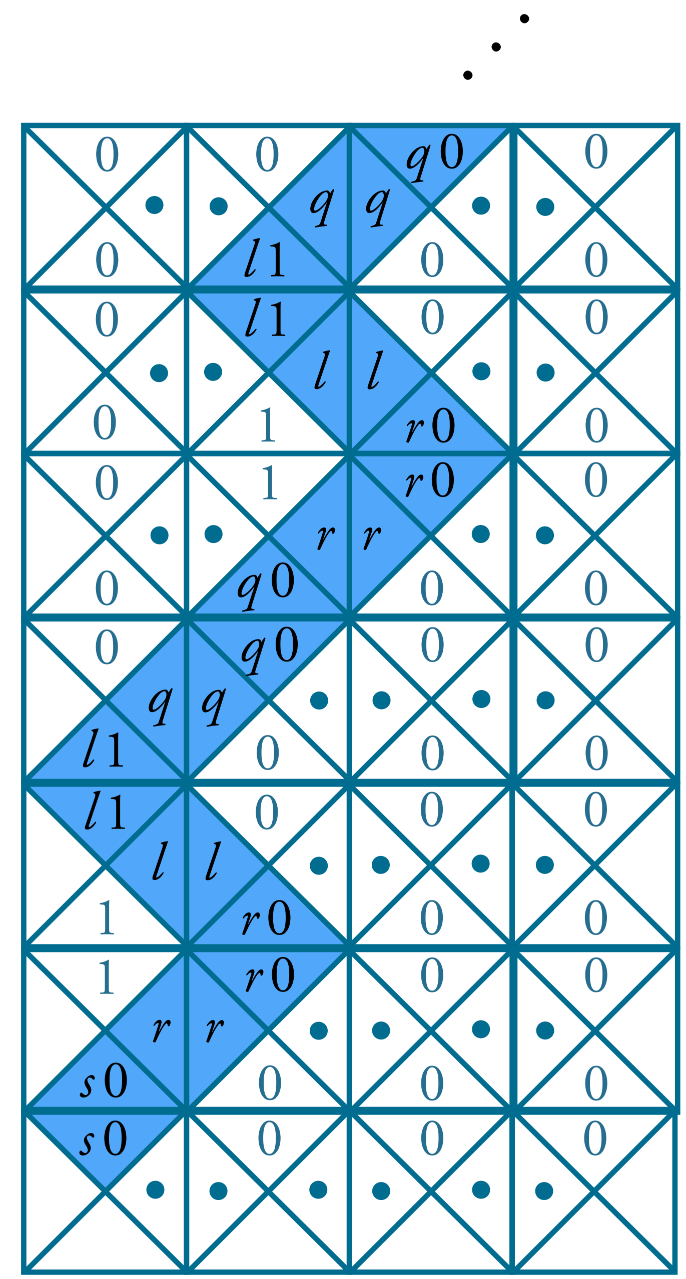
The (undecidable) Domino problem

Domino - Why is it undecidable?

It can easily encode *halting* computations of Turing machines:



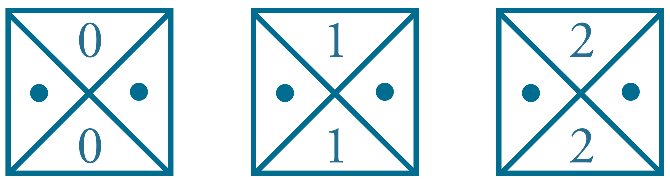
(head is elsewhere,
symbol is not modified)



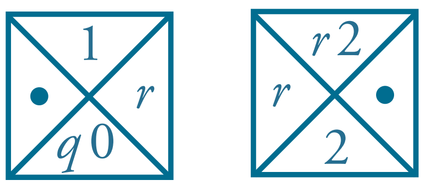
The (undecidable) Domino problem

Domino - Why is it undecidable?

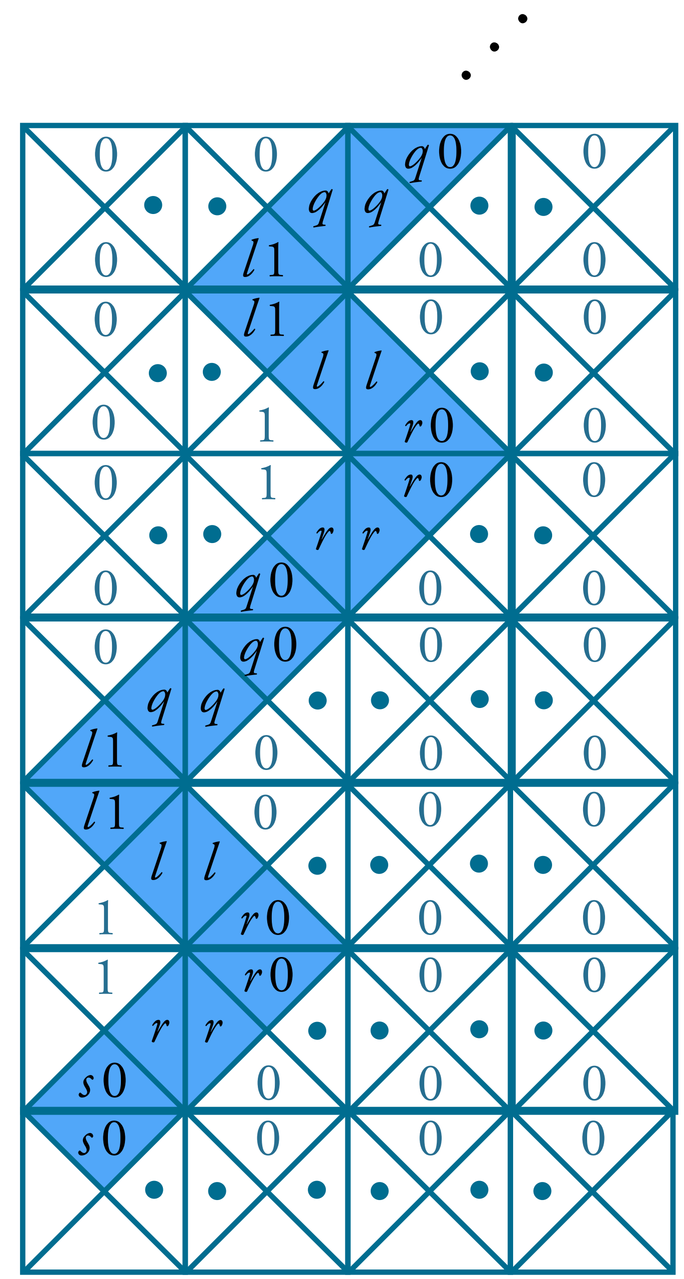
It can easily encode *halting* computations of Turing machines:



(head is elsewhere, symbol is not modified)



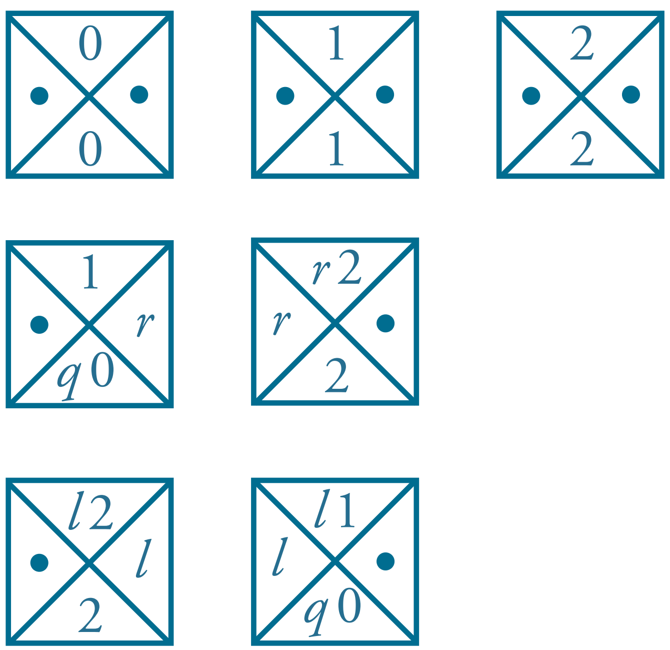
(head is here, symbol is rewritten, head moves right)



The (undecidable) Domino problem

Domino - Why is it undecidable?

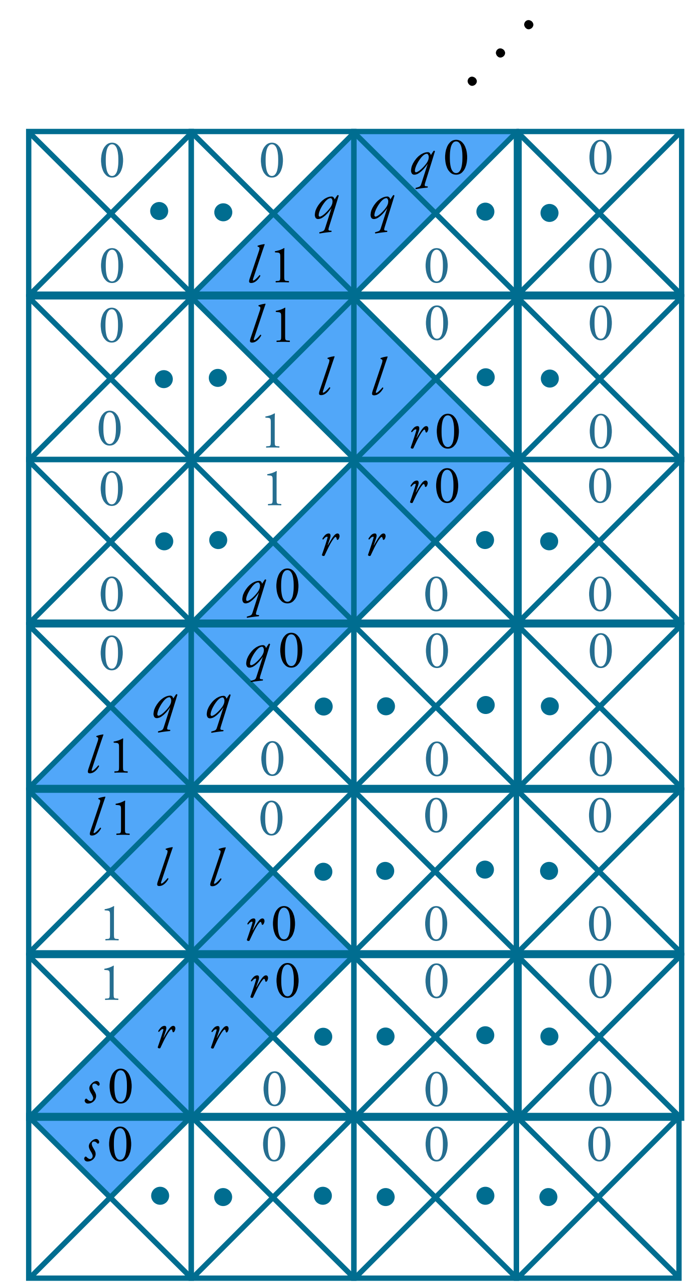
It can easily encode *halting* computations of Turing machines:



(head is elsewhere, symbol is not modified)

(head is here, symbol is rewritten, head moves right)

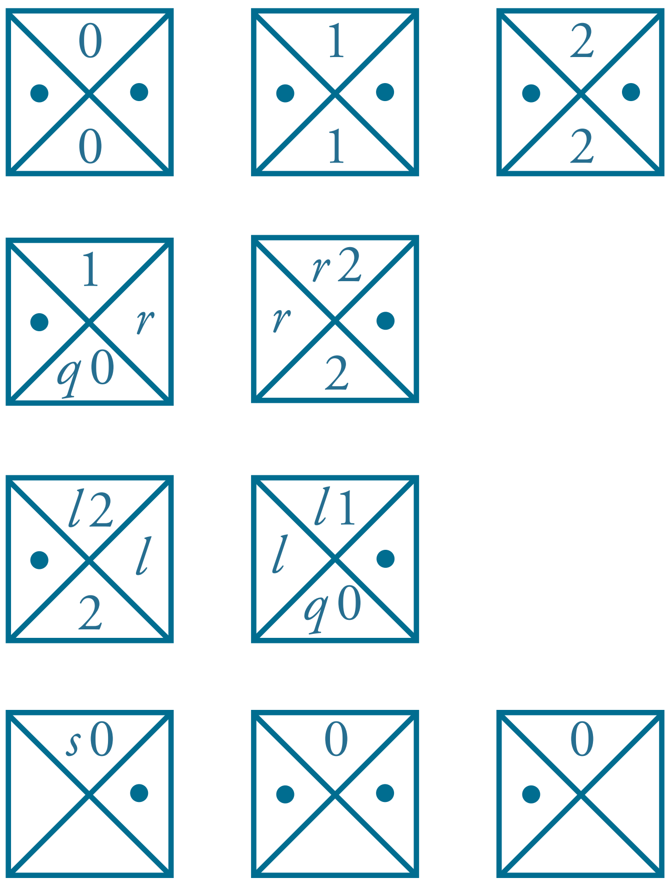
(head is here, symbol is rewritten, head moves left)



The (undecidable) Domino problem

Domino - Why is it undecidable?

It can easily encode *halting* computations of Turing machines:

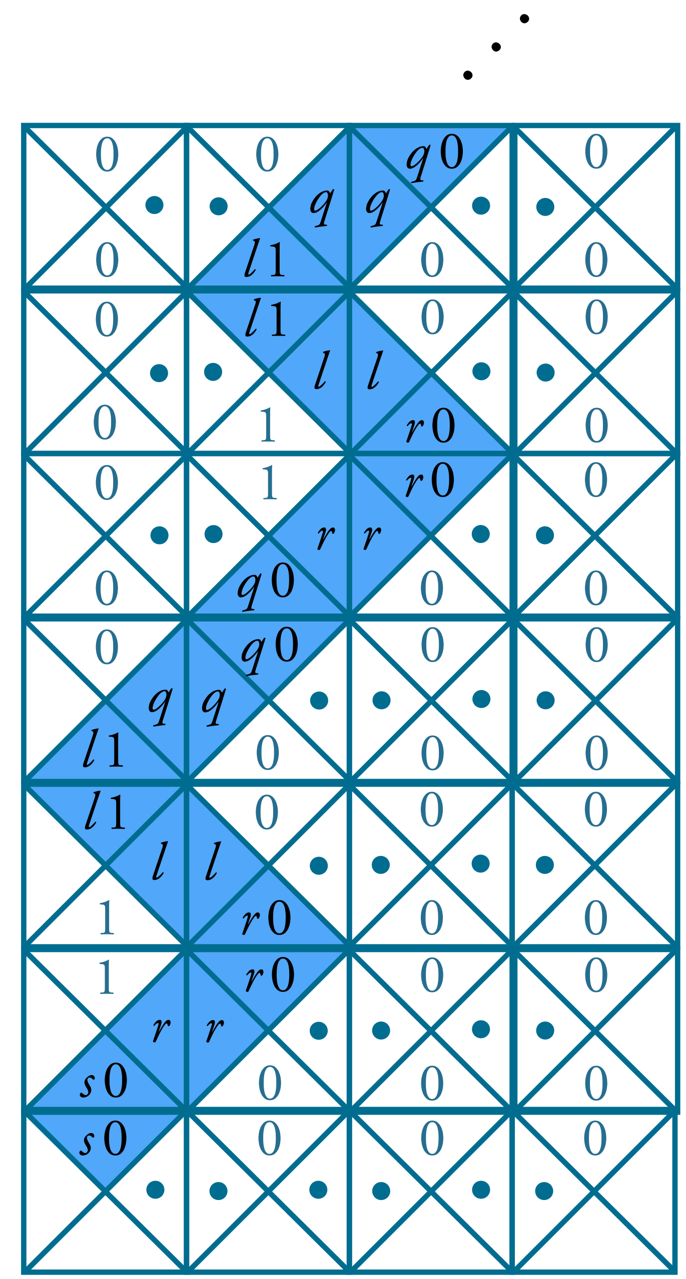


(head is elsewhere, symbol is not modified)

(head is here, symbol is rewritten, head moves right)

(head is here, symbol is rewritten, head moves left)

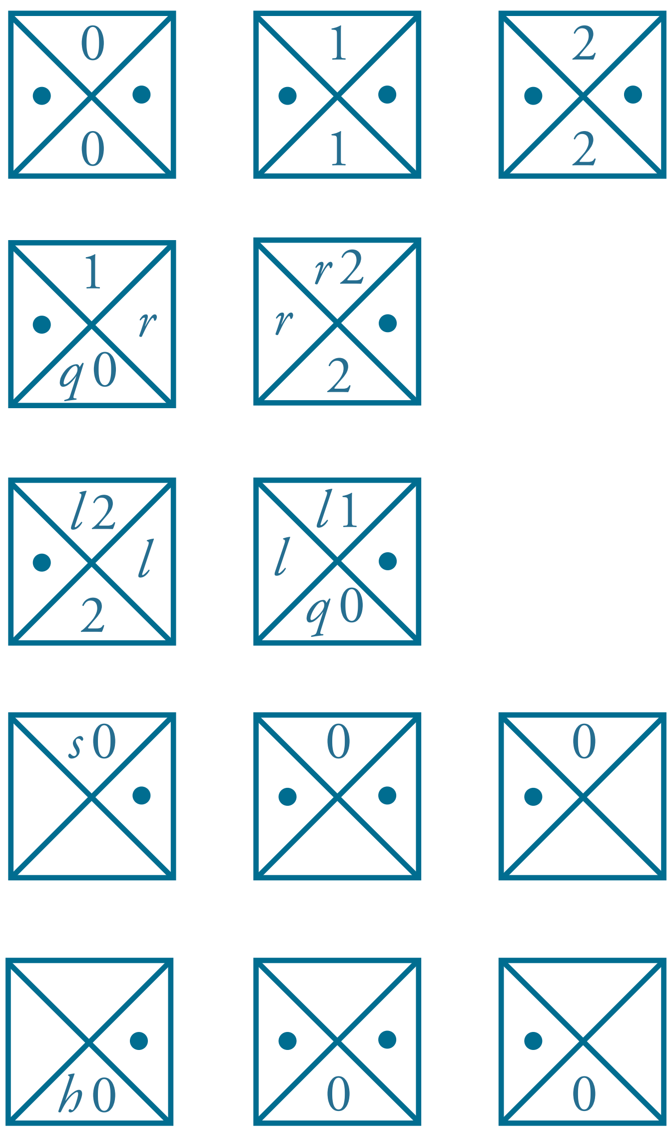
(initial configuration)



The (undecidable) Domino problem

Domino - Why is it undecidable?

It can easily encode *halting* computations of Turing machines:



(head is elsewhere, symbol is not modified)

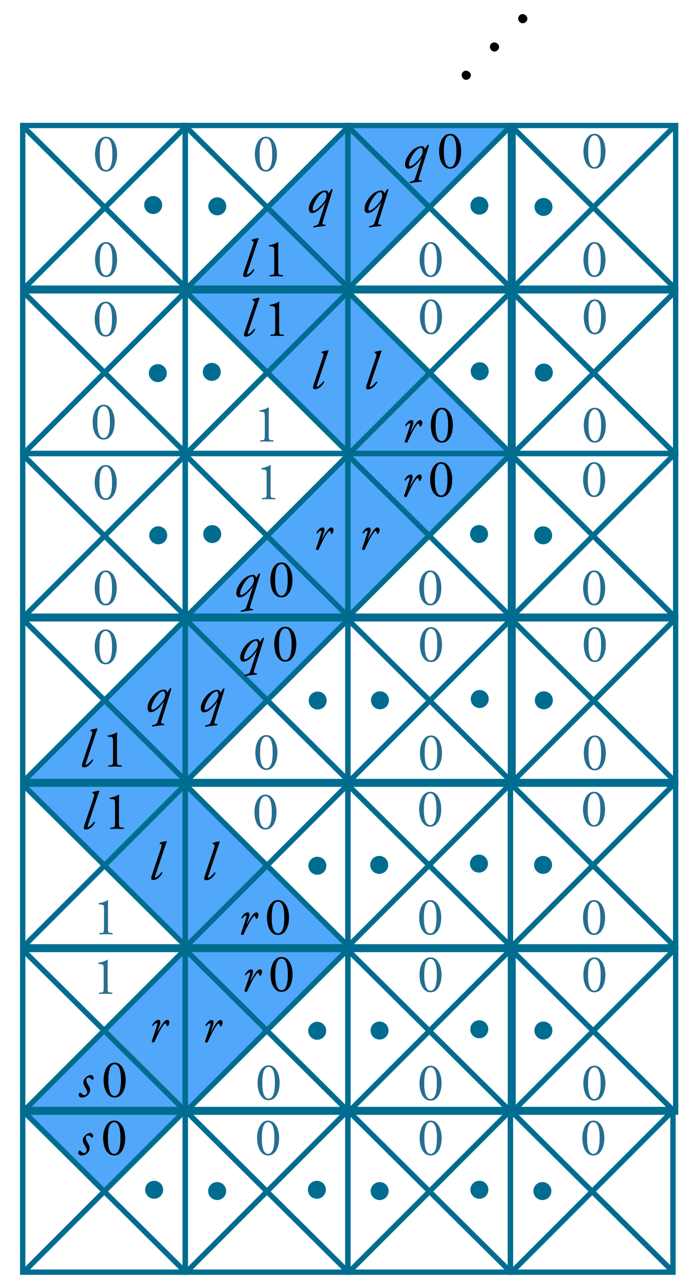
(head is here, symbol is rewritten, head moves right)

(head is here, symbol is rewritten, head moves left)

(initial configuration)

(halting configuration)

. . .

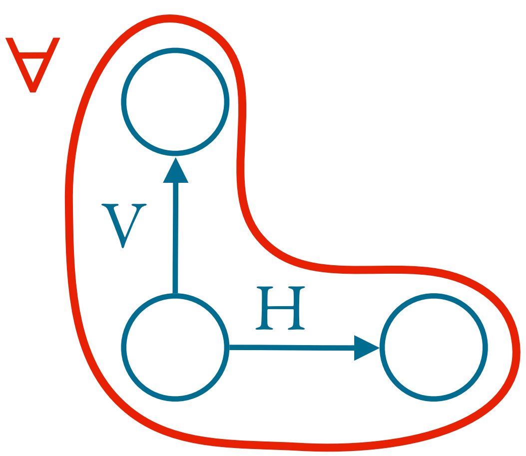


Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** $H(,)$ and $V(,)$ are relations representing bijections such that…

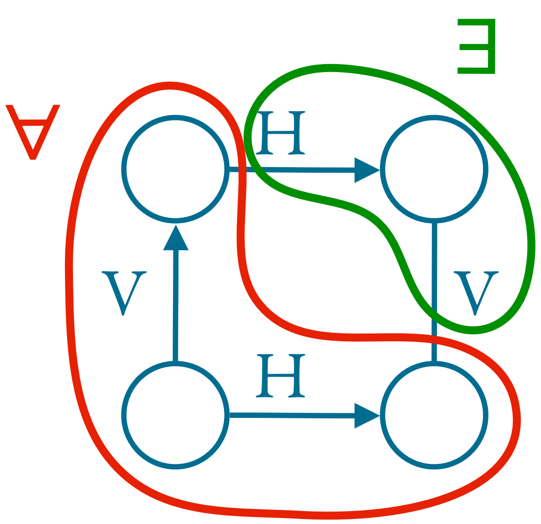
Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…



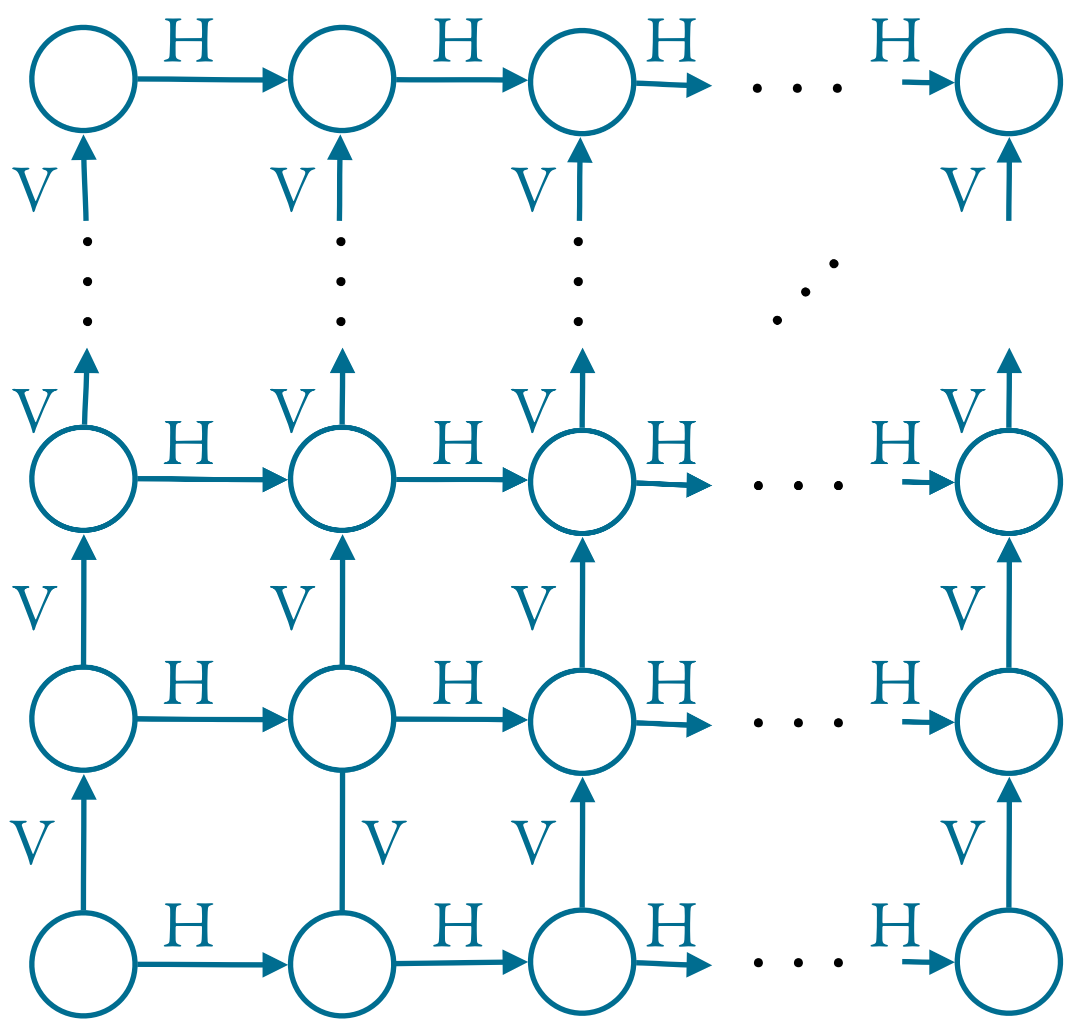
Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…



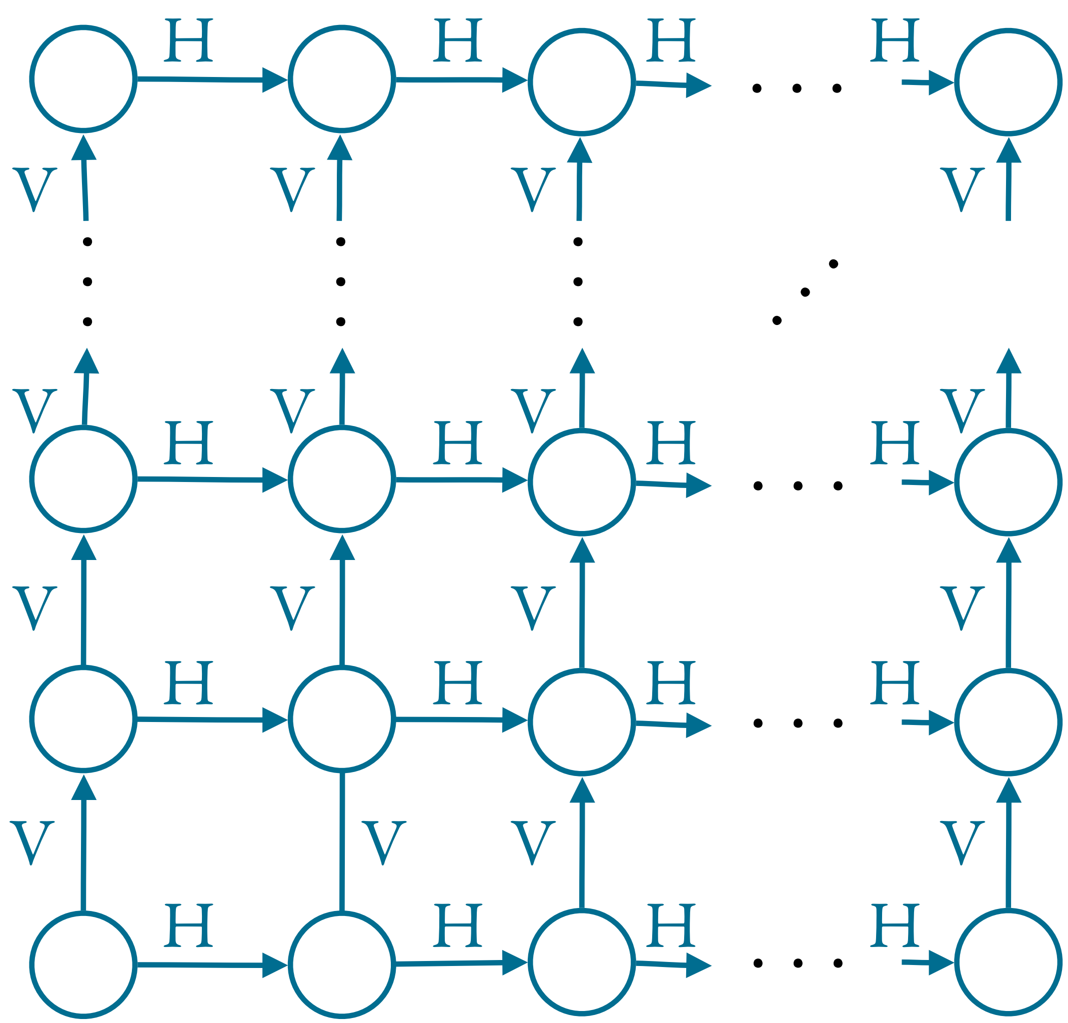
Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…



Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…



2. **Assign one domino to each node:**

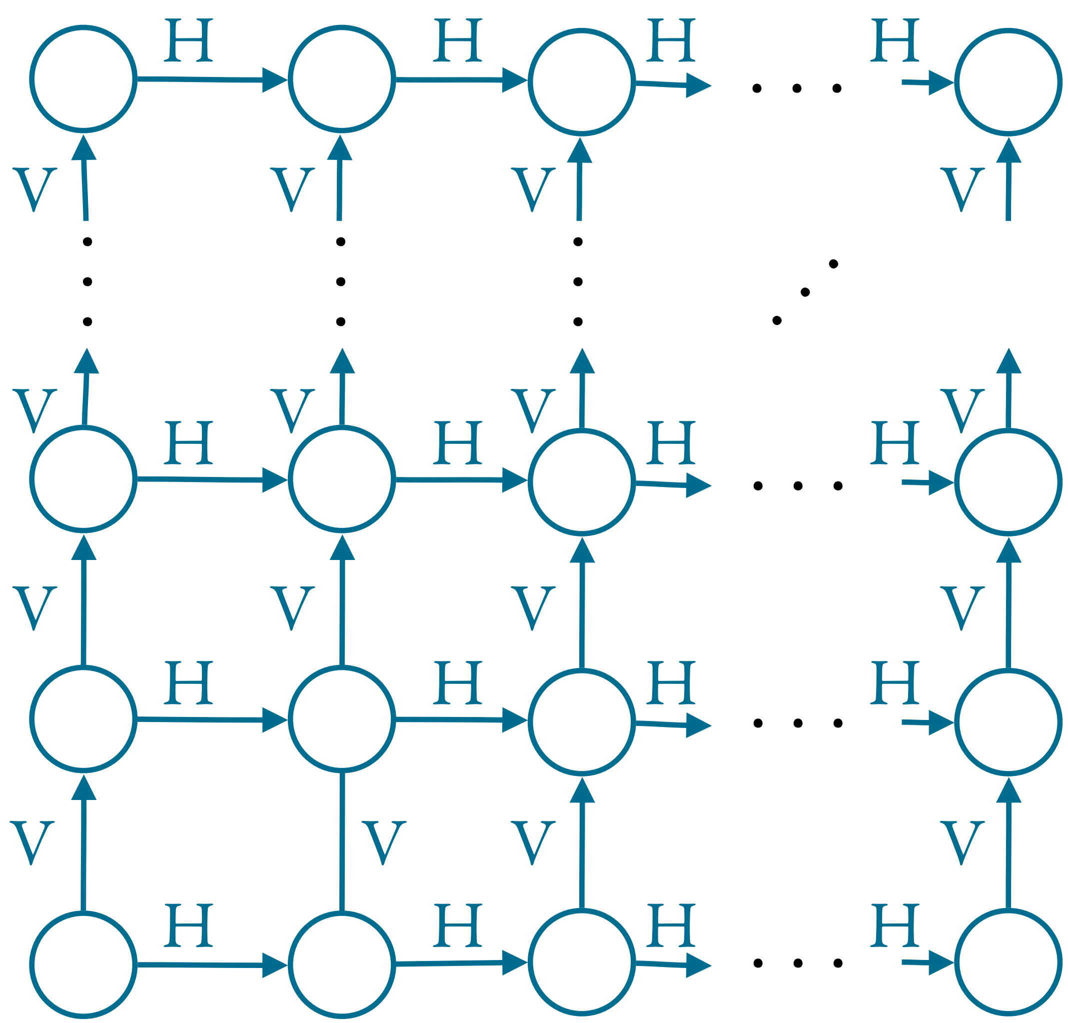
a unary relation

$$\mathbf{D}_{\boxtimes}(\mathbf{x})$$

for each domino $\boxtimes$

Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that...



2. **Assign one domino to each node:**

a unary relation

$$D_{\boxtimes}(x)$$

for each domino ⊠

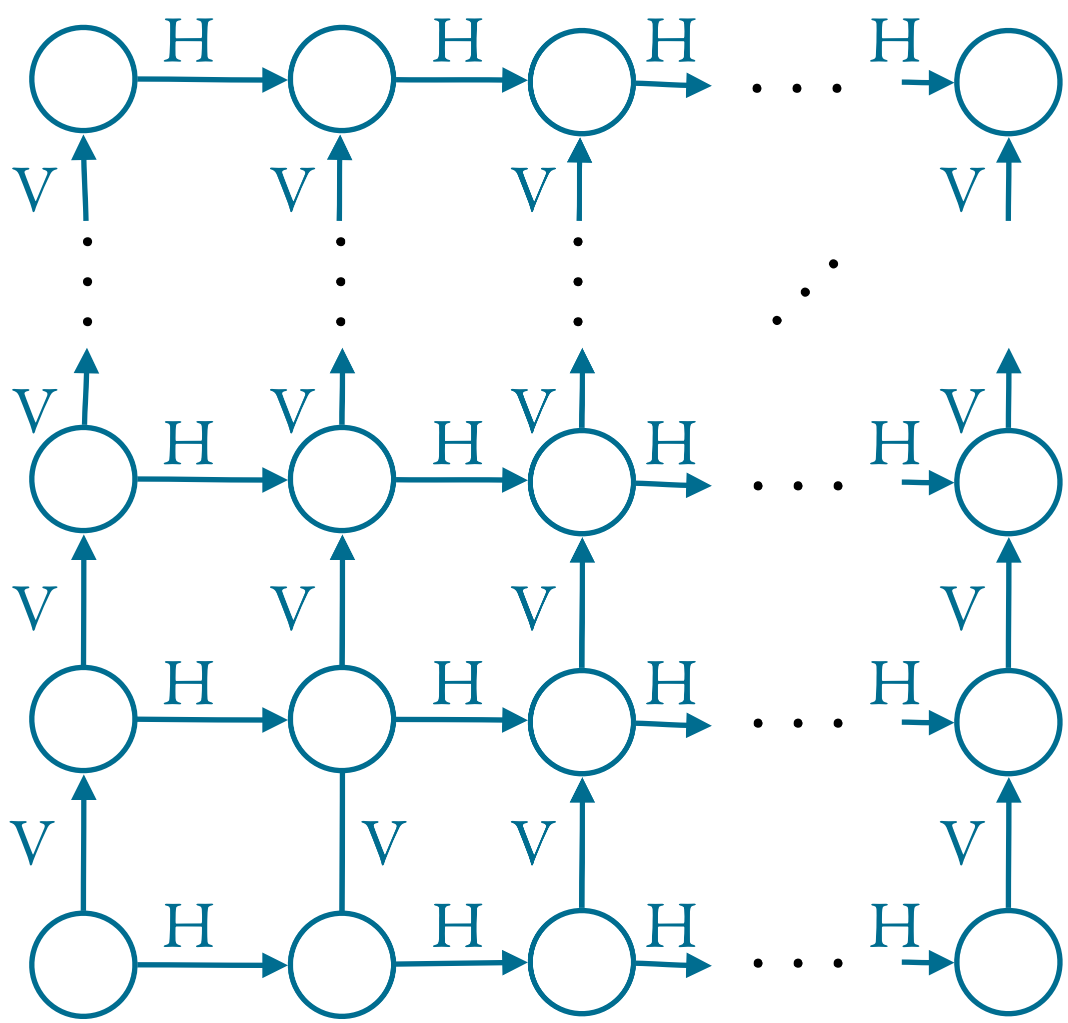
3. **Match the sides** $\forall x,y$

if $H(x,y)$, then $D_a(x) \wedge D_b(y)$

for some dominos **a,b** that 'match' horizontally (Idem vertically)

Domino ⇝ Sat-FO (domino has a solution iff ϕ satisfiable)

1. **There is a grid:** H(,) and V(,) are relations representing bijections such that…



2. **Assign one domino to each node:**

a unary relation

$$D_{\boxtimes}(x)$$

for each domino $\boxtimes$

3. **Match the sides** ∀x,y

if H(x,y), then $D_a(x) \wedge D_b(y)$

for some dominos **a,b** that 'match' horizontally (Idem vertically)

4. **Borders are white.**

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, ..., x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{finite}$

Equivalence problem: Given FO formulae ϕ,ψ, is

$G \models_\alpha \phi$ iff $G \models_\alpha \psi$

for all graphs G and bindings α?

💀 UNDECIDABLE ⇝ by reduction to the satisfiability problem

Algorithmic problems for FO

Equivalence problem: Given FO formulae ϕ,ψ, is

$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

for all graphs G and bindings α?

💀 UNDECIDABLE ⇝ by reduction to the satisfiability problem

Algorithmic problems for FO

ϕ is **satisfiable** iff ϕ is **not equivalent** to $\bot$

Satisfiability problem undecidable $\rightsquigarrow$ Equivalence problem undecidable

Equivalence problem: Given FO formulae ϕ,ψ, is

$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

for all graphs G and bindings α?

💀 UNDECIDABLE $\rightsquigarrow$ by reduction to the satisfiability problem

Algorithmic problems for FO

ϕ is **satisfiable** iff ϕ is **not equivalent** to $\bot$

Satisfiability problem undecidable ⇝ Equivalence problem undecidable

Actually, there are reductions in both senses:

$\phi(x_1,\ldots,x_n)$ and $\psi(y_1,\ldots,y_m)$ are **equivalent** iff

- $n=m$
- $(x_1=y_1) \wedge \cdots \wedge (x_n=y_n) \wedge \phi(x_1,\ldots,x_n) \wedge \neg\psi(y_1,\ldots,y_n)$ is unsatisfiable
- $(x_1=y_1) \wedge \cdots \wedge (x_n=y_n) \wedge \psi(x_1,\ldots,x_n) \wedge \neg\phi(y_1,\ldots,y_n)$ is unsatisfiable

Equivalence problem: Given FO formulae ϕ,ψ, is

$$G \models_\alpha \phi \quad \text{iff} \quad G \models_\alpha \psi$$

for all graphs G and bindings α?

💀 UNDECIDABLE ⇝ by reduction to the satisfiability problem

Algorithmic problems for FO

Evaluation problem: Given a FO formula $\phi(x_1, \ldots, x_n)$, a graph G, and a binding α, does $G \models_\alpha \phi$?

DECIDABLE ⇝ foundations of the database industry

Satisfiability problem: Given a FO formula ϕ, is there a graph G and binding α, such that $G \models_\alpha \phi$?

💀 UNDECIDABLE ⇝ both for $\models$ and $\models_{\text{finite}}$

Equivalence problem: Given FO formulae ϕ,ψ, is

$G \models_\alpha \phi$ iff $G \models_\alpha \psi$

for all graphs G and bindings α?

💀 UNDECIDABLE ⇝ by reduction to the satisfiability problem

Evaluation problem for FO

Input:

$$\left(\begin{array}{l} \phi(x_1,\dots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\dots,x_n\} \longrightarrow V \end{array} \right.$$

Output: $G \models_\alpha \phi$?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\dots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\dots,x_n\} \longrightarrow V \end{cases}$

Output: $G \models_\alpha \phi$?

Encoding of $G = (V, E)$

- each node is coded with a bit string of size $\log(|V|)$,
- edge set is encoded by its tuples, e.g. (100,101), (010, 010), ...

Cost of coding: $||G|| = |E| \cdot 2 \cdot \log(|V|) \approx |V|$ (mod a polynomial)

Evaluation problem for FO

Input:
$$\begin{cases} \phi(x_1,\dots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\dots,x_n\} \longrightarrow V \end{cases}$$

Output: $G \models_\alpha \phi$?

Encoding of $G = (V, E)$

- each node is coded with a bit string of size $\log(|V|)$,
- edge set is encoded by its tuples, e.g. (100,101), (010, 010), …

Cost of coding: $||G|| = |E| \cdot 2 \cdot \log(|V|) \approx |V|$ (mod a polynomial)

Encoding of $\alpha = \{x_1,\dots,x_n\} \longrightarrow V$

- each node is coded with a bit string of size $\log(|V|)$,

Cost of coding: $||\alpha|| = n \cdot \log(|V|)$

Evaluation problem for FO

Input: $\left(\begin{array}{l} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{array} \right.$

Output: $G \models_\alpha \phi$?

Evaluation problem for FO

Input: $\left(\begin{array}{l} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{array} \right.$

Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \land \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$

Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
- If $\phi(x_1,\ldots,x_n) = \exists y . \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$ Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

  use 4 pointers ⇝ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$

- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$ Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
  use 4 pointers $\rightsquigarrow$ LOGSPACE
- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
  $\rightsquigarrow$ MAX(SPACE($G \models_\alpha \psi$), SPACE($G \models_\alpha \psi'$))
- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
- If $\phi(x_1,\ldots,x_n) = \exists y . \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$ Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
  use 4 pointers ⇝ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
  ⇝ MAX(SPACE($G \models_\alpha \psi$)), SPACE($G \models_\alpha \psi'$)))

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
  ⇝ SPACE($G \models_\alpha \psi$))

- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\left(\begin{array}{l} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{array} \right.$

Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$

  use 4 pointers $\rightsquigarrow$ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$

  $\rightsquigarrow$ MAX(SPACE($G \models_\alpha \psi$)), SPACE($G \models_\alpha \psi'$)))

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$

  $\rightsquigarrow$ SPACE($G \models_\alpha \psi$))

- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.

  $\rightsquigarrow$ $2 \cdot \log(|G|)$ + SPACE($G \models_{\alpha'} \psi$)

Question:
How much space
does it take?

Evaluation problem for FO

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$ Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
  use 4 pointers ⇝ LOGSPACE
- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
  ⇝ MAX(SPACE($G \models_\alpha \psi$)), SPACE($G \models_\alpha \psi'$)))
- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
  ⇝ SPACE($G \models_\alpha \psi$))
- If $\phi(x_1,\ldots,x_n) = \exists y \,.\, \psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$
  we have $G \models_{\alpha'} \psi$.
  ⇝ $2 \cdot \log(|G|)$ + SPACE($G \models_{\alpha'} \psi$)

Question:
How much space does it take?

$\underbrace{2\cdot\log(|G|) + \cdots + 2\cdot\log(|G|)}_{\leq |\phi| \text{ times}} + k\cdot\log(|\alpha|+|G|)$ space

Evaluation problem for FO in PSPACE

Input: $\begin{cases} \phi(x_1,\ldots,x_n) \\ G = (V,E) \\ \alpha = \{x_1,\ldots,x_n\} \longrightarrow V \end{cases}$

Output: $G \models_\alpha \phi$?

- If $\phi(x_1,\ldots,x_n) = E(x_i,x_j)$:
  answer YES iff $(\alpha(x_i),\alpha(x_j)) \in E$
  use 4 pointers ⇝ LOGSPACE

- If $\phi(x_1,\ldots,x_n) = \psi(x_1,\ldots,x_n) \wedge \psi'(x_1,\ldots,x_n)$:
  answer YES iff $G \models_\alpha \psi$ and $G \models_\alpha \psi'$
  ⇝ $\text{MAX}(\text{SPACE}(G \models_\alpha \psi)), \text{SPACE}(G \models_\alpha \psi'))\ )$

- If $\phi(x_1,\ldots,x_n) = \neg\psi(x_1,\ldots,x_n)$:
  answer NO iff $G \models_\alpha \psi$
  ⇝ $\text{SPACE}(G \models_\alpha \psi))$

- If $\phi(x_1,\ldots,x_n) = \exists y\,.\,\psi(x_1,\ldots,x_n,y)$:
  answer YES iff for some $v \in V$ and $\alpha' = \alpha \cup \{y \mapsto v\}$ we have $G \models_{\alpha'} \psi$.
  ⇝ $2 \cdot \log(|G|) + \text{SPACE}(G \models_{\alpha'} \psi\ )$

Question:
How much space does it take?

$$\underbrace{2 \cdot \log(|G|) + \cdots + 2 \cdot \log(|G|)}_{\leq |\phi| \text{ times}} + k \cdot \log(|\alpha|+|G|) \text{ space}$$

Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A database of size 10^6

A query of size 100

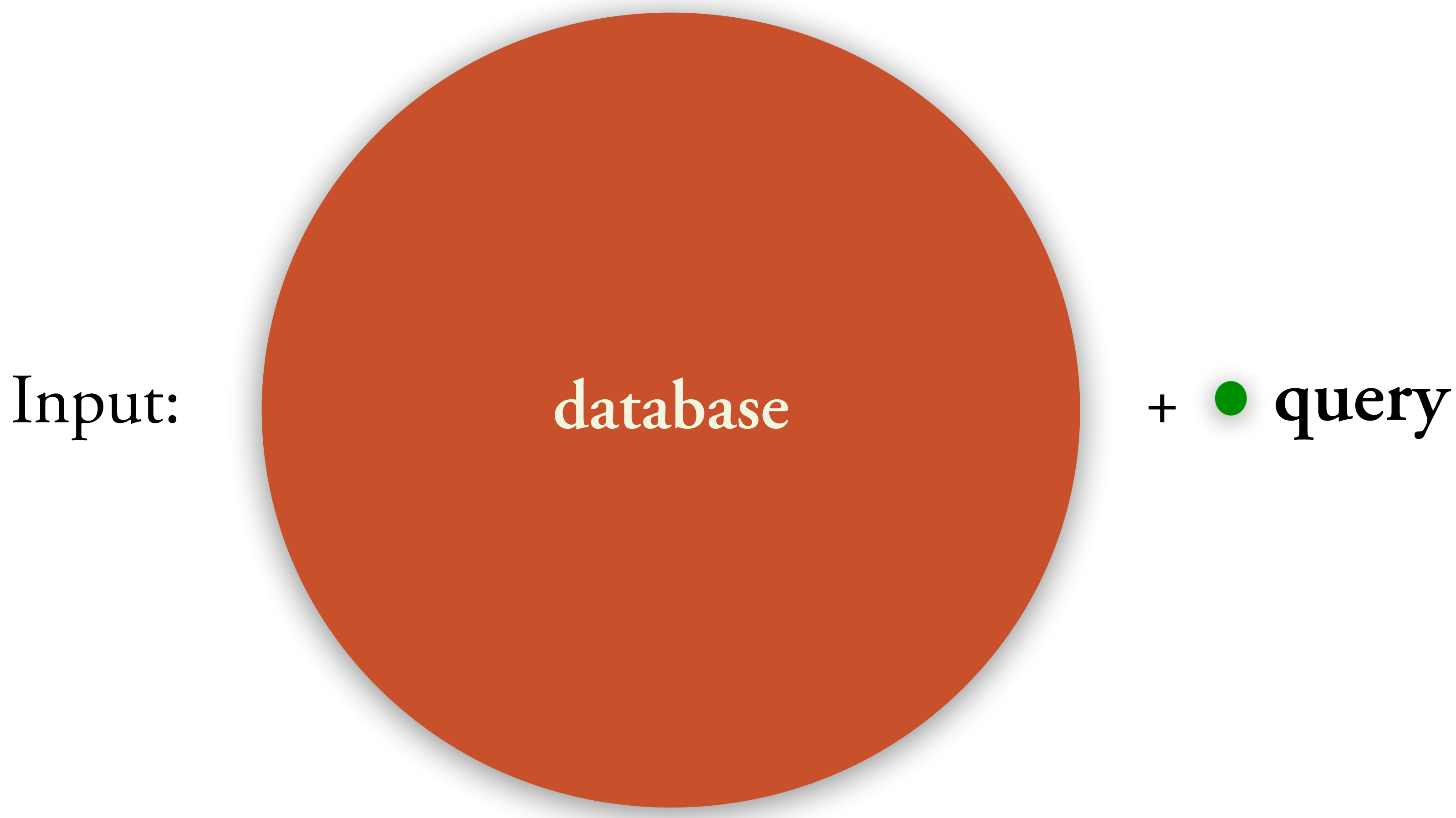
Combined, Query, and Data complexities

[Vardi, 1982]

Problem: Usual scenario in database

A **database** of size 10^6

A **query** of size 100



Combined, Query, and Data complexities

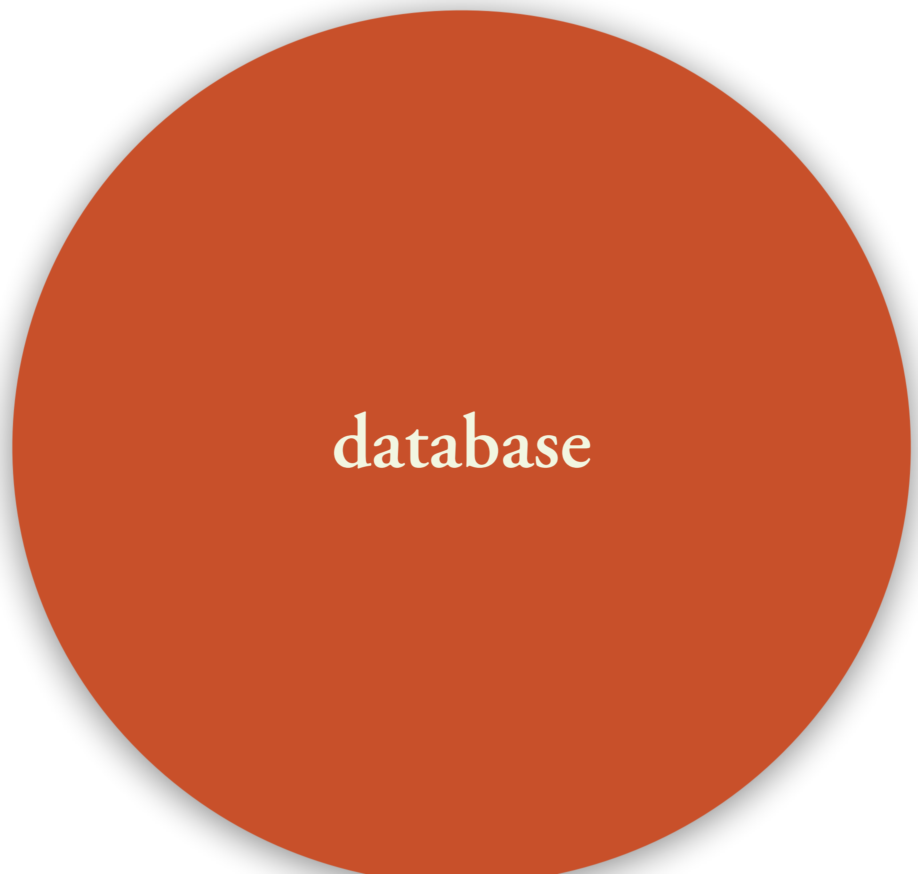
[Vardi, 1982]

Problem: Usual scenario in database

A database of size 10^6

A query of size 100

Input:



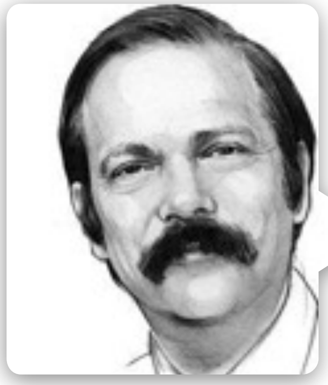
+ • **query**

But we don't distinguish this in the analysis:

$$\mathrm{TIME}(2^{|\text{query}|} + |\text{data}|) = \mathrm{TIME}(|\text{query}| + 2^{|\text{data}|})$$

Combined, Query, and Data complexities

[Vardi, 1982]



Query and data play very **different roles.**

Separation of concerns: How the resources grow with respect to

- the size of the data
- the query size

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed): input size is |query|

Data complexity (|query| fixed): input size is |data|

Combined, Query, and Data complexities

Combined complexity: input size is |query| + |data|

Query complexity (|data| fixed): input size is |query|

Data complexity (|query| fixed): input size is |data|

$O(2^{|query|} + |data|)$ is
- exponential in **combined** complexity
- exponential in **query** complexity
- linear in **data** complexity

$O(|query| + 2^{|data|})$ is
- exponential in **combined** complexity
- linear in **query** complexity
- exponential in **data** complexity

Question

What is the data, query and combined complexity for the evaluation problem for FO?

Remember:

data complexity, input size: |data|

query complexity, input size: |query|

combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha|+|G|)$ space

Question

What is the data, query and combined complexity
for the evaluation problem for FO?

Remember: **data** complexity, input size: |data|

query complexity, input size: |query|

combined complexity, input size: |data| + |query|

$|\phi| \cdot 2 \cdot \log(|G|) + k \cdot \log(|\alpha|+|G|)$ space

O(log(|data|)·|query|) space

PSPACE combined and query complexity

LOGSPACE data complexity

Evaluation pb for FO is PSPACE-complete (combined complexity)

PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

Evaluation pb for FO is PSPACE-complete (combined complexity)

PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

$\exists p\ \forall q\ .\ (p \vee \neg q)$ where p,q range over {T,F}

Evaluation pb for FO is PSPACE-complete

(combined complexity)

PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

$\exists p \, \forall q \, . \, (p \lor \neg q)$ where p,q range over {T,F}

Theorem: Evaluation for FO is PSPACE-complete (combined c.)

Evaluation pb for FO is PSPACE-complete (combined complexity)

PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

$\exists p \, \forall q \, . \, (p \vee \neg q)$ where p,q range over {T,F}

Theorem: Evaluation for FO is PSPACE-complete (combined c.)

Polynomial reduction **QBF** $\rightsquigarrow$ FO :

1. Given $\psi \in$ QBF, let $\psi'(x)$ be the replacement of each 'p' with 'p=x' in ψ.

2. Note: $\exists x \, \psi'$ holds in a 2-element graph iff ψ is QBF-satisfiable

3. Test if $G \models_{\emptyset} \psi'$ for G=({v,v'},{})

Evaluation pb for FO is PSPACE-complete (combined complexity)

PSPACE-complete problem: **QBF**

(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

$\exists p\ \forall q\ .\ (p \lor \neg q)$ where p,q range over {T,F}

Theorem: Evaluation for FO is PSPACE-complete (combined c.)

Polynomial reduction **QBF** ⇝ FO :

$\psi'(x) = \exists p\ \forall q\ .\ (\ (p=x) \lor \neg(q=x)\ )$

1. Given $\psi \in$ QBF,
let $\psi'(x)$ be the replacement
of each 'p' with 'p=x' in ψ.

2. Note: $\exists x\ \psi'$ holds in a 2-element graph iff ψ is QBF-satisfiable

3. Test if $G \models_{\emptyset} \psi'$ for G=({v,v'},{})

Evaluation pb for FO is PSPACE-complete (combined complexity)

PSPACE-complete problem: **QBF**
(satisfaction of Quantified Boolean Formulas)

QBF = a boolean formula with quantification over the truth values (T,F)

$\exists p\ \forall q\ .\ (p \lor \neg q)$ where p,q range over {T,F}

Theorem: Evaluation for FO is PSPACE-complete (combined c.)

Polynomial reduction **QBF** $\rightsquigarrow$ FO :

$\psi'(x) = \exists p\ \forall q\ .\ (\ (p=x) \lor \neg(q=x)\ )$

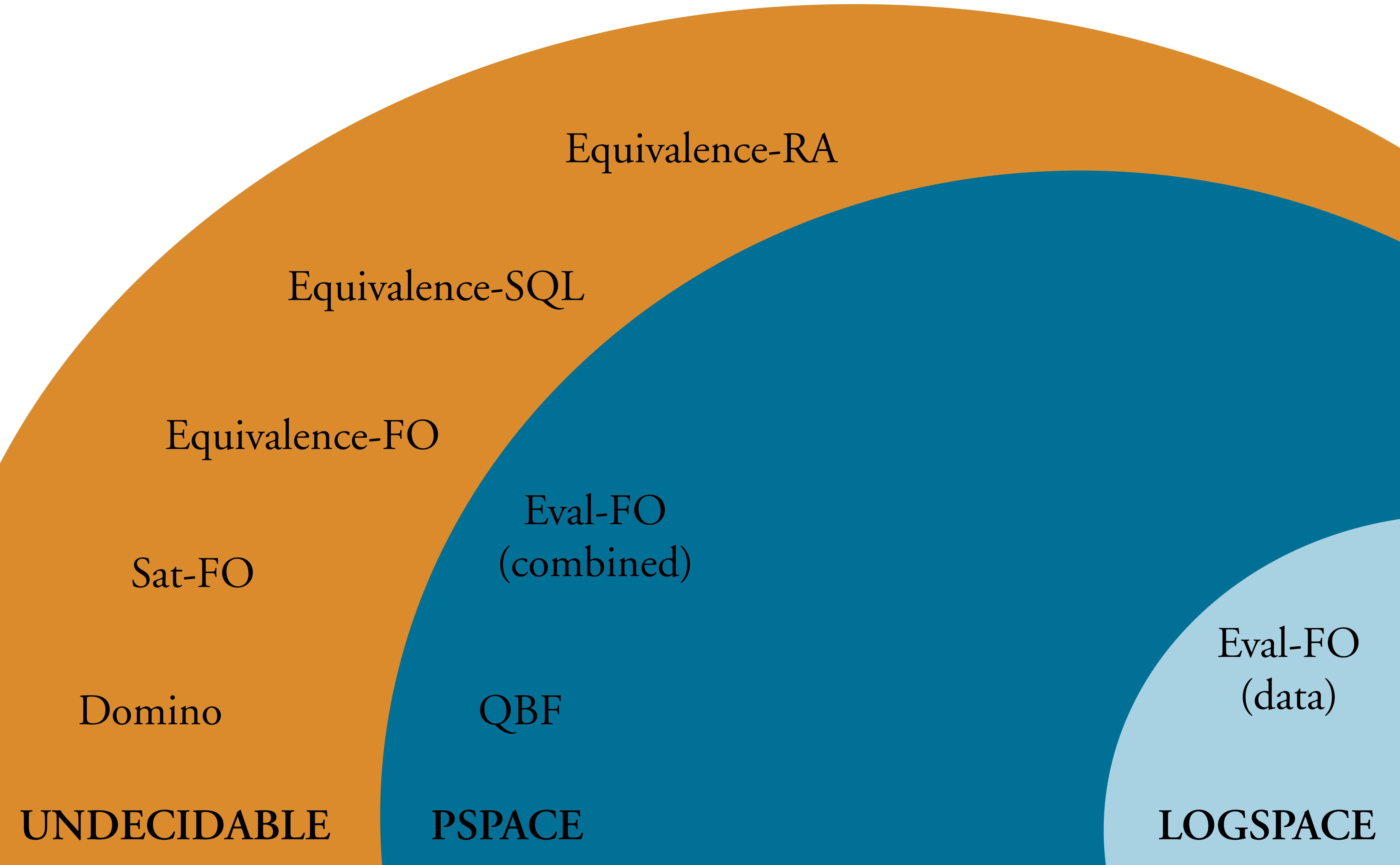
$\exists x\ \exists p\ \forall q\ .\ (\ (p=x) \lor \neg(q=x)\ )$

1. Given $\psi \in$ QBF, let $\psi'(x)$ be the replacement of each 'p' with 'p=x' in ψ.

2. Note: $\exists x\, \psi'$ holds in a 2-element graph iff ψ is QBF-satisfiable

3. Test if $G \models_{\emptyset} \psi'$ for $G=(\{v,v'\},\{\})$

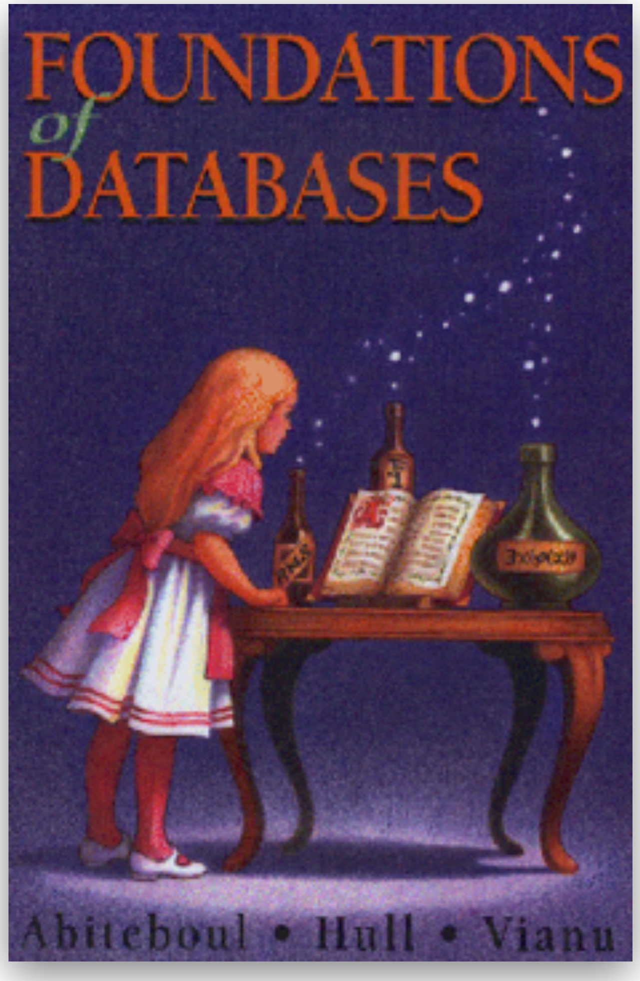
Recap


Equivalence-RA
Equivalence-SQL
Equivalence-FO
Eval-FO
(combined)
Sat-FO
Eval-FO
(data)
Domino
QBF
UNDECIDABLE
PSPACE
LOGSPACE

Bibliography

Abiteboul, Hull, Vianu, “Foundations of Databases”, Addison-Wesley, 1995.

(freely available at http://webdam.inria.fr/Alice/)



Chapters 1, 2, 3