Operations, Collections and Sets within a General Axiomatic Framework*

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dedicated to the memory of Ennio De Giorgi
"There are more things in heaven and earth
... than are dreamt of in your philosophy"

Introduction

This paper is part of a general research programme on the Foundations of Mathematics, Logic and Computer Science, carried out since the early eighties at the the Seminar directed by Ennio De Giorgi at the Scuola Normale Superiore in Pisa. In this context, Foundations are not intended to provide *safe and unquestionable grounds* to scientific activity, but rather to provide *conceptual environments* where this activity can be carried out naturally and without artificial constraints. Earlier proposals of such foundational theories appear in [3, 1] (see also [11]). Further investigations, along theses lines, have been carried out by various mathematicians, logicians and computer scientists since Spring 1994, starting from the "Basic Theories" introduced in [4] (see e.g. [17, 6, 5, 7, 12, 13]).

The principles which inspire and inform the foundational programme of E. De Giorgi are:

- *non-reductionism*: the fact that there are many kinds of "qualitatively different" objects and concepts should be taken seriously. All codings are ultimately artificial and opaque, and they can undermine the conceptual clarity of the system. In particular, blurring the distinction between *natural mathematical concepts* and their usual set-theoretic implementations, e.g. reducing *natural numbers* to *Von Neumann ordinals*, *ordered pairs* to *Kuratowski doubletons*, *binary relations* and *simple operations* to *graphs* makes it difficult, even impossible at times, to formulate appropriate axioms and conjectures. The intuitive notion of *operation* brings about the *non-extensional* concept of computation process, and so operations cannot be accounted for only in terms of graphs. Similarly, conceiving *collections*

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as *truth-valued operations* forces unnecessary commitments on the definition of collection, and yet it does not make apparent the crucial property of collections which is *extensionality.* Taking *natural numbers* as primitives saves us from having to fix priorities among different implementations, e.g. Frege-Russell cardinals, Von Neumann Ordinals, Church Numerals, etc..

- *selfdescription*: the most relevant operations and relations which a foundational theory utilizes should themselves be objects of the theory. The principal relations and operations on collections (*membership, inclusion, union, relative complement, cartesian product*, etc.) should be introduced, as "first class" objects; similarly *application*, *abstraction* and *composition* of operations should themselves be operations. Moreover, various *qualities* should be introduced in order to classify the different *species* of objects, including operations,collections and qualities themselves. al theory should be open to extensions. The introduction of qualitatively new notions, both of mathematical character and of other kind, should be always possible. A foundational theory should be a framework suitable for accommodating most of classical and modern theories arising in Mathematics, Logic, Computer Science, and possibly other sciences (Economics, Linguistics, etc.). Any sufficiently clear concept should be "engraftable" (*innestabile*) in a natural way in it. For example, in [6, 8], metamathematical notions, such as *formula, proposition*, and *interpretation* are engrafted in the open foundational theories of [4] and [7] respectively, by introducing suitable kinds of objects together with relations and operations involving them. In [5], the concept of *variable* of classical Mathematical Physics and of Economics is engrafted, in the same style, in the theory of [4]. Following [7], this *open* character of foundational theories can be epitomized, by the words of Hamlet: "There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy" (Hamlet, Act I, Sc.V, vv.166-167).

- *semiformal axiomatization*: a foundational theory should be expounded in a rigorous yet informal style and investigated using the axiomatic method of traditional mathematics. Although many first-order formalizations can be given (e.g. [15]) none can be satisfactorily taken as definitive. In fact, any formalization misses part of the intended meaning of the original theory. Moreover, in order to keep the metatheoretic requirements to a minimum, a foundational theory should be finitely axiomatizable.

The present paper focuses on the concepts of *collection*, *correlation* (or *correspondence*), *set*, *operation* and *function.* As it is to be expected, many interesting alternatives arise in the process of axiomatizing these concepts. Besides giving the basic list of axioms which capture the traditional concepts of classes, functions and sets, we propose also several attractive strengthenings. Correspondingly, interesting problems of relative consistency also arise.

The paper is organized as follows. In Section 1, we introduce the first fundamental concepts of the theory, namely *qualities* and *relations*, as in [7]. In Section 2 we introduce the concept of *operation* and develop a simplified version of the theory Oper of [13]. In Section 3 we develop the general theory of *collections* and *correlations*. In Section 4 we deal with the concepts of *set* and *function*. The theory presented in Sections 3-4 is a reformulation of the theory TBCN of [12], within the axiomatic framework of Section 1-2. Final remarks appear in Section 5. In the Appendix, we outline the construction of models of the whole theory axiomatized in Sections 1-4, which is thus proved to be relatively consistent with respect to ZF.

The general approach of this paper originates with Ennio De Giorgi (see [2]). The authors are grateful for many useful discussions to all the participants in the Seminar on the Foundations of Mathematics at the Scuola Normale Superiore in Pisa, and in particular to Ennio De Giorgi, Giacomo Lenzi and Vincenzo M. Tortorelli. In the spirit of all the papers arising from the foundational research originated by Ennio De Giorgi, we hope that also this paper might foster reactions, further contributions or just comments, from researchers in various areas of Logic, Methodology and Philosophy of Science.

1 Fundamental Qualities and Relations

Mathematics, Logic and Computer Science, as well as most disciplines in the Sciences and Humanities, deal, ultimately, with *several qualitatively different* objects and study *relations* over them. It seems therefore appropriate to propose, as starting point, a general axiomatic framework consisting of a few fundamental *qualities* and *relations*. Additional qualities and relations peculiar to each specific field can then be naturally introduced within this framework.

Following [7], we isolate the "pre-mathematical" notions of *quality* (or *property)* and of *binary, ternary, quaternary relation*. Therefore we assume as *primitive*, i.e. not reducible to preceding concepts, the following notions:

- *the object q is a quality*;
- *the object r is a binary relation*;
- *the object s is a ternary relation*;
- *the object t is a quaternary relation*;
- *the object x has the quality q* (written as $q\ x$);
- *the objects x, y are in the binary relation r* (written as $r\ x, y$);
- *the objects x, y, z are in the ternary relation s* (written as $s\ x, y, z$);
- *the objects x, y, z, w are in the quaternary relation t* (written as $t\ x, y, z, w$).

In accordance to the general principle of *selfdescription* we introduce and axiomatize distinguished qualities and relations corresponding to the primitive notions above:[1]

- the *classifying qualities*: *Q_qual, Q_rel_b, Q_rel_t, Q_rel_q*, which correspond to the "reification" of the fundamental kinds of objects considered above. Hence we postulate:

[1]As an aid to recalling the intended meaning of the distinguished objects we use *suggestive names* containing fragments of the corresponding *Latin* word, e.g. qualities have prefix *Q_*, relations *R_*, qualities and relations concerning qualities or relations usually have suffix or infix *qual* and *rel* respectively. Moreover basic operations or collections have upper case initials and objects involving them have suffix or infix *op* or *coll*.

G.1 *Q_qual, Q_rel_b, Q_rel_t, Q_rel_q are qualities:*
1. no object has simultaneously two of the above qualities;
2. x is a quality if and only if Q_qual x;
5. x is a binary relation if and only if Q_rel_b x;
4. x is a ternary relation if and only if Q_rel_t x;
5. x is a quaternary relation if and only if Q_rel_q x.

- the *fundamental relations R_qual, R_rel_b, R_rel_t*, which correspond to the "reification" of the behaviour of qualities and relations. Hence we postulate:

G.2 *R_qual is a binary relation:*
1. if $R\_qual\ x, y$ *then* $Q\_qual\ x$*;*
2. if $Q\_qual\ q$ *then* $R\_qual\ q, x \Longleftrightarrow q\ x$.

G.3 *R_rel_b is a ternary relation:*
1. if $R\_rel\_b\ x, y, z$ *then* $Qrel\_b\ x$*;*
2. if $Q\_rel\_b\ r$ *then* $R\_rel\_b\ r, x, y \Longleftrightarrow r\ x, y$.

G.4 *R_rel_t is a quaternary relation:*
1. if $R\_rel\_t\ x, y, z, w$ *then* $Q\_rel\_t\ x$*;*
2. if $Q\_rel\_t\ s$ *then* $R\_rel\_t\ s, x, y, z \Longleftrightarrow s\ x, y, z$.

We do *not* postulate *extensionality* for qualities and relations. We rather introduce the binary relation of *inclusion R_incl* and that of *extensional equivalence R_ext_eq*, whose behaviours are given by the following axiom:

G.5 *R_incl and R_ext_eq are binary relations:*
1. $\forall xy\ (R\_ext\_eq\ x, y \Longleftrightarrow R\_incl\ x, y \wedge R\_incl\ y, x)$*;*
2. If $Q\_qual\ q$*,* $Q\_qual\ p$ *then* $R\_incl\ p, q \Longleftrightarrow \forall x(q\ x \Longrightarrow p\ x)$*;*
3. If $Q\_rel\_b\ r$*,* $Q\_rel\_b\ s$ *then* $R\_incl\ r, s \Longleftrightarrow \forall xy(s\ x, y \Longrightarrow r\ x, y)$*;*
4. If $Q\_rel\_t\ s$*,* $Q\_rel\_t\ t$ *then* $R\_incl\ s, t \Longleftrightarrow \forall xyz(t\ x, y, z \Longrightarrow s\ x, y, z)$*;*
5. If $Q\_rel\_q\ t$*,* $Q\_rel\_q\ u$ *then* $R\_incl\ t, u \Longleftrightarrow \forall xyzw(u\ x, y, z, w \Rightarrow t\ x, y, z, w)$.

The action of *R_incl* and *R_ext_eq* is not restricted just to qualities and relations. The behaviour of *R_incl* and *R_ext_eq* will be extended step by step, as soon as we will introduce further kinds of objects. We shall adopt the usual set-theoretic notation of inclusion, namely $x \subseteq y$ (or $y \supseteq x$) for $R\_incl\ y, x$, and the notation of [13] for extensional equivalence, namely $x \cong y$ for $R\_ext\_eq\ x, y$.

We introduce also an *identity* relation *R_id* between objects of arbitrary kind:

G.6 *R_id is a binary relation.*
For all x, y $R\_id\ x, y$ *if and only if x and y are the same object.*

Since *R_id* is the *identity relation* between arbitrary objects, we will write $x = y$ in place of $R\_id\ x, y$.

We conclude this Section by introducing the quality *Q_un* of being an *univalent* object, which will be useful in formulating the axioms of the following sections. In the following axiom we consider only *univalent* relations. Univalent objects of other kinds will be axiomatized later.

G.7 *Q_un is a quality:*
1. if $Q\_rel\_b\ r$ *then* $Q\_un\ r \iff \forall xyz\ (r\ x,y\ \wedge\ r\ x,z \implies y = z)$;
2. if $Q\_rel\_t\ s$ *then* $Q\_un\ s \iff \forall xyzw\ (s\ x,y,z\ \wedge\ s\ x,y,w \implies z = w)$;
3. if $Q\_rel\_q\ t$ *then* $Q\_un\ t \iff \forall xyzvw\ (t\ x,y,z,v\ \wedge\ t\ x,y,z,w \implies v = w)$.

2 Operations

We present in this section an axiomatization of the primitive notion of *operation*. It builds solely on the intuition of an operation as an object which *acts* (operates) on one, or two, objects and possibly produces a *result*. This presentation can be viewed as an adaptation of the theory Oper introduced in [13] to the foundational framework outlined in Section 1.

According to the pattern which will be followed methodically in engrafting mathematical and logical concepts in the framework of Section 1, we introduce suitable qualities which classify the objects under consideration and suitable relations which describe their behaviour. Here we introduce the qualities Q_op_s and Q_op_b of being respectively a *unary* and a *binary* operation and the corresponding relations R_op_s and R_op_b. The *functionality* of operations is expressed by postulating that the relations R_op_s and R_op_b are univalent. Hence we postulate:

O.1 *Q_op_s is a quality, R_op_s is a ternary univalent relation.*
1. If $R\_op\_s\ x,y,z$ *then* $Q\_op\_s\ x$;
2. If $Q\_op\_s\ f,\ Q\_op\_s\ g$ *then*
$$R\_incl\ f,g \iff \forall xy\ (R\_op\_s\ g,x,y \implies R\_op\_s\ f,x,y).$$

O.2 *Q_op_b is a quality, R_op_b is a quaternary univalent relation.*
1. If $R\_op\_b\ x,y,z,w$ *then* $Q\_op\_b\ x$;
2. If $Q\_op\_b\ g,\ Q\_op\_b\ h$ *then*
$$R\_incl\ g,h \iff \forall xyz\ (R\_op\_b\ h,x,y,z \implies R\_op\_b\ g,x,y,z).$$

Given the axioms above, we can adopt unambiguously the standard notations:
- if $Q\_op\_s\ f$ then $y = fx$ stands for $R\_op\_s\ f,x,y$;
- if $Q\_op\_b\ g$ then $z = gxy$ stands for $R\_op\_b\ g,x,y,z$.
If there exists an object y such that $y = fx$ we say that "y is the result of f applied to x", that "f is defined at x", or that "fx exists"; moreover we denote such y by fx. If no such y exists, then we say that "f is undefined (or not defined) at x". Similarly, if there exists z such that $z = gxy$ we say that "z is the result of g applied to x,y", that "g is defined at x,y", or that "gxy exists"; moreover, we denote such z by gxy. If no such z exists, we say that "g is undefined (or not defined) at x,y".

We have *not* postulated *extensionality* for operations, since we do not want to rule out the possibility that there exist operations, acting on the *same* objects and giving the *same* result on each object, which are nonetheless *different*.

The fact that we have considered only *simple* and *binary* operations is somewhat arbitrary. Since we intend to focus on the theory of *simple* operations, we could have done away even with *binary* operations, by encoding them as "iterated" unary operations. But we prefer to keep both notions since, in the field of Mathematics, Logic and Computer Science, many operations appear naturally as *binary*: e.g. addition of real numbers, composition of operations, conjunction of formulæ. However, we shall not try to achieve a theory of binary operations in any sense "complete", we shall guarantee only that any binary operation can be reduced to a simple one. On the other hand, we intend to achieve a significant completeness at the level of simple operations. To this end, we introduce the simple operation K (which generates simple constant operations), the binary operations $Comp$ (composition) and S (substitution *à la* Schönfinkel), the simple operation Cur (abstraction *à la* Curry). In order to strengthen the "self-description of operations" we introduce also the binary operation App (application) which amounts to an operational counterpart of the relation R_op_s. We postulate the following:

O.3 *K is a simple operation. For all x, Kx exists and is a simple operation such that $\forall xy.\,(Kx)y = x$.*

O.4 *$Comp$ is a binary operation. If f, g are simple operations, then $Comp\ fg$ exists and is a simple operation such that*

$$\forall xy\ ((Comp\ fg)x = y \iff \exists z\ (gx = z \wedge fz = y)).$$

O.5 *S is a binary operation. If f, g are simple operations, then Sfg exists and is a simple operation such that*

$$\forall xy\ ((Sfg)x = y \iff \exists hz\ (gx = z \wedge fx = h \wedge hz = y)).$$

O.6 *Cur is a unary operation. For all binary operations g, $Cur\ g$ exists and is a simple operation. For all x, $(Cur\ g)x$ exists and is a simple operation such that $\forall xyz\ (((Cur\ g)x)y = z \iff gxy = z)$.*

O.7 *App is a binary operation. If f is a simple operation, then*

$$\forall xy\ (App\ fx = y \iff fx = y).$$

Following [13] we introduce also five operations Eq, Neq, Bun, $Univ$ and $Exist$, inspired by the logical operators of predicate calculus:

O.8 *Eq and Neq are binary operations.*
1. For all x,y,z, $Eq\ xy = z$ if and only if $x = y = z$.
2. For all x,y,z, $Neq\ xy = z$ if and only if $x \neq y$ and $y = z$.

O.9 *Bun is a binary operation. If f, g are simple operations, then $Bun\ fg$ exists and is a binary operation such that*
1. $\forall xy\ ((Bun\ fg)x = y \implies fx = y \vee gx = y)$
2. $\forall x\ (\exists y\ ((Bun\ fg)\ x = y) \iff \exists y\ (fx = y \vee gx = y))$.

O.10 *$Univ$ is a simple operation. If f is a simple operation, then $Univ\ f$ exists and is a simple operation such that $\forall yz\ ((Univ\ f)y = z \iff \forall x.(fx)y = z)$.*

O.11 *Exist is a simple operation. If f is a simple operation, then Exist f exists and is a simple operation such that:*
1. $\forall y\ (\exists z.(Exist\ f)y = z \iff \exists xz.(fx)y = z)$;
2. $\forall yz\ ((Exist\ f)y = z \implies \exists x.(fx)y = z)$.

The Axioms O.1-11 yield a rather rich theory of simple operations. Using simple operations, one can implement a powerful functional programming language as in [13], and show, for instance, the existence of all simple *finite operations*, and that simple operations are closed under *composition, intersection, inversion.* One can also prove that simple operations are a model of the *comprehension* and *selection* schemes of Feferman (see [9]).

3 Collections and Correlations

We present in this section an axiomatization of the primitive notions of *collection* and *correlation* (or *correspondence*). Both notions are "extensional" in nature, in the sense that *extensionally equivalent* collections or correlations are *identical.* Our engrafting of collections is intended to capture the general concept underlying the notions of *class* and *set*, conceived by Frege and Cantor. In ordinary theories of sets and classes, *graphs* (i.e. classes of pairs) play the rôle of relations and operations. In our setting, it seems more appropriate to account for this "binary behaviour" of graphs by introducing *correlations*, a new kind of *binary objects.* In some sense one could draw the "conceptual ratio":

$$\frac{\text{Qualities}}{\text{Collections}} = \frac{\text{Binary Relations}}{\text{Correlations}}.$$

Therefore we introduce the qualities *Q_coll* and *Q_corr* of being respectively a *collection* and a *correlation* (or *correspondence*), together with the relations *R_coll* and *R_corr*. It seems appropriate to introduce also the quality *Q_corr_un* of being a *univalent (*or *functional) correlation*, in order to single out an extensional counterpart of simple operations.

C.1 *Q_coll, Q_corr and Q_corr_un are qualities, R_coll is a binary relation, R_corr is a ternary relation.*
1. If $R\_coll\ x, y$ *then* $Q\_coll\ x$.
2. If $R\_corr\ x, y, z$ *then* $Q\_corr\ x$.
3. $Q\_corr\_un\ x$ *if and only if* $Q\_corr\ x$ *and* $Q\_un\ x$. *Moreover, if* $Q\_corr\ G$, *then* $Q\_un\ G \iff \forall xyz\ (R\_corr\ G, x, y \wedge R\_corr\ G, x, z \implies y = z)$.
4. If C, D *are collections then* $R\_incl\ C, D \iff \forall x(R\_coll\ D, x \implies R\_coll\ C, x)$.
5. If F, G *are correlations then*
$R\_incl\ F, G \iff \forall xy(R\_corr\ G, x, y \implies R\_corr\ F, x, y)$.

We can now give the axiom of extensionality for collections and correlations:

C.2 *If* X, Y *are either collections or correlations, then* $X \cong Y \implies X = Y$.

When C is a collection we use the standard notation $x \in C$ for $R\_coll\ C, x$ and $x \notin C$ if it is not the case that $R\_coll\ C, x$. When G is a correlation we adopt the notation of relations and write $G\ x, y$ for $R\_corr\ G, x, y$; if G is univalent then we use also the "functional notation" and write $G(x) = y$ for $R\_corr\ G, x, y$.

First of all we introduce the *identity correlation* Id:

C.3 *Id is a univalent correlation such that* $\forall xy\ (Id\ (x) = y \Longleftrightarrow x = y)$.

We proceed now to enrich the universe by introducing six *distinguished operations* which produce collections and correlations, so as to obtain the counterparts of Von Neumann-Bernays-Gödel classes.

- The binary operation *Sing_corr* (the generator of *singular correlation*):

C.4 *Sing_corr is a binary operation defined at all objects x,y, and giving as result a correlation such that* $\forall zw\ ((Sing\_corr\ xy)\ z, w \Longleftrightarrow x = z \wedge y = w)$.

The correlation *Sing_corr xy* is denoted by $\binom{x}{y}$, and called the *singular correlation* of *x,y*. Notice that, if G is a correlation, then $\binom{x}{y} \subseteq G \Longleftrightarrow G\ x, y$.

- The simple operation *Inv* (*inversion*):

C.5 *Inv is a simpe operation. If X is a correlation, then Inv X exists and is a correlation such that* $\forall uv\ ((Inv\ X)\ u, v \Longleftrightarrow X\ v, u)$.

- The simple operation *Dom* (*domain*):

C.6 *Dom is a simple operation. If G is a correlation then Dom G exists and is a collection such that* $\forall x\ (x \in Dom\, G \Longleftrightarrow \exists y. G\ x, y)$.

- The binary operation *Diff* (*relative complement*):

C.7 *Diff is a binary operation. If X, Y are both collections, then Diff XY exists and is a collection such that* $\forall t\ (t \in Diff\ X\ Y \Longleftrightarrow t \in X \wedge t \notin Y)$.

- The binary operation *Cart_b* (*binary cartesian product*):

C.8 *Cart_b is a binary operation. If C, D are collections, then Cart_b CD exists and is a correlation such that* $\forall xy\ ((Cart\_b\ CD)\ x, y \Longleftrightarrow x \in C \wedge y \in D)$.

- The binary operation *Trans_b* (*transposition of two correlations*):

C.9 *Trans_b is a binary operation. If F and G are correlations then there exists $Trans\_b\ FG$ and it is a correlation such that*

$$\forall xy\ ((Trans\_b\ FG)\ x, y \Longleftrightarrow \exists zw(F\ x, z \wedge G\ x, w \wedge y = \tbinom{z}{w})).$$

In order to enhance readability, we adopt the usual set-theoretic notation and put: $Inv\ F = F^{-1}$, $Diff\ X\ Y = X \setminus Y$, $Cart\_b\ (X, Y) = X \times Y$. Since the operation $Tras\_b$ corresponds to the operation $*$ of [11], we shall denote $Tras\_b\ FG$ by $F \star G$.

We intend correlations as *mere binary counterparts* of collections, hence we introduce the one-to-one correspondence *Gr_sing* (*singular graph*) between correlations and collections of singular correlations, and the functional correlation *Coll_corr* between collections and correlations:

C.10 *Gr_sing and Coll_corr are univalent correlations.*
1. $X \in Dom\ Gr\_sing \Longleftrightarrow Q\_corr\ X$;
2. $Gr\_sing\ X = Y \Longrightarrow Q\_coll\, Y \ \wedge\ \forall x\ (x \in Y \Longleftrightarrow \exists yz (x = \binom{y}{z} \subseteq X))$.
3. $X \in Dom\ Coll\_corr \Longleftrightarrow Q\_coll\ X$;
4. $Coll\_corr\ X = Y \Longrightarrow Q\_corr\, Y \ \wedge\ \forall xy\ (\binom{x}{y} \subseteq Y \Longleftrightarrow \binom{x}{y} \in X)$.

Some remarks are in order:

i) Axioms C. 3 - 10 yield at once the existence of the *universal collection* $V = Dom\ Id$, of the *empty collection* $\emptyset = V \setminus V$, of the *collection of all collections* $Coll = Dom\ Coll\_corr$, and of the *collection of all correlations* $Corr = Dom\ Gr\_sing$. Moreover there exist the *singleton collection* $\{x\} = Dom\ \binom{x}{y}$ for every object x, and the *intersection* $X \cap Y = X \setminus (X \setminus Y)$ and the *union* $X \cup Y = V \setminus ((V \setminus X) \setminus Y)$ of any pair of collections X, Y. Finally, given a collection C and a correlation G, there exists also the collection *image of C under G* defined by $\widehat{G}(C) = Dom\ (G^{-1} \cap (V \times C))$.
Notice the intriguing chain: $\{V\} \in V \in \{V\} \in Coll \in V \in Coll \in \{Coll\}$.

ii) The existence of the one-to-one correspondence *Gr_sing* implies that also correlations are closed under *intersection, union* and *difference.*

iii) The operation $Trans\_b$ allows to define the functional correlations *first* and *second projections*, namely: $P_1 = (Id \star (V \times V))^{-1}$, $P_2 = ((V \times V) \star Id)^{-1}$. Iterating suitable transpositions of $(V \times V), Id, P_1, P_2$ we can construct as correlations many "combinatorial" operations such as permutations, diagonalizations, groupings, concatenations, etc. Finally, correlations are closed under *composition*: $G \circ H \ = \ Dom(H \star G^{-1})$.

iv) In the present context, singular correlations play the rôle of *ordered pairs.* Since also "iterated pairs" are useful, we put $< x, y, z >= (\substack{\binom{x}{y} \\ z})$ and in general for any "concrete" n, $< x_1, \ldots, x_n, x_{n+1} >= (\substack{<x_1,\ldots,x_n> \\ x_{n+1}})$. A precise connection with Kuratowski's encoding could be given by the following correlations (whose existence follows from the axioms of Section 4):
- *Doub*, which associates to any singular correlation $\binom{x}{y}$ the corresponding *doubleton* collection $\{x, y\}$;
- *Kur*, the Kuratowski's *encoding of pairs* which associates to any singular correlation $\binom{x}{y}$ the *Kuratowski pair* $\{\{x, y\}, \{x\}\}$.
The cartesian product *à la Kuratowski* of two collections C and D could then be obtained by putting $C \times_K D = \widehat{Kur}(Gr\_sing\ (C \times D))$.

Since we have operations and correlations corresponding to all "Gödel Operations" we can prove a "Bernays-like" *Comprehension Principle* for collections. To this end we introduce the first order language $\mathcal{L}$, with two sequences of constants c_i, s_i (intended to denote respectively collections and correlations):
- the atomic formulæ of $\mathcal{L}$ are: $x = y$, $x = \binom{y}{z}$, $x \in c_i$ and $s_i\ x, y$;
- if ϕ and ψ are in $\mathcal{L}$ then also $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $\neg\phi$ are in $\mathcal{L}$;
- if ϕ is in $\mathcal{L}$ then also $\exists x \phi$ and $\forall x \phi$ are in $\mathcal{L}$.

First we prove that each formula of $\mathcal{L}$ has a "graph":

Theorem 1 *Let $\phi(x_1, \ldots, x_n, c_1, \ldots, c_h, s_1, \ldots, s_k)$ be a formula in $\mathcal{L}$ whose free variables are among $x_1, \ldots, x_n$ and whose constants are among $c_1, \ldots, c_h$,*

$s_1, \ldots, s_k$. *Then, for any assignment of collections* $C_1, \ldots, C_h$ *to the constants* $c_1, \ldots, c_h$ *and of correlations* $S_1, \ldots, S_k$ *to the constants* $s_1, \ldots, s_k$, *there exists a collection* Y *such that*

$$t \in Y \Longleftrightarrow t = <\omega_1, \ldots, \omega_n> \wedge\ \phi[\omega_1, \ldots, \omega_n, C_1, \ldots, C_h, S_1, \ldots, S_k].$$

Proof The proof is by induction on the length of the formula.

i) (atomic formulæ) All cases are straightforward but for $x_i = <x_j, x_k>$. By way of example if $n = 2$ and ϕ is $x_1 = <x_1, x_2>$ then $Y = Dom\ (P_1 \cap\ Id)$.

ii) (connectives) Without loss of generality we restrict ourselves to $\wedge$ and $\neg$: both cases are immediately dealt with since collections are stable under binary intersection and difference.

iii) (quantifiers) We show only the case $\phi \equiv \exists x_{n+1}\psi$. Putting

$$Z = \{<\omega_1, \ldots, \omega_n, \omega_{n+1}> \mid \psi[\omega_1, \ldots, \omega_n, \omega_{n+1}, C_1, \ldots, C_h, S_1, \ldots, S_k]\}$$

we have $Y = Dom\ Z$. **Q.E.D.**

Now we can state the "comprehension theorem"

Theorem 2 *Let* $\phi(x_1, \ldots, x_n, z, c_1, \ldots, c_h, s_1, \ldots, s_k)$ *be a formula in* $\mathcal{L}$ *whose free variables are among* $x_1, \ldots, x_n, z$ *and whose constants are among* $c_1, \ldots, c_h$, $s_1, \ldots, s_k$. *Then for any assignment of objects* $\eta_1, \ldots, \eta_n$ *to the variables* $x_1, \ldots$, x_n, *of collections* $C_1, \ldots, C_h$ *to the constants* $c_1, \ldots, c_h$ *and of correlations* $S_1, \ldots, S_k$ *to the constants* $s_1, \ldots, s_k$, *there exists a collection* Z *s. t.:*

$$t \in Z \Longleftrightarrow \phi[\eta_1, \ldots, \eta_n, t, C_1, \ldots, C_h, S_1, \ldots, S_k].$$

Proof Consider the collection

$$Y = \{<\omega_1, \ldots, \omega_n, \omega_{n+1}> \mid \phi[\omega_1, \ldots, \omega_n, \omega_{n+1}, C_1, \ldots, C_h, S_1, \ldots, S_k]\}$$

given by Theorem 1, and let $G = Coll\_corr\ Y$. Then clearly

$$Z = Dom\,(G \cap (\{<\eta_1, \ldots \eta_n>\} \times V))^{-1}.$$ **Q.E.D.**

In the usual theories, *functions*, *relations*, *operations*, etc. are *identified* with their "graphs". An unpleasant consequence of this "ultra-reductionist" attitude is that many fundamental relations and operations cannot be objects of the theory. We have chosen instead the maximal selfdescriptive capability, and we have introduced from the very beginning many selfreferential objects, such as *R_qual, R_coll, R_incl.* Therefore we cannot consistently postulate that any object has a graph. Russell's argument yields in fact:

Theorem 3 *There are binary relations without graph. In particular the inclusion relation R_incl and the membership relation R_coll have no correlations corresponding to them.*

An immediate consequence of Theorem 3 is that, even by postulating further axioms, we cannot obtain analogues of Theorems 1, 2 for the language $\mathcal{L}$ expanded by allowing any of the atomic formulæ $x \in y$ or $x \subseteq y$. A weaker extension, however, is provided by the language $\mathcal{L}_1$ of the next section. In the present setting we cannot prove even that collections are stable under *unary union* or *unary intersection*, or that the collections $\mathcal{SC}(X)$ (of all *subcollections* of the collection X) and X^Y (of all functional correlations with domain Y and taking values in X) always exist. *A fortiori* we cannot prove the existence of "cartesian products" of arbitrary functional correlations of collections, and of the "transpositions" of arbitrary functional correlations of functional correlations. Since these constructions play an important rôle in many areas of Mathematics, Logic and Semantics, it is interesting to postulate the existence of operations *Un, Int, Sub_coll, Cart, Trans* that carry out the intended tasks. Hence we give the axioms:

C.11 (Union) *Un is a simple operation. If X is a collection of collections and $Un\ X = \bigcup X$ exists, then it is a collection such that*

$$t \in \bigcup X \Longleftrightarrow \exists x \in X (t \in x).$$

C.12 (Intersection) *Int is a simple operation. If X is a collection of collections and $Int\ X = \bigcap X$ exists, then it is a collection such that*

$$t \in \bigcap X \Longleftrightarrow \forall x \in X (t \in x).$$

C.13 (Subcollections) *Sub_coll is a simple operation. If X is a collection and $Sub\_coll\ X = \mathcal{SC}(X)$ exists, then it is a collection such that*

$$Y \in \mathcal{SC}(X) \Longleftrightarrow Q\_coll\ Y \wedge Y \subseteq X.$$

C.14 (Cartesian Product) *Cart is a simple operation. If F is a functional correlation of collections, and $Cart\ F = X$ exists, then it is collection of correlations such that*

$$G \in X \Longleftrightarrow (Dom\ G = Dom\ F) \wedge \forall x \in Dom\ G.(Gx \in Fx).$$

Notice that the *power* of collections can be obtained from the cartesian product by putting $X^Y = Cart\ (Y \times \{X\})$.

C.15 (Transposition) *Trans is a simple operation. If F is a functional correlation of functional correlations and $Trans\ F = {}^tF$ exists, then it is a functional correlation of non-empty functional correlations such that*

$$\forall x \in Dom\ F.\ \forall yz\ (Fxy = z \Longleftrightarrow ({}^tFy)x = z).$$

The *non-emptyness* clause characterizes *uniquely* the transposition and it implies that transpositions of correlations of"small size" are small. In order to grasp the action of the operation *Trans*, it is convenient to visualize a special case which makes use of a "functional" notion of n-tuple. Taking $(x_1, \ldots, x_n)$ to be the *univalent correlation* assigning x_i to i, and viewing a matrix $m \times n$ as an n-tuple of m-tuples, then its *transposition* is an m-tuple of n-tuples, corresponding exactly to the *transposed matrix.* Noteworthy is the transposition of *n-tuples of univalent correlations with the same domain*, which gives a sort of *fibred product*, namely: ${}^t(f_1, \ldots, f_n)x = (f_1x, \ldots, f_nx)$.

The axioms C.11-15 are formulated *prudentially.* They assert only that, *if the operations involved are defined*, then they behave according to their intendend meaning. They do not enforce the operations to be defined anywhere. In order use collections in place of *sets* in Mathematics, Logic, and Semantics, the following "problematic" axiom could then be necessary:

Coll *The operations Un and Int are defined for every collection of collections, Sub_coll is defined for every collection, Cart is defined for every functional correlation of collections, and Trans is defined for every functional correlation of functional correlations.*

4 Sets and Functions

As remarked at the end of the previous section, we cannot perform on arbitrary collections some of the manipulations carried out in ordinary mathematical practice. The Theory of Sets originated by Cantor, and later axiomatized by Zermelo, is an attempt of isolating as *sets* those *collections which can be freely manipulated* as mathematical objects, the *Limitation of Size Principle* being the basic criterion of "sethood". In our view, the *naïve* concept of *set* mediates two contrasting concepts: that of "extension of an arbitrary property" and that of "finite list". The former is captured in our theory by the notion of *collection*, while the latter has not yet been introduced. We could follow the plan underlying the theory TBCN of [12], but we prefer not to deal with the problematic notion of *finiteness* at such an early stage in the development of our theory. We shall suggest possible axiomatizations of *natural numbers* and *finite lists* in Section 5. However, we take as guidelines to the axiomatization of *sets* and *functions* the main intuitive properties of finite lists.

We introduce the notion of *set* in our framework by postulating the existence of the collection *Ins* of *sets*[2]. Similarly to what we did at the level of collections, we introduce also the collections *Sys* (of *systems*[3]) and *Fun* (of *functions*) according to the "conceptual ratio":

$$\frac{\text{Sets}}{\text{Collections}} = \frac{\text{Systems}}{\text{Correlations}} = \frac{\text{Functions}}{\text{Functional Correlations}}.$$

Thus we postulate:

I.1 *Ins, Sys and Fun are collections.*
1. $Ins \subseteq Coll$;
2. $s \in Sys \Longleftrightarrow Q\_corr\ s\ \wedge\ Gr\_sing\ (s) \in Ins$;
3. $f \in Fun \Longleftrightarrow Q\_corr\_un\ f\ \wedge\ f \in Sys$;

In order to have some sets to start with, we postulate:

I.2 *For all objects x, y the collection $\{x, y\}$ is a set.*

A first axiom expressing the *manipulability* of sets and functions is obtained by suitably restricting to them the problematic Axiom Coll:

I.3 *The operations Un and Int are defined for every collection of sets and every set of collections, Sub_coll is defined for every set, Cart is defined for every collection-valued function, and Trans is defined for every correlation-valued function. Moreover the cartesian product of a set-valued function is a set.*

The "Limitation of Size Principle" is embodied in the *Axiom of Replacement*, which we formulate relatively to *quaternary* relations, since these regulate the behaviour of all other kinds of objects of our theory.

[2]The word *Ins* has been chosen in accordance with the Latin *insimul* (meaning *simultaneously*) which is the root of the word for *sets* in some Neo-Latin languages, e.g. *insieme* (Italian), *ensemble* (French).

[3]The word *System* has been chosen in [1] to stress the fact that such a notion is a *system* or *method* of attaching objects to objects e.g. car plates to cars, labels to wine bottles, etc.

I.4 (Replacement-union) *Let* X, Y, Z *be sets and let* t *be a quaternary relation. If for all* $x \in X,\ y \in Y,\ z \in Z$ *the collection* $\{w \mid t\ x,y,z,w\}$ *exists and is a set, then also the collection* $\{w \mid \exists x \in X\ \exists y \in Y\ \exists z \in Z.\ t\ x,y,z,w\}$ *exists and is a set.*

Axiom I.4 subsumes the traditional axiom of *union* together with an "internal" version of *replacement.* Namely, if E is a set of sets, then $\bigcup E$ can be obtained by taking $t = R\_rel\_t$, $X = \{R\_rel\_b\}$, $Y = \{R\_coll\}$ and $Z = E$. The classical formulation of the Replacement Axiom in Gödel-Bernays-Von Neumann theory, i.e.

$$Q\_corr\_un\ F\ \wedge\ E \in Ins \implies \widehat{F}(E) \in Ins,$$

can be easily derived by taking $t = R\_rel\_t$, $X = \{R\_corr\}$, $Y = \{F\}$ and $Z = E$. Such a formulation is definitely weaker than the one in I.4: for instance, it does not imply that the image of a set under an operation is a set. Similarly one obtains an "internal" version of *separation*, in particular any subcollection of a set is a set. From Axioms I.3-4 one derives also the usual *Powerset Axiom.*

The idea that the "internal structure" of sets and functions is "simple" and "controlled" is expressed by the following axiom, which introduces the *correlations Memb_ins* and *Incl_sys* corresponding to the relations of *membership* between sets and objects and of *inclusion* between systems:

I.5 *Memb_ins and Incl_sys are correlations, such that:*
1. $\forall xy(Memb\_ins\ x,y \iff x \in Ins\ \wedge\ y \in x)$*;*
2. $\forall xy(Incl\_sys\ x,y \iff x,y \in Sys\ \wedge\ y \subseteq x)$*.*

Strong *set-theoretic comprehension principles* can now be stated for the language $\mathcal{L}_1$, obtained by adding the atomic formulæ "$x \in y \in Ins$" and "$\binom{x}{y} \subseteq z \in Sys$" to the language $\mathcal{L}$ of Section 3.

Using the correlations $Memb\_ins$ and $Incl\_sys$, we can prove the exact counterparts of Theorems 1 and 2 of Section 3 for the extended language $\mathcal{L}_1$. Thus we obtain the usual "external" or "logical" versions of the axioms of *separation* and *replacement.* We also obtain the existence of *union* and *intersection* of every *collection of sets*, and of the collection of all sub*sets* of any collection.

Up to now we have left undetermined *which* collections are sets. We can only derive the "sethood" of all *intuitively finite* collections, and deduce from Theorem 3 that there are collections which are not sets. Various alternative choices can be made as to what we want to be a set. In view of the Limitation of Size Principle, the most liberal attitude is expressed by Von Neumann's axiom:

Ins.1 (Von Neumann) *A collection* X *is a set* $\iff \forall F \in Corr\_un.\ \widehat{F}(X) \neq V$.

Axiom Ins.1 implies the "internal" version of Replacement, given above. It allows to derive also the existence of a one-to-one correlation between *Ord*, the collection of all *Von Neumann's ordinals*, and the universe V.

On the other hand, a strong *finitistic* attitude is expressed by the axiom:

Ins.2 (Finitism) *All sets are Dedekind-finite.*

This axiom greatly simplifies the structure of sets and functions, still allowing for fruitful engraftings of many mathematical theories, as well as of large parts of the applied Sciences. However, assuming Ins.2, the natural set-theoretic model of Peano's Axioms is Ord itself (which is not a set). Thus Mathematical Analysis cannot be developed in "Cantor's style". The usual development of Arithmetic can still be carried out, in "Peano's style", since there are correlations corresponding to *Addition*, *Multiplication* and *Ordering*; moreover Theorem 2 for $\mathcal{L}_1$ gives a sort of "arithmetical comprehension principle". The choice of postulating Ins.2 could still allow a substantial development of modern Mathematics, provided one could consistently assume the strong axiom Coll which makes available unions, intersections and powers of arbitrary collections. More classically one could postulate:

Ins.3 (Cantor) *There exists a Dedekind-infinite set.*

We conclude by mentioning a different kind of "finitistic" axiom arising from a "Skolem's principle", namely:

Ins.4 (Skolem) *There exists an univalent object e enumerating V.*

The natural understanding of an *enumerating object* (i.e. a simple operation e defined, say, only on finite ordinals, and whose codomain is V) is obviously inconsistent with Axiom Ins. 3. An interesting feature of this theory is that we can still maintain both Axioms Ins. 3 and Ins. 4 by simply giving a different engrafting of the "*naïve* concept of enumeration" (for instance a *univalent* object e subject to the action of *no* binary, ternary or quaternary relation).

5 Final Remarks

Why stop at quaternary relations? - The motivation for considering only *binary, ternary* and *quaternary* relations is twofold. One could have considered simply *binary* relations and used some "internal" encoding of n-tuples: but we preferred not to have to encode *natural, primitive* concepts such as *simple* and *binary* operations. Hence we introduced a ternary and a quaternary relation in order to express their behaviours. We did not carry on introducing relations of higher and higher arity, as we did in [12, 13], so as not to have to rely on the possibly problematic notion of *standard natural number*. The decision to stop at arity 4 seems to be not completely satisfactory from the point of view of *self-description*. However many methods are available to describe internally the behaviour of quaternary relations. E.g. we can introduce a *binary* relation R_rel_q connecting each quaternary relation t with all 4-tuples $< x, y, z, w >$ such that $t\ x, y, z, w$. An axiomatization of R_rel_q should be

G.8 *R_rel_q is a binary relation.*
1. If $R\_rel\_q\ x, y$, then $Q\_rel\_q\ x$ and $\exists uvwz\ (y =< u, v, w, z >)$
2. If $Q\_rel\_q\ t$ then $\forall xyzw\ (t\ x, y, z, w \iff R\_rel\_q\ t, < x, y, z, w >)$.

Engrafting Logic - In order to engraft elementary logic, we can follow [7, 8], and introduce the qualities Q_g_prop and Q_cl_prop of being a *general* and a *classical proposition*, together with the *logical* operations Neg (negation), Et (conjunction), and Vel (disjunction), acting on *propositions*, and $Univ$ and $Exist$ (universal and existential quantifiers), acting on *proposition-valued operations*, subject to appropriate axioms. We can also introduce an operation Gap which *generates atomic propositions* by iterated applications: e.g. if t is a quaternary relation, then $((((Gap\ t)x)y)z)w$ is the proposition asserting that "x, y, z, w are in the relation t".

One should also introduce the quality Q_t_ver of being a *totally* (or *absolutely) true proposition*, which behaves classically on classical propositions. This engrafting gives an alternative solution to the problem of describing internally the behaviour of quaternary relations. In fact one has:

$$\forall xyzw\ (t\ x, y, z, w \Longleftrightarrow Q\_tver\ ((((Gap\ t)x)y)z)w).$$

Notice that one cannot assume simultaneously that *all atomic propositions* are *classical* and that *classical propositions* are *stable under logical operations*, since this would yield an analogue of the "Liar paradox" (see [7, 8]).

Engrafting Natural Numbers and Finite Sets and Systems - As we remarked in the Introduction, it is not in the spirit of this theory to rely on artificial codings (like Von Neumann Ordinals) in defining important concepts such as *natural numbers* and *finite sets* (or *lists*). As suggested in [7, 8], one should rather introduce the quality Q_nat of being a *true (standard) natural number* together with the operations Add and $Mult$ (*Addition* and *Multiplication*) and the *Order* relation R_ord, subject to Peano's Axioms. Then one has to take into account that the *Induction Principle* can be formulated in many substantially different ways in theories where many different kinds of objects are present. E.g. the mere assumption

Ind.1 *Non-empty collections of natural numbers have a least element*

seems rather weak. Taking into account the fundamental rôle played by quaternary relations, a better formulation of induction would be

Ind.2 *Let t be a quaternary relation and let x,y,z be arbitrary objects. If there exists a natural number n such that $t\ x, y, z, n$ (or $t\ x, y, n, z$), then there exists a least such n.*

If Logic has been engrafted as above, one should postulate the stronger axiom

Ind.3 *Let f be a proposition-valued operation. If there exists a natural number n such that $Q\_tver\ fn$, then there exists a least such n.*

For any natural number n we can consider the collection V^n of all *functions with domain* $\{1, \ldots, n\}$. Then we can introduce the qualities Q_fins, Q_fsys of being a *finite set* and a *finite system* (or *list*) by postulating (see [12]):

Fin *Q_fins and Q_fsys are qualities.*
1. $\forall X\ (Q\_fins\ X \Longleftrightarrow \exists nx\ (Q\_nat\ n \wedge x \in V^n \wedge X = Dom\ x^{-1})$.
2. $\forall X\ (Q\_fsys\ X \Longleftrightarrow \exists nxy\ (Q\_nat\ n \wedge x, y \in V^n \wedge X = y \circ x^{-1})$.

Arithmetic can be carried out in various ways in this setting. One could focus directly on the quality Q_nat, but this approach would be rather cumbersome because qualities cannot be manipulated. Arithmetic, possibly non-standard, could be developed more easily by considering *collections* $\mathcal{N}$ including all *true natural numbers* and stable under (suitable *extensions* of) the operations *Add* and *Mult*. In this case it would be useful to postulate the existence of a *least such collection* $\mathbb{N}$. A strong axiom of "standardness" would be

Ind.4 $\forall x\ (x \in \mathbb{N} \Longleftrightarrow Q\_nat\ x)$.

Comparison with other foundational theories - We propose in this paper the *framework theory* GOCI, consisting of the axioms G.1-8, O.1-11, C.1-16 and I.1-5. A model of GOCI can be obtained by combining the techniques used in [12, 13] for modeling the theory of collections and sets TBCN and the theory of operations Oper. In the Appendix we outline such a construction and obtain models of GOCI + Ins. 1 + Corr_op. The latter is the axiom which asserts that every functional correlation corresponds to a simple operation, thus strengthening Axiom GrOp of [13]. We have:

Theorem 4 *1.* Con (ZF) $\Longrightarrow$ Con (GOCI + Corr_op + Ins. 1,2,4);
2. Con (ZF) $\Longrightarrow$ Con (GOCI + Corr_op + Ins. 1,3);
3. Con (ZF+$\exists M\, transitive.\, M \models$ZF) $\Longrightarrow$ Con (GOCI + Corr_op + Ins.1,3,4).

The theory GOCI deals only with objects of *arities* $1, 2, 3$, and 4. Even without introducing objects of arbitrary *arity*, one can still find in GOCI suitable objects corresponding to n-ary objects. For instance, one could use the "external" notion of n-tuple $< x_1, \ldots, x_n >$ introduced in Section 3, or else use the "internal" notion introduced above, or even define n-tuples as functions whose domain is the *Von Neumann's* integer n. Using such an "internal" notion of n-tuple it is possible to define n-ary operations as simple operations acting on n-tuples and give a *simultaneous interpretation* of Oper and TBCN in GOCI.

The problematic Axiom Coll **-** The relative consistency of Coll with GOCI is problematic. The inductive construction of the model in the Appendix does not allow to show the validity of the Axiom Coll, not even with respect to union, say, in that new collections arise at each level. As a matter of fact, theories where the union of "large classes" is always possible, such as Quine's New Foundations NF, or the Generalized Positive Comprehension GPC (see e.g. [14]), or theories *à la* Feferman (see e.g. [9]), fail to satisfy some key axioms of GOCI. For instance, GPC lacks complementation, Feferman's theories are non-extensional, and NF lacks many fundamental correlations, e.g. the singleton map. We conjecture the consistency of GOCI + Coll − C. 2 (extensionality).

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Appendix

In this Appendix we sketch the construction of a model of the theory GOCI. This construction combines those carried out in [12] and in [13]. We use standard set-theoretical notation and we freely refer to basic results in Set Theory (see [16, 10]). In particular we use standard set-theoretic reductions of pairs (Kuratowski doubletons) and functions (sets of Kuratowski doubletons). In order to operate more naturally, we work in the theory $\mathsf{ZF}_0^-\mathsf{C}+\mathsf{X}_i(U)$, i.e. a Zermelo-Frænkel-like Set Theory, where Extensionality is weakened so as to allow for a set U of *Urelementen* and Foundation is replaced by the *injective Free Construction Axiom*[4]

$\mathsf{X}_i(\mathbf{U})$. *Let f:* $A \longrightarrow \mathcal{P}(A) \cup U$ *be injective. Then there exists a transitive set* T *and a bijective function g:* $A \longrightarrow T$ *, such that*

$$g(x) = \begin{cases} f(x) & \textit{if } f(x) \in U \\ \{g(y) \mid y \in f(x)\} & \textit{otherwise} \end{cases}$$

For sake of simplicity we assume that $|U| = \kappa$ is *strongly inaccessible.* Thus we can work with *sets* rather than with *definable proper classes.*

The procedure we follow in defining the model reflects the "stratified" hierarchy of the fundamental kinds of objects. *Sets, systems* and *functions* lie at the bottom level, and are controlled by *collections* and *correlations* which constitute the next level up. On a still higher level are placed *operations.* Finally, at the top, we have *relations* involving all kinds of objects, including *relations themselves.* Having only a descriptive value, *qualities* can be considered separately.

In order to define the universe of the model we proceed according to the following steps:

- put $A = \{x \in \Pi_\kappa(U) \mid \ |TC(x)| < \kappa\}$: the intended rôle of the elements of $A \setminus U$ is that of *sets*;
- let $U_0, U_1, \ldots, U_{10}$, be pairwise disjoint subsets of U of size κ, whose intended rôle is that of the different domains of primitive objects, which are not sets, namely:
  - elements of U_0 stand for *proper collections*;
  - elements of $U_1 \cup U_2 \cup U_3$ stand for *correlations*;
  - elements of $U_4 \cup U_5 \cup U_6$ stand for *operations*;
  - elements of $U_7 \cup U_8 \cup U_9$ stand for *relations*;
  - elements of U_{10} stand for *qualities*;
- pick a bijection $J_1 : A \times A \longrightarrow U_1$: the intended rôle of the elements of U_1 is that of all *singular correlations*;
- pick a bijection $J_2 : \mathcal{P}_\kappa(U_1) \longrightarrow U_1 \cup U_2$, such that

  $$J_1(\{\{x,y\},\{x\}\}) = J_2(\{J_1(\{\{x,y\},\{x\}\})\}) :$$

  the intended rôle of the elements of U_2 is that of all *non singular systems*;

[4]One can also consider a *class version* $\overline{\mathsf{X}}_i(U)$, where A and U can be *Proper classes.* The relative consistency of $\mathsf{X}_i(U)$ w.r.t. ZFC can be established by the arguments of [10]

- pick an injection $J_3 : U_3 \longrightarrow U_0$ and a surjection $J_4 : U_0 \longrightarrow U_1 \cup U_2 \cup U_3$ such that, for all $x \in U_1 \cup U_2 \cup U_3$, $\mid J_4^{-1}(x) \mid = \kappa$ and, for all $x \in U_3$, $(J_4 \circ J_3)(x) = x$: the intended meaning of $J_3(x)$ is the *singular graph* of x, and that of $J_4(x)$ is $Coll\_corr(x)$, hence $J_3 \cup J_2^{-1}$ and J_4 are the "Kuratowski graphs" of Gr_sing and $Coll\_corr$ respectively;

- let $J_5 = \{(x,x) \mid x \in A\}$, $J_6 = \{(x,y) \in (A \setminus U) \times A \mid y \in x\}$, and $J_7 = \{(x,y) \in (U_1 \cup U_2) \times (U_1 \cup U_2) \mid J_2^{-1}(x) \supseteq J_2^{-1}(y)\}$: these are the "Kuratowski graphs"of the correlations Id, $Memb\_ins$, and $Incl\_sys$ respectively;

- in order to "activate" the elements of U_0 as proper collections we define inductively on $\alpha \leq \kappa$ sets $B_\alpha \subseteq \mathcal{P}(A)$ and injective mappings $\psi_\alpha : B_\alpha \longrightarrow U_0 \cup (A \setminus U)$ as follows:
  - put $B_0 = (A \setminus U) \cup \{U_i \mid 0 \leq i \leq 10\} \cup \{J_i \mid 1 \leq i \leq 7\}$
    and let ψ_0 be an injection such that:
    - $\psi_0(x) = x$ for $x \in A \setminus U$ (sets are their own extensions)
    - $\psi_0(U_1) \in cod\ J_3$ (since U_1 stands for $Gr\_sing(V \times V)$) and
    - $J_4(\psi_0(U_i)) = J_4(\psi_0(J_j)) = J_2(\emptyset)$ for all $i \neq 1, j > 0$;
  - put $B_{\alpha+1} = B_\alpha \cup \{X \setminus Y,\ X \times Y,\ X * Y \mid X, Y \in B_\alpha\} \cup$
    $\{dom\ X,\ X^{-1},\ \bigcup_{x \in X} \psi_\alpha^{-1}(x),\ \bigcap_{x \in X} \psi_\alpha^{-1}(x) \mid X \in B_\alpha\} \cup$
    $\{^t(\psi_\alpha^{-1} \circ F),\ \Pi_{y \in a} \psi_\alpha^{-1}(F(y)) \mid F \in (B_\alpha)^a,\ a \in A \setminus U\} \cup$
    $\{(x, \psi_\alpha(X)) \mid x \in X \in B_\alpha\}$
    and take $\psi_{\alpha+1}$ any injection extending ψ_α such that:
    - $\mid U_0 \setminus cod\ \psi_{\alpha+1} \mid = \kappa$,
    - $\mid J_4^{-1}(x) \setminus cod\ \psi_{\alpha+1} \mid = \kappa$ for all $x \in U_1 \cup U_2 \cup U_3$,
    - $\mid \widehat{J_3}(U_3) \setminus cod\ \psi_{\alpha+1} \mid = \kappa$,
    - $\forall X, Y \in B_{\alpha+1}.(X \cap U_1 = Y \cap U_1 \Longrightarrow J_4(\psi_{\alpha+1}(X)) = J_4(\psi_{\alpha+1}(Y)))$,
    - $\forall X \in B_{\alpha+1}.(X \subseteq U_1 \Longrightarrow \psi_{\alpha+1}(X) \in \widehat{J_3}(U_3)$;
  - for limit λ, put $B_\lambda = \bigcup_{\alpha < \lambda} B_\alpha$ and $\psi_\lambda = \bigcup_{\alpha < \lambda} \psi_\alpha$.

At this stage we could define a model for GOCI with universe A, where membership between to avoid the nuisance of having to use *two* different "memberships", we define a *transitive* structure by means of the Axiom $\mathsf{X}_i(U)$.

Define the injective function $f : A \longrightarrow \mathcal{P}(A) \cup U$ by

$$f(x) = \begin{cases} \psi_\kappa^{-1}(x) & \text{if } x \in U_0 \\ x & \text{otherwise} \end{cases}$$

and pick a transitive set M and a bijective function $g : A \longrightarrow M$, such that

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in U \\ \{g(y) \mid y \in f(x)\} & \text{otherwise.} \end{cases}$$

Now we are ready to define a transitive model $\mathcal{M}$ of GOCI with universe M. ¿From the above discussion, it should be clear which elements of M have ¿to be taken as *sets*, *collections*, *correlations*, *¿systems* respectively. E.g. $Coll = \widehat{g}((A \setminus U) \cup U_0)$, $Corr = U_1 \cup U_2 \cup U_3$.

In order to complete the construction of the model we still have to deal with the interpretation of *operations*, *qualities*, and *relations*. We begin by dealing with *operations*.

- We pick *eleven* elements g_{Comp}, g_S, g_{Bun}, g_{Def}, $g_=$, $g_{\neq}$, g_{Diff}, g_{Sing_corr}, $g_\times$, $g_\star$, $g_{App} \in U_4$ to represent the binary operations *Comp, S, Bun, Def, Eq, Neq, Diff, Sing_corr, Cart_b, Trans_b, App* respectively. The remaining elements of U_4 are taken to be *empty binary operations*.
- Next we take an injective mapping $\chi : U_5 \longrightarrow U_1 \cup U_2 \cup U_3$, whose codomain is the subcollection of all *univalent correlations*: the intended meaning of $f \in U_5$ is the simple operation whose graph is the same as that of $\chi(f)$, so as to satisfy Axiom Corr_op.
- In order to deal with the remaining simple operations we begin by taking *eleven* elements u_K, u_{Cur}, $u_\forall$, $u_\exists$, u_{Inv}, u_{Dom}, $u_\cup$, $u_\cap$, $u_{\mathcal{SC}}$, u_{Cart}, $u_{Trans} \in U_6$, in order to represent the fundamental simple operations K, Cur, $Univ$, $Exist$, Inv, Dom, Un, Int, Sub_coll, $Cart$, $Trans$ respectively.
- For each $g \in U_4$ we pick new urelements $u_{Cur\,g}, u_{\forall g}, u_{\exists g} \in U_6$, whose intended meaning is clear from the index.
- For each $g \in U_4$ and each $x \in M$ we pick a new urelement $u_{g,x} \in U_6$, whose intended meaning is the value of $Cur\ g$ applied to x.
- We pick four injective mappings $\phi_{Comp}, \phi_S, \phi_{Bun}, \phi_{Def} : (U_5 \cup U_6)^2 \longrightarrow U_6$ whose ranges are pairwise disjoint and disjoint from the previously used elements of U_6. The intended meaning of $\phi_{Comp}(f,g), \phi_S(f,g), \phi_{Bun}(f,g), \phi_{Def}(f,g)$ is the *result* of applying respectively *Comp, S, Bun, Def* to the operations f, g.

We assume a fixed well ordering of M, in order to deal with the "non-deterministic" operations $Exist$ and Bun. We "activate" the elements of $U_4 \cup U_5 \cup U_6$ by defining inductively sets:

$$F_\alpha = \{(f,x,y) \mid f \in U_5 \cup U_6, fx = y \text{ at level } \alpha\},$$
$$G_\alpha = \{(g,x,y,z) \mid g \in U_4, gxy = z \text{ at level } \alpha\}.$$

- At level 0 we put:
  - $(\chi^{-1}(u), x, y) \in F_0 \iff u \in Corr\_un$ and $\binom{y}{x} \subseteq u$;
  - $(u_K, x, y) \in F_0 \iff y = \chi^{-1}(M \times \{x\})$;
  - $(u_{Cur}, g, u_{Cur\,g}) \in F_0$ for $g \in U_4$;
  - $(u_{Cur\,g}, x, u_{g,x}) \in F_0$ for $g \in U_4$ and $x \in M$;
  - $(u_\forall, g, u_{\forall g}) \in F_0$ for $g \in U_4$;
  - $(u_\exists, g, u_{\exists g}) \in F_0$ for $g \in U_4$;
  - $(u_{Inv}, X, Y) \in F_0 \iff X, Y \in Corr$ and $Y = X^{-1}$;
  - $(u_{Dom}, X, Y) \in F_0 \iff X \in Corr$ and $Y = Dom\,X$;
  - $(u_\cup, X, Y) \in F_0 \iff X, Y \in Coll$ and $Y = \bigcup X$;
  - $(u_\cap, X, Y) \in F_0 \iff X, Y \in Coll$ and $Y = \bigcap X$;
  - $(u_{\mathcal{SC}}, X, Y) \in F_0 \iff X, Y \in Coll$ and $Y = \mathcal{SC}X$;
  - $(u_{Cart}, X, Y) \in F_0 \iff X \in Fun, Y \in Coll$ and $Y = Cart\,X$;
  - $(u_{Trans}, X, Y) \in F_0 \iff X \in Fun, Y \in Corr$ and $Y = Trans\,X$;
  - $(g_{Comp}, x, y, \phi_{Comp}(x,y)) \in G_0$ for $x, y \in U_5 \cup U_6$;
  - $(g_S, x, y, \phi_S(x,y)) \in G_0$ for $x, y \in U_5 \cup U_6$;

- $(g_{Bun}, x, y, \phi_{Bun}(x,y)) \in G_0$ for $x, y \in U_5 \cup U_6$;
- $(g_{Def}, x, y, \phi_{Def}(x,y)) \in G_0$ for $x, y \in U_5 \cup U_6$;
- $(g_{=}, x, x, x) \in G_0$ for $x \in M$;
- $(g_{\neq}, x, y, y) \in G_0$ for all $x, y \in M$ such that $x \neq y$;
- $(g_{Diff}, X, Y, Z) \in G_0 \iff X, Y, Z \in Coll$ and $Z = X \setminus Y$;
- $(g_{Sing\_corr}, x, y, z) \in G_0 \iff x, y \in M$, $z \in U_1$ and $z = \binom{x}{y}$;
- $(g_{\times}, X, Y, Z) \in G_0 \iff X, Y \in Coll$, $Z \in Corr$ and $Z = X \times Y$;
- $(g_{\star}, X, Y, Z) \in G_0 \iff X, Y, Z \in Corr$ and $Z = X \star Y$;

- At level $\alpha + 1$ we put $F_\alpha \subseteq F_{\alpha+1}$, $G_\alpha \subseteq G_{\alpha+1}$ and
  - $(\phi_{Comp}(f,g), x, y) \in F_{\alpha+1} \iff \exists z.(g,x,z),(f,z,y) \in F_\alpha$;
  - $(\phi_S(f,g), x, y) \in F_{\alpha+1} \iff \exists z, w.(f,x,w),(g,x,z),(w,z,y) \in F_\alpha$;
  - $(\phi_{Bun}(f,g), x, y) \in F_{\alpha+1} \iff$either $(\phi_{Bun}(f,g), x, y) \in F_\alpha$ or y is the least element z such that $(f,x,z) \in F_\alpha \vee (g,x,z) \in F_\alpha$;
  - $(\phi_{Def}(f,g), x, y) \in F_{\alpha+1} \iff \exists z.(f,x,z),(g,x,y) \in F_\alpha$;
  - $(u_{\forall g}, x, y) \in F_{\alpha+1} \iff \forall z.(g,x,z,y) \in G_\alpha$;
  - $(u_{\exists g}, x, y) \in F_{\alpha+1} \iff$either $(u_{\exists g}, x, y) \in F_\alpha$ or z is the least element such that $(g,x,z,y) \in G_\alpha$;
  - $(u_{g,x}, y, z) \in F_{\alpha+1} \iff (g,x,y,z) \in G_\alpha$;
  - $(g_{App}, f, x, y) \in G_{\alpha+1} \iff (f,x,y) \in F_\alpha$.

- At limit λ we put $F_\lambda = \bigcup_{\alpha<\lambda} F_\alpha$ and $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$.

- Let ν be the least ordinal such that $F_\nu = F_{\nu+1}$ and $G_\nu = G_{\nu+1}$. Then
  - for $g \in U_4$ we put $gxy = z \iff (g,x,y,z) \in G_\nu$,
  - for $f \in U_5 \cup U_6$ we put $fx = y \iff (f,x,y) \in F_\nu$.

Having dealt completely with operations, we proceed to define the interpretation of *qualities* and *relations*. We pick *ten* elements of U_{10}, say q_{qual}, q_{rel_b}, q_{rel_t}, q_{rel_q}, q_{op_s}, q_{op_b}, q_{coll}, q_{corr}, q_{corr_un}, q_{un}, whose intended meaning is clear, and we assume that all the remaining elements of U_{10} are "empty qualities". We pick *six* elements of U_7, say r_{qual}, r_{rel_q}, r_{coll}, r_{id}, r_{incl}, and r_{ext_eq}, *three* elements of U_8, say r_{rel_b}, r_{op_s}, r_{corr}, and *two* elements of U_9, say r_{rel_t}, r_{op_b}, whose intended meaning is clear. We assume that all the remaining elements of U_7, U_8, U_9 are respectively binary, ternary, quaternary "empty relations".

The extension of all qualities is immediate from the definition, except for q_{un}, since we have not yet determined the action of relations. However, in accordance to Axiom G.7, we can decide that the only "non-empty" univalent relations are *R_id*, *R_op_s*, and *R_op_b*.

The action of the relations r_{qual}, r_{op_s}, r_{op_b}, r_{coll}, r_{corr}, r_{id} is determined by the previous definitions. The action of r_{incl} and r_{ext_eq} is already determined on objects which are not relations. In accordance to Axiom G.5, we complete the definition of r_{incl} and r_{ext_eq} by imposing that "empty relations" are extensionally equivalent to each other and included in any relation, and that "non-empty relations" are included only in themselves, except for $r_{ext\_eq} \subseteq r_{incl}$.

In order to "activate" the relations r_{rel_b}, r_{rel_t}, and r_{rel_q} we define inductively sets R_n, S_n, T_n as follows:

- at level 0 we put
  - $R_0 = \{(r, x, y) \mid r = r_{qual}, r_{coll}, r_{id}, r_{incl}, r_{ext\_eq} \wedge r\ x, y\}$
  - $S_0 = \{(s, x, y, z) \mid s = r_{op\_s}, r_{corr} \wedge s\ x, y, z\}$
  - $T_0 = \{(r_{op\_b}, x) \mid x =< g, u, v, w > \wedge g \in U_4 \wedge guv = w\}$;
- at level $n+1$ we put
  - $R_{n+1} = R_n \cup \{(r_{rel\_q}, x, y) \mid (x, y) \in T_n\}$
  - $S_{n+1} = S_n \cup \{(r_{rel\_b}, x, y, z) \mid (x, y, z) \in R_n\}$
  - $T_{n+1} = T_n \cup \{(r_{rel\_t}, x) \mid x =< t, u, v, w > \wedge (t, u, v, w) \in S_n\}$.
- Finally we put:
  - $r_{rel\_b}\ x, y, z \iff (x, y, z) \in \bigcup_{n \in \omega} R_n$,
  - $r_{rel\_t}\ x, y, z, w \iff (x, y, z, w) \in \bigcup_{n \in \omega} S_n$,
  - $r_{rel\_q}\ x, y \iff (x, y) \in \bigcup_{n \in \omega} T_n$.

The model $\mathcal{M}$ being now completely defined, we can state:

Theorem 5 *$\mathcal{M}$ is a model of* GOCI + Corr_op + Ins. 1*. Moreover:*
1. if $\kappa = \omega$, then $\mathcal{M} \models$ Ins.2 +(Ins.4 $\wedge$ *Q_corr_un e);*
2. if $\kappa > \omega$, then $\mathcal{M} \models$ Ins.3 .

Working with *definable* classes within a Gödel-Bernays-Von Neumann-type Class Theory with a *proper class U* of urelements, GBVN(U), and assuming the class version $\overline{\mathsf{X}}_i(U)$ of the injective free construction principle, we can give up the assumption of the existence of an inaccessible cardinal, by taking $\kappa = Ord$ and defining a class model $\mathcal{M}_\infty \models$ GOCI + Corr_op + Ins. 1,3.

Moreover, if we define the class model $\mathcal{M}_\infty$ inside a countable transitive model $\mathcal{A}$ of GBVN(U)+$\overline{\mathsf{X}}_i(U)$, we have from the outside $\mid Ord^{\mathcal{A}} \mid = \omega$ hence also $\mid M_\infty \mid = \omega$. Then we can pick a fresh urelement e and state that e is an *enumeration of the Universe*, which satisfies Ins.4. From Theorem 5 and the above considerations we obtain directly Theorem 4 of Section 5.

In the construction of the model $\mathcal{M}$, we left enough *open space* for accommodating many new kinds of objects. For instance, if *natural numbers* and *finite sets* and *systems* are engrafted in the theory GOCI as in Section 5, we can extend the model $\mathcal{M}$ in the following way. Fix fresh urelements q_{nat}, q_{fins}, $q_{fsys} \in U_{10}$, g_{Add}, $g_{Mult} \in U_4$, $f_{tuple} \in U_6$, and $r_{ord} \in U_7$ to interpret the qualities *Q_nat*, *Q_fins*, *Q_fsys*, the operations *Add* and *Mult*, the simple operation associating V^n to n and the order relation *R_ord* respectively. Then we can suitably extend the action of the fundamental relations so as to satisfy the Axioms Fin and Ind.1-4. Similarly one could model engraftings of *variables* (see [5]), *propositions* (see [7, 8]), *etc.*.

As far as properties of operations are concerned, the model $\mathcal{M}$ is rather "syntactical" in nature. Clearly, the theory GOCI is inconsistent with extensionality of operations, similarly to Oper (see [13]). However several forms of "mild extensionality" could be forced in the model using the technique of [13]. For instance, we could obtain that the collection U_5 of the operations corresponding to correlations is closed under the fundamental operations, hence e.g. composition is associative. We could obtain furthermore that union is idempotent, commutative and associative, that $\chi(SKK) = Id$, $\chi(Cur\ App) = Id \cap (U_5 \cup U_6)^2$, and so on.